**BAULKHAM HILLS HIGH SCHOOL**

**TRIAL 2013**

**YEAR 12 TASK 4**

# Mathematics Extension 2

**General Instructions**

- Reading time – 5 minutes
- Working time – 180 minutes
- Write using black or blue pen
- Board-approved calculators may be used
- Show all necessary working in Questions 11-16
- Marks may be deducted for careless or badly arranged work

**Total marks – 100**
**Exam consists of 11 pages.**

This paper consists of TWO sections.

**<u>Section 1</u> – Page 2-4 (10 marks)**
**Questions 1-10**

- Attempt Question 1-10
- Allow about 15 minutes for this section

**<u>Section II</u> – Pages 5-10 (90 marks)**

- Attempt questions 11-16
- Allow about 2 hours 45 minutes for this section

**Table of Standard Integrals is on page 11**

## Section I - 10 marks

**Allow about 15 minutes for this section**

**Use the multiple choice answer sheet for question 1-10**

**1.** Which of the following is equal to $\cos\theta$ ?

(A) $\dfrac{\sin\theta}{\tan\theta}$

(B) $\sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2}$

(C) $2\cos^2\theta - 1$

(D) $2\cos^2\frac{\theta}{2} + 1$

**2.** In Cartesian form $2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ is

(A) $-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

(B) $-i$

(C) $-\sqrt{2}(1-i)$

(D) $\sqrt{2}(1-i)$

**3** Using an appropriate substitution

$\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2 x}{(1+\tan x)^2}\,dx$ is equivalent to:

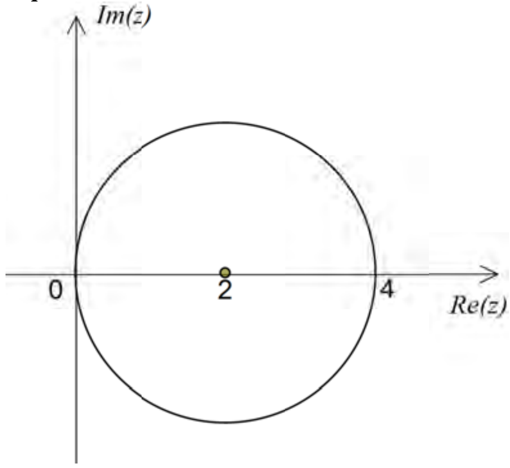
(A) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{u}\,du$

(B) $\displaystyle\int_0^2 \frac{u^2}{(1+u)^3}\,du$

(C) $\displaystyle\int_0^2 \frac{1}{u^2}\,du$

(D) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{u^3}\,du$

**4.** Which of the following is the equation of the circle shown below?




(A) $(z+2)(\bar{z}+2)=4$

(B) $(z-2)(\bar{z}-2)=4$

(C) $(z+2i)(\bar{z}-2i)=4$

(D) $(z+2)(\bar{z}-2)=4$

**5.** Using implicit differentiation on the equation $y^3 = x^2 + xy$, then $\frac{dy}{dx}$ would equal

(A) $\frac{3y^2-2x}{x}$

(B) $\frac{2x+y}{3y^2-x}$

(C) $\frac{2x-y}{3y^2+y}$

(D) $\frac{2x}{3y^2+y}$

**6.** A satellite in a circular orbit around Earth, at a distance of 12000 km from Earth's centre makes 12 revolutions per day. Find the tangential speed of the satellite in km/h.

(A) π

(B) $\frac{72000}{\pi}$

(C) $12000\pi$

(D) $12000\pi^2$

**7.** If $\alpha, \beta$ and $\gamma$ are the roots of the equation $x^3 - 3x + 4 = 0$
Then the cubic with roots $\alpha^2, \beta^2$ and $\gamma^2$ *is*

(A) $8x^3 - 9x + 4 = 0$

(B) $x^3 - 6x^2 + 9x - 16 = 0$

(C) $x^3 + 9x^2 - 12x + 4 = 0$

(D) $8x^3 + 4x^2 - 9x + 16 = 0$

**8.** Given $(2i + 1)$ is a root of the equation $x^3 - 4x^2 + 9x - 10 = 0$ then another root is

(A) 2

(B) 5

(C) $2i - 1$

(D) 10

**9.** $\tan(\cos^{-1} x)$ is equal to

(A) $-\frac{\sqrt{1-x^2}}{x}$

(B) $-\frac{x}{\sqrt{1-x^2}}$

(C) $\frac{\sqrt{1-x^2}}{x}$

(D) $\frac{x}{\sqrt{1-x^2}}$

**10.** $\int x\sqrt{1-x}\,dx$ equals

(A) $-\frac{1}{3}x^2(1-x)^{\frac{3}{2}} + c$

(B) $\frac{1}{3}x^2(1-x)^{\frac{3}{2}} + c$

(C) $-\frac{2}{5}x(1-x)^{\frac{5}{2}} + \frac{2}{3}(1-x)^{\frac{3}{2}} + c$

(D) $\frac{2}{5}(1-x)^{\frac{5}{2}} - \frac{2}{3}(1-x)^{\frac{3}{2}} + c$

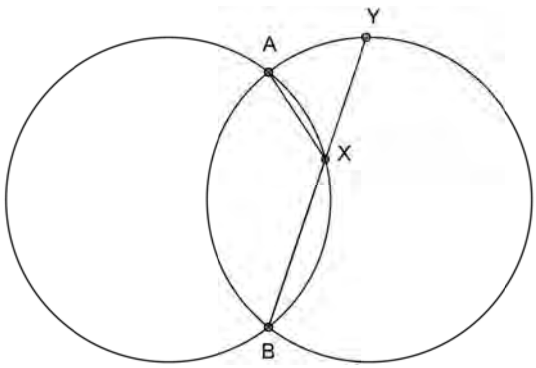
**End of Section 1**

## Section II – Extended Response

**Attempt questions 11-16.**
**Answer each question on a SEPARATE PAGE. Clearly indicate question number.**
**Each piece of paper must show your name.**
**All necessary working should be shown in every question.**

| **Question 11 (15 marks)** | | **Marks** |
|---|---|---|
| a) | Let $z_1 = 3 - 4i$ and $z_2 = -3 + 2i$ | |
| | (i) $z_1 - \overline{z_2}$ | **1** |
| | (ii) $\frac{z_1}{z_2}$ | **2** |
| b) | Given that $(1-2i)^2 = -3-4i$, solve $z^2 - 5z + (7+i) = 0$. | **2** |
| c) | On an Argand diagram, shade the region specified by the conditions $|z - 6 + 5i| \leq 3$ and $\text{Re}(z) \leq 6$. | **2** |
| d) | If $z = a(\cos\theta + i\sin\theta)$ when $a$ and $\theta$ are real, show that $\frac{z}{z^2+a^2}$ is equivalent to $\frac{1}{2a\cos\theta}$ | **3** |
| e) | (i) Prove that if $y = \left(x + \sqrt{1+x^2}\right)^m$ then $\frac{1}{y} \cdot \frac{dy}{dx} = \frac{m}{\sqrt{1+x^2}}$ | **2** |
| | (ii) Show $\frac{d^2y}{dx^2} = \frac{m^2y\sqrt{1+x^2} - myx}{(1+x^2)\sqrt{1+x^2}}$ | **2** |
| | (iii) Prove that $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2y = 0$ | **1** |
| | End of Question 11 | |

| **Question 12 (15 marks)** | | **Marks** |
|---|---|---|
| a) | Find $\int \frac{dx}{(x+1)(x^2+2)}$ | **3** |
| b) | (i) Show that $\log_{ab} x = \frac{\log_a x}{1+\log_a b}$ | **2** |
| | (ii) Hence show that $\log_2 5 = \frac{1-\log_{10} 2}{\log_{10} 2}$ | **1** |
| c) | Consider the curves $$\frac{x^2}{16}+\frac{y^2}{7}=1 \quad \text{and} \quad x^2-\frac{y^2}{8}=1$$ | |
| | (i) Show that both curves have the same focii. | **3** |
| | (ii) Find the equation of the circle that passes through the points of intersection of these two curves. | **3** |
| d) | (i) In how many distinct ways can the letters of the word **A N G L E** be arranged. | **1** |
| | (ii) If these arrangements are listed in alphabetical order, in which place (ie. 1st, 2nd, 3rd, etc... ) is the word ANGLE . | **2** |
| | **End of Question 12** | |

| **Question 13 (15 marks)** | | **Marks** |
|---|---|---|
| a) | $$I_n = \int_0^{\pi} \sin^n x \, dx$$ | |
| | (i) Prove that $I_n = \frac{n-1}{n} I_{n-2}$ | 3 |
| | (ii) Hence evaluate $I_5$ | 2 |
| b) | When a polynomial $P(x)$ is divided by $(x-3)$ and $(x-7)$ the respective remainders are 3 and 5. Find the remainder when $P(x)$ is divided by $(x-3)(x-7)$. | 3 |
| c) | Two circles of equal radii intersect at $A$ and $B$.<br><br>$X$ is a point on the circle between $A$ and $B$ and $BX$ is produced to meet the second circle at $Y$.<br><br><br><br>Copy the diagram in your booklet and prove that $AX = AY$, showing any necessary constructions. | 3 |
| d) | Find the volume of the solid of revolution generated when the area enclosed between the curve $y = 4 - x^2$ and the lines $y = 4$ and $x = 2$ is rotated about the line $x = 2$. | 4 |
| | **End of Question 13** | |

| **Question 14 (15 marks)** | | **Marks** |
|---|---|---|
| a) | A body of unit mass falls under gravity through a resistive medium.<br>The body falls from rest from a cliff 50 metres above the ground.<br>The resistance to its motion is $\frac{v^2}{100}$ where v m $s^{-1}$ is the speed of the body when it has fallen a distance of $x$ metres. | |
| | (i) Show that the equation of the motion is $\ddot{x} = g - \frac{v^2}{100}$ | 1 |
| | (ii) Show that the terminal velocity $V$ of the body is given by<br>$$V = 10\sqrt{g}\ \text{ms}^{-1}$$ | 1 |
| | (iii) Show that $v^2 = V^2\left(1 - e^{-\frac{x}{50}}\right)$. | 3 |
| | (iv) How far has the body fallen when it reaches a velocity of $\frac{V}{2}$. | 2 |
| | (v) Find the velocity reached in terms of the terminal velocity when the body hits the ground. | 2 |
| | (vi) If $v = v_1$ when $x = d$ and $v = v_2$ when $x = 2d$, show that<br>$$v_2^2 = v_1^2\left(2 - \frac{v_1^2}{V^2}\right)$$ | 2 |
| b) | The equation $x^4 - 5x^3 - 9x^2 + ax + b = 0$ has a triple root.<br>Given that this root is an integer: | |
| | (i) find the triple root. | 2 |
| | (ii) find the value of $b$. | 2 |
| | **End of Question 14** | |

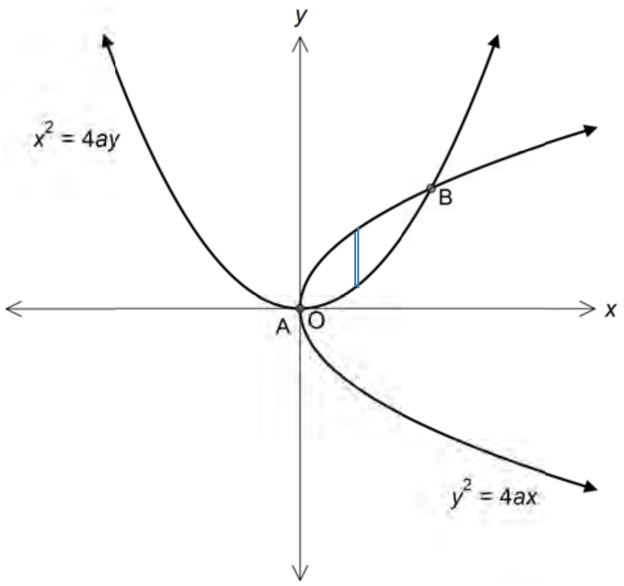
**Question 15 (15 marks)** | **Marks**

a) (i) Prove by mathematical induction that $(1 + x)^n - 1$ is divisible by $x$ for all integers $n \geq 1$. **3**

(ii) By factorising $35^n - 7^n - 5^n + 1$ and using part (i), prove that $35^n - 7^n - 5^n + 1$ is divisible by 24. **2**

b) Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{5 + 4\cos x}$ **4**

c) The base of a certain solid is the region bounded by the curve $y^2 = 4ax$ and $x^2 = 4ay$ and cross sections to the plane perpendicular to the $x$-axis are semi circles.




(i) Show that the two curves intersect at $A(0,0)$ and $B(4a, 4a)$. **1**

(ii) Show that the cross sectional area, $A$, of a typical slice is $A = \frac{\pi}{2}\left(\sqrt{ax} - \frac{x^2}{8a}\right)^2$. **2**

(iii) Hence find the volume of the solid formed. **3**

**End of Question 15**

**Question 16 (15 marks)**

a) $P$ is a point $\left(p, \frac{1}{p}\right)$ on the rectangular hyperbola $xy = 1$.
The line $PO$ is produced to point Q also on the rectangular hyperbola.
A circle centre $P$ and radius $PQ$ is drawn to cut the hyperbola at $A, B, C$ and $Q$.

(i) Prove that the parameters of the points of intersection of the circle and the hyperbola are given by the equation **3**

$$p^2t^4 - 2p^3t^3 - 3(p^4 + 1)t^2 - 2pt + p^2 = 0$$

(ii) Deduce that $t_A + t_B + t_C = 3p$
where $t_A$, $t_B$ and $t_C$ are the parameters at $A, B$ and $C$ **2**

b) (i) Show that

$$(1 + i\tan\theta)^n + (1 - i\tan\theta)^n = \frac{2\cos n\theta}{\cos^n\theta}$$ **2**

where $\cos\theta \neq 0$ and $n$ is a positive integer.

(ii) Hence show that if $z$ is a purely imaginary number, the roots of $(1 + z)^4 + (1 - z)^4 = 0$ are $z = \pm i\tan\frac{\pi}{8}, \pm i\tan\frac{3\pi}{8}$. **3**

c) Consider the sequence defined by

$$V_k = \frac{1}{2k+1} + \frac{1}{2k+2} + \cdots + \frac{1}{3k}$$

where $k$ is a positive integer

(i) Show that $V_k < \frac{1}{2}$ **1**

(ii) Given that $p < x < p + 1$, where $x$ is a real number and $p$ is a positive integer **1**

$$\text{show that } \frac{1}{p+1} < \int_p^{p+1} \frac{dx}{x} < \frac{1}{p}$$

(iii) Hence show that **2**

$$\int_{2k+1}^{3k+1} \frac{dx}{x} < V_k < \int_{2k}^{3k} \frac{dx}{x}$$

(iv) Hence find the limit of $V_k$ as $k \to \infty$ **1**

**End of Paper**

# STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \ if \ \ n < 0$$

$$\int \frac{1}{x} dx = \ln x, \qquad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \ a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln(x + \sqrt{x^2 - a^2}), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2})$$

NOTE: $\ln x = \log_e x, \quad x > 0$

1 A  2 C  3 C  4 B  5 B

6 C  7 B  8 A  9 C  10 D

**Question 11**

a) (i) $z_1 - \overline{z_2}$

$= (3-4i) - (-3-2i)$

$= 6 - 2i$ 1

(ii) $\frac{z_1}{z_2} = \frac{3-4i}{-3+2i} \times \frac{-3-2i}{-3-2i}$ 1

$= \frac{(-9-8)+i(12-6)}{3^2+2^2}$

$= \frac{-17}{13} + \frac{6i}{13}$ 1

b) $z^2 - 5z + (7+i) = 0$

$z = \frac{5 \mp \sqrt{25-4(7+i)}}{2}$

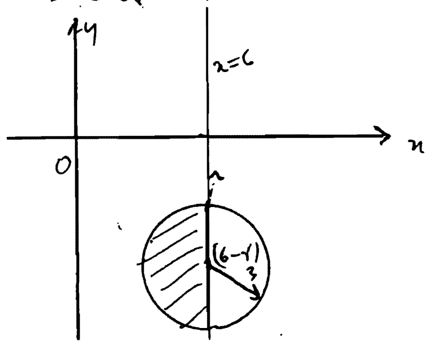
$= \frac{5 \mp \sqrt{-3-4i}}{2}$ 1

$= \frac{5 \mp (1-2i)}{2}$  (as $\sqrt{-3-4i} = 1-2i$)

$= \frac{6-2i}{2}$ or $\frac{4+2i}{2}$

$= 3-i$ or $2+i$ 1

c)






d) $\frac{z}{z^2+a^2} = \frac{a(\cos\theta + i\sin\theta)}{a^2(\cos\theta+i\sin\theta)^2+a^2}$

$= \frac{a(\cos\theta+i\sin\theta)}{a^2(\cos2\theta+i\sin2\theta)+a^2}$

$= \frac{1}{a}\frac{(\cos\theta+i\sin\theta)}{(\cos2\theta+1+i\sin2\theta)}$

1 for using De Moivres or expanding.

now $\cos2\theta = 2\cos^2\theta - 1$

$\therefore \cos2\theta + 1 = 2\cos^2\theta$.

$\therefore \frac{z}{z^2+a^2} = \frac{1}{a}\cdot\frac{(\cos\theta+i\sin\theta)}{(2\cos^2\theta+2i\sin\theta\cos\theta)}$ −1

$= \frac{1}{a}\cdot\frac{(\cos\theta+i\sin\theta)}{2\cos\theta(\cos\theta+i\sin\theta)}$

$= \frac{1}{2a\cos\theta}$ 1

e) i) $y = (x+\sqrt{1+x^2})^m$

$\ln y = m\ln(x+\sqrt{1+x^2})$ 1

$\frac{1}{y}\frac{dy}{dx} = \frac{m}{x+\sqrt{1+x^2}}\cdot\left(1+\frac{x}{\sqrt{1+x^2}}\right)$

$= \left(\frac{m}{x+\sqrt{1+x^2}}\right)\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\right)$

$\therefore \frac{1}{y}\frac{dy}{dx} = \frac{m}{\sqrt{1+x^2}}$ 1

ii) $\frac{dy}{dx} = \frac{my}{\sqrt{1+x^2}}$

$\therefore \frac{d^2y}{dx^2} = \frac{m\frac{dy}{dx}\sqrt{1+x^2} - my\frac{x}{\sqrt{1+x^2}}}{1+x^2}$ 1

$= \frac{m\cdot\frac{my}{\sqrt{1+x^2}}\cdot\sqrt{1+x^2} - \frac{myx}{\sqrt{1+x^2}}}{1+x^2}$

$$= \frac{n^2 y\sqrt{1+x^2} - myx}{\sqrt{1+x^2}\,(1+x^2)}$$

$=$

ii) $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - m^2 y = (1+x^2)\left(\frac{m^2 y\sqrt{1+x^2} - myx}{\sqrt{1+x^2}\,(1+x^2)}\right) + \frac{mxy}{\sqrt{1+x^2}} - m^2 y$

$$= m^2 y - \frac{myx}{\sqrt{1+x^2}} + \frac{myx}{\sqrt{1+x^2}} - m^2 y$$

$$= 0$$

Question 12

a) $\int \frac{dx}{(x+1)(x^2+1)}$

let $\frac{1}{(x+1)(x^2+1)} = \frac{a}{x+1} + \frac{bx+c}{x^2+2}$

$a(x^2+2) + (x+1)(bx+c) = 0$

$x=-1 \Rightarrow \quad 3a = 1$

$a = \frac{1}{3}$ *

coeff of $x^2$ $\quad a+b=0$

$\therefore b = -\frac{1}{3}$ *

$x=0 \quad 2a + c = 1$

$\therefore c = \frac{1}{3}$ *

$$I = \frac{1}{3}\int\frac{dx}{x+1} - \frac{1}{3}\int\frac{x\,dx}{x^2+2} + \frac{1}{3}\int\frac{dx}{x^2+2}$$

$$= \frac{1}{3}\ln(x+1) - \frac{1}{6}\ln(x^2+2) + \frac{1}{3\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + c$$

b) i) $\log_{ab} x = \frac{\log_a x}{\log_a ab}$

$$= \frac{\log_a x}{\log_a a + \log_a b}$$

$$= \frac{\log_a x}{1 + \log_a b}$$

ii) let $a=2$ $b=5$ $x=2$

$$\therefore \log_{10} 2 = \frac{\log_2 2}{1+\log_2 5}$$

$$1 + \log_2 5 = \frac{1}{\log_{10} 2}$$

$$\log_2 5 = \frac{1}{\log_{10} 2} - 1 = \frac{1-\log_{10} 2}{}$$

12c

4(c)

1) for ellipse $a^2=16$ $b^2=7$

$\therefore b^2=a^2(1-e^2)$

$7=16(1-e^2)$

$16e^2=9$

$e=\frac{3}{4}$ — 1

foci $(\mp ae, 0)$

$\therefore (\mp 4\cdot\frac{3}{4}, 0)$

$\therefore (\mp 3, 0)$ — 1

for hyp $a^2=1$ $b^2=8$

$\therefore b^2=a^2(e^2-1)$

$8=e^2-1$

$e^2=9$

$e=3$ — 1

foci is $(\mp ae, 0)$

i.e. $(\pm 3, 0)$

$\therefore$ both foci same

ii) $\frac{x^2}{16}+\frac{y^2}{7}=1$ — ①

$x^2-\frac{y^2}{8}=1$ — ②

$7x^2+16y^2=16\times 7$ — ③

$8x^2-y^2=8$ — ④

④ × 16 ⟹ $16\times 8x^2-16y^2=8\times 16$ — ⑤

③ + ⑤ ⟹ $(7+16\times 8)x^2=15\times 16$

$\therefore 135x^2=15\times 16$

$9x^2=16$

$x^2=\frac{16}{9}$ — 1

subst → ② $\therefore \frac{y^2}{8}=\frac{16}{9}-1$

$y^2=\frac{7}{9}\times 8$

$=\frac{56}{9}$ — 1

by symmetry ⊙ $x^2+y^2=r^2$

$x^2+y^2=\frac{16}{9}+\frac{56}{9}$

$=\frac{72}{9}$ — 1

$x^2+y^2=8$

d i) no of ways $= 5! = 120$ 1

ii) let A = 1, E = 2, G = 3, L = 4, N = 5

∴ ANGLE ≡ 15342

there must be after all numbers starting with 1 and 2nd no 1, 2, 3, 4

$= 3 \times 3! = 18$

1 mark for progression to answer.

∴ AFTER THIS NUMBERS ARE 15234
15243
15324
15342

i.e. 4 more

∴ number is 22

∴ ANGLE IS IN 22nd place 2

13a) let $u = \sin^{n-1}x$ $v' = \sin x$

$u' = (n-1)\sin^{n-2}x\cos x$ $V = -\cos x$

$\therefore I_n = \left[-\cos x \sin^{n-1}x\right]_0^{\pi} + (n-1)\int_0^{\pi} \sin^{n-2}x\cos^2 x\,dx$ 1

$= 0 + (n-1)\int_0^{\pi} \sin^{n-2}x(1-\sin^2 x)\,dx$ 1

$= (n-1)\int_0^{\pi}(\sin^{n-2}x - \sin^n x)\,dx$

$= (n-1)I_{n-2} - (n-1)I_n$

$\therefore I_n = \frac{n-1}{n} I_{n-2}$ 1

$I_5 = \frac{4}{5} I_3$

$I_3 = \frac{2}{3} I_1$

$I_1 = \int_0^{\pi} \sin x\,dx = 2$ 1

$\therefore I_5 = \frac{4}{5} \cdot \frac{2}{3} \cdot 2 = \frac{16}{15}$ 1

b) $P(3) = 3$

$P(7) = 5$

$P(x) = Q(x)(x-3)(x-7) + ax + b$ 1

$P(3) = 0 + 3a + b = 3$

$P(7) = 0 + 7a + b = 5$ 1

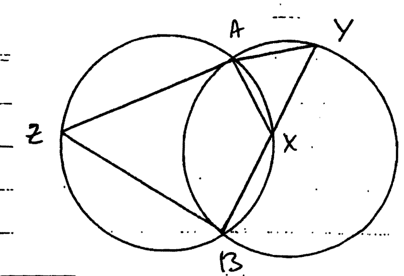
$\therefore 4a = 2$

$a = \frac{1}{2}$ 1

$b = \frac{3}{2}$

$\therefore R(x) = \frac{x}{2} + \frac{3}{2}$

c)




Construct AZ and ZB where Z is on circle

$\angle AZB = \angle AXY$ (EXTERIOR ANGLE OF A CYCLIC QUAD = THE INTERIOR OPPOSITE) 1

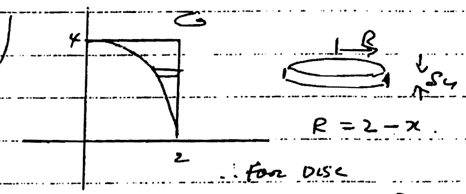
$\angle AZB = \angle AYB$ (ANGLES SUBTENDED BY EQUAL ARCS IN CIRCLES OF EQUAL RADII)

$\therefore \angle AXY = \angle AYB$ 1

∴ △AXY IS ISOSCELES

∴ AX = AY (SIDES OPPOSITE EQUAL ANGLES IN ISOSCELES TRIANGLE) 1

d)




$R = 2 - x$

∴ for disc

$\delta V = \pi(2-x)^2\delta y$ 1

$= \pi(4 - 4x + x^2)\,\delta x$

now $x^2 = 4 - y$

$\therefore \delta V = \pi(4 - 4\sqrt{4-y} + 4 - y)$

$= \pi(8 - y - 4\sqrt{4-y})$ 1

$V = \sum_0^4 \delta V$

$= \pi\int_0^4 (8 - y - 4\sqrt{4-y})\,dy$

$= \pi\left[8y - \frac{y^2}{2} + \frac{8}{3}(4-y)^{\frac{3}{2}}\right]_0^4$

$= \frac{8\pi}{3}$ cu units 1

14a

i) $ma = mg - \frac{mv^2}{100}$ ↓ mg ↑ $\frac{mv^2}{100}$

$\therefore a = \ddot{x} = g - \frac{v^2}{100}$ 1

ii) TV when $a = 0$

$\therefore \frac{V^2}{100} = g$

$\therefore V = 10\sqrt{g}\ ms^{-1}$. 1

iii) $\frac{v\,dv}{dx} = g - \frac{v^2}{100}$

$\int \frac{100v}{100g - v^2}\,dv = \int dx$ $\quad V^2 = 100g$.

$\therefore x = -50\ln(V^2 - v^2) + c$ 1

at $t = 0$ $v = 0$

$\therefore c = 50\ln V^2$

$\therefore \frac{-x}{50} = \ln\left(\frac{V^2 - v^2}{V^2}\right)$

$\frac{V^2 - v^2}{V^2} = e^{-\frac{x}{50}}$ 1

$V^2 - v^2 = V^2 e^{-\frac{x}{50}}$

$\therefore v^2 = V^2 - V^2 e^{-\frac{x}{50}}$

$v^2 = V^2\left(1 - e^{-\frac{x}{50}}\right)$ 1

iv) subst $v = \frac{V}{2}$ into $v^2$

$\therefore \frac{V^2}{4} = V^2\left(1 - e^{-\frac{x}{50}}\right)$ 1

$\therefore e^{-\frac{x}{50}} = \frac{3}{4}$

$\therefore \frac{-x}{50} = \ln\frac{3}{4}$

$\therefore x = 50\ln\frac{4}{3}$. 1

v) subst $x = 50$ in $v^2$ 1

$\therefore v^2 = V^2(1 - e^{-1})$

$\therefore v = V\sqrt{\frac{e-1}{e}}$. 1

vi) $v_1^2 = V^2\left(1 - e^{-\frac{d}{50}}\right)$ $\quad v = v_1\ x = d$

$v_2^2 = V^2\left(1 - e^{-\frac{2d}{50}}\right)$ $\quad v = v_2\ x = 2d$

$= V^2\left(1 - \left(e^{-\frac{d}{50}}\right)^2\right)$

$\therefore \frac{v_2^2}{v_1^2} = \frac{V^2\left(1 - e^{-\frac{d}{50}}\right)\left(1 + e^{-\frac{d}{50}}\right)}{V^2\left(1 - e^{-\frac{d}{50}}\right)}$ 1

$\therefore v_2^2 = v_1^2\left(1 + e^{-\frac{d}{50}}\right)$

now $\frac{v_1^2}{V^2} = 1 - e^{-\frac{d}{50}}$

$\therefore e^{-\frac{d}{50}} = 1 - \frac{v_1^2}{V^2}$

$\therefore v_2^2 = v_1^2\left(1 + 1 - \frac{v_1^2}{V^2}\right)$

$= v_1^2\left(2 - \frac{v_1^2}{V^2}\right)$ 1

(4b) $P(x) = x^4 - 5x^3 - 9x^2 + ax + b$

$P'(x) = 4x^3 - 15x^2 - 18x + a$

$P''(x) = 12x^2 - 30x - 18$. 1

for triple root $P''(x) = 0$

$\therefore 6(2x+1)(x-3) = 0$

$\therefore x = 3$ is triple root 1

now $\sum\alpha = 3 + 3 + 3 + \beta = 5$ $\therefore \beta = -4$ 1

$\alpha^3\beta = b$

$\therefore 3^3 \cdot -4 = b$

$\therefore b = -108$. 1

15a): Test for $n = 1$

LHS $= (1+x)^1 - 1$

$= x$ which is divisible by $x$.

$\therefore$ true for $n = 1$

Assume that $(1+x)^k - 1$ is divisible by $x$ for $n = k$.

ie assume $(1+x)^k - 1 = x\,P(x)$ where $P(x)$ is a polynomial

must prove that $(1+x)^{k+1} - 1$ is divisible by $x$.

$(1+x)^{k+1} - 1 = (1+x)(1+x)^k - 1$

$= (1+x)^k + x(1+x)^k - 1$

$= \left[(1+x)^k - 1\right] + x(1+x)^k$

as $\left[(1+x)^k - 1\right]$ is divisible by $x$ by assumption

and $x(1+x)^k$ is " " ".

then $(1+x)^{k+1} - 1$ is divisible by $x$.

$\therefore$ true for $n = k+1$.

Since the result is true for $n = 1$

from assumption it is true for $n = 1+1 = 2$

and then for $n = 3$ and so on for all $n \geq 1$.

Marking: 4 components

- test $n=1$
- uses assumption
- determines divisibility
- correct structure

All 4 correct 3 marks

3 correct 2 marks

2 correct 1 mark

0 otherwise

15a ii)

$35^n - 7^n - 5^n + 1$

$= 7^n \cdot 5^n - 7^n - 5^n + 1$

$= 7^n(5^n - 1) - (5^n - 1)$

$= (7^n - 1)(5^n - 1)$. 1

for $(7^n - 1)$ let $x = 6$

$\therefore ((1+6)^n - 1)$ is divisible by 6 from i)

similar $5^n$ is divisible by 4 for ii)

$\therefore (7^n - 1)(5^n - 1)$ is divisible by $6 \times 4 = 24$. 1

15b) $\int_0^{\frac{\pi}{2}} \frac{dx}{5 + 4\cos x}$ let $t = \tan \frac{\theta}{2}$ 1

$\therefore dx = \frac{2dt}{1+t^2}$

$\therefore I = \int_0^1 \frac{dx}{5 + \frac{4(1-t^2)}{1+t^2}} \cdot \frac{2dt}{1+t^2}$ 1 when $x = \frac{\pi}{2}$ $t = 1$, $x = 0$ $t = 0$

$= \int_0^1 \frac{2dt}{5+5t^2+4-4t^2}$.

$= \int_0^1 \frac{2}{9+t^2} dt$

$= \frac{2}{3} \tan^{-1} \frac{t}{3}$ 1

$= \frac{2}{3} \tan^{-1} \frac{1}{3} - \frac{2}{3} \tan^{-1} 0$ 1

$= \frac{2}{3} \tan^{-1} \frac{1}{3}$.

15 c) i) $x^2 = 4ay$

$y^2 = 4ax$

$\therefore \frac{x^4}{16a^2} = 4ax$

$x^4 - 64a^3x = 0$

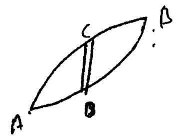
$x(x^3 - 64) = 0$

$\therefore x = 0$ or $4a$

$y = 0$ or $4a$

$\therefore$ A is $(0,0)$ B is $(4a, 4a)$ 1

ii)



$CD = 2\sqrt{ax} - \frac{x^2}{4a}$ (diameter of semi ⊙) 1

$\therefore A(x) = \frac{\pi r^2}{2}$

where $r = \frac{CD}{2}$

$\therefore A = \frac{\pi}{2}\left(\sqrt{ax} - \frac{x^2}{8a}\right)^2$ 1

iii) $\delta V = A\,\delta x$

$\delta V = \frac{\pi}{2}\left(\sqrt{ax} - \frac{x^2}{8a}\right)^2 \delta x$

$\therefore V = \sum_{0}^{4a} \delta V$

$= \frac{\pi}{2}\int_0^{4a}\left(\sqrt{ax} - \frac{x^2}{8a}\right)^2 dx$

$= \frac{\pi}{2}\int_0^{4a}\left(ax - \frac{x^{5/2}}{4a^{1/2}} + \frac{x^4}{64a^2}\right) dx$ 1

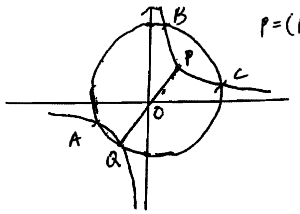
$= \frac{\pi}{2}\left[\frac{ax^2}{2} - \frac{x^{7/2}}{14a^{1/2}} + \frac{x^5}{5 \times 64a^2}\right]_0^{4a}$ 1

$= \frac{\pi}{2}\left[8a^3 - \frac{64 \cdot 2a^3}{14} + \frac{4^3 \times 4^2 a^3}{5 \times 64}\right] - 0$

$= \frac{\pi}{2} a^3\left[8 - \frac{64}{7} + \frac{16}{5}\right]$ 1

$= \frac{36}{35}\pi a^3$ cu unit

(a)




$P=(p,\frac{1}{p})$

) $y=\frac{1}{x}$ is an odd function.
It has 'half turn' symmetry.

$\therefore OP = OQ$

$Q = (-p, -\frac{1}{p})$

$r = PQ = 2OP$

$= 2\sqrt{p^2+\frac{1}{p^2}}$

$= 2\sqrt{\frac{p^4+1}{p^2}}$

$r^2 = \frac{4(p^4+1)}{p^2}$

equation of circle centre P, radius = r is

$(x-p)^2+(y-\frac{1}{p})^2 = 4\left(\frac{p^4+1}{p^2}\right)$ 1 mark

If a point $(x,y)$ on the circle also lies on the hyperbola $y=\frac{1}{x}$ then $(x,y) = (t,\frac{1}{t})$, for some values of t and $(t,\frac{1}{t})$ also satisfies the equation of the circle.

$\therefore (t-p)^2+(\frac{1}{t}-\frac{1}{p})^2 = \frac{4(p^4+1)}{p^2}$ 1 mark

$t^2+p^2-2tp+\frac{1}{t^2}+\frac{1}{p^2}-\frac{2}{tp} = \frac{4(p^4+1)}{p^2}$

$t^2-2pt+(p^2+\frac{1}{p^2})-\frac{2}{pt}+\frac{1}{t^2} = 4\left(\frac{p^4+1}{p^2}\right)$

$t^2-2tp-3\left(\frac{p^4+1}{p^2}\right)-\frac{2}{pt}+\frac{1}{t^2}=0$

$\times p^2 t^2$,

$p^2t^4-2p^3t^3-3(p^4+1)t^2-2pt+p^2=0$ 1 mark

) $t_A+t_B+t_C+t_Q = \frac{2p^3}{p^2}$ 1 mark

$t_A+t_B+t_C-p = 2p$

$t_A+t_B+t_C = 3p$ 1 mark

(i) LHS $= (1+\frac{i\sin\theta}{\cos\theta})^n+(1-\frac{i\sin\theta}{\cos\theta})^n$

$= \frac{(\cos\theta+i\sin\theta)^n}{\cos^n\theta}+\frac{(\cos\theta-i\sin\theta)^n}{\cos^n\theta}$ 1 mark

$= \frac{\cos n\theta+i\sin n\theta}{\cos^n\theta}+\frac{(\cos(-\theta)+i\sin(-\theta))^n}{\cos^n\theta}$

$= \frac{\cos n\theta+i\sin n\theta}{\cos^n\theta}+\frac{\cos(-n\theta)+i\sin(-n\theta)}{\cos^n\theta}$

$= \frac{\cos n\theta+i\sin n\theta+\cos n\theta-i\sin n\theta}{\cos^n\theta}$

$= \frac{2\cos n\theta}{\cos^n\theta}$ $\cos\theta\neq0$ 1 mark

(ii) If $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$
the range of $\tan\theta$ is all real values
Hence any imaginary number z can be written as $i\tan\theta$, $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$

$(1+z)^4+(1-z)^4=0$

$\Leftrightarrow (1+i\tan\theta)^4+(1-i\tan\theta)^4=0$

$\Leftrightarrow \frac{2\cos4\theta}{\cos^4\theta}=0$ 1 mark

if $\cos4\theta=0$ $-2\pi<4\theta<2\pi$

$4\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$ 1 mark

$\theta = \pm\frac{\pi}{8}, \pm\frac{3\pi}{8}$

$\therefore z = i\tan\pm\frac{\pi}{8}$, $i\tan\pm\frac{3\pi}{8}$

$\tan-\frac{\pi}{8} = -\tan\frac{\pi}{8}$

$\tan-\frac{3\pi}{8} = -\tan\frac{3\pi}{8}$

$\therefore z = \pm i\tan\frac{\pi}{8}$, $\pm i\tan\frac{3\pi}{8}$ 1 mark

C(i) $V_k = \frac{1}{2k+1}+\frac{1}{2k+2}+\cdots+\frac{1}{3k}$

$2k<2k+1<2k+2<\cdots<3k$

$\Rightarrow \frac{1}{3k}<\frac{1}{3k-1}<\cdots<\frac{1}{2k+2}<\frac{1}{2k+1}<\frac{1}{2k}$

$\therefore V_k < \frac{1}{2k}+\frac{1}{2k}+\cdots+\frac{1}{2k}$ (k terms)

$= \frac{1}{2k}\times k$

$= \frac{1}{2}$ 1 mark.

(ii) $0<p<x<p+1$

$\Rightarrow \frac{1}{p+1}<\frac{1}{x}<\frac{1}{p}$

$\therefore \int_p^{p+1}\frac{1}{p+1}dx < \int_p^{p+1}\frac{1}{x}dx < \int_p^{p+1}\frac{1}{p}dx$

$\left[\frac{x}{p+1}\right]_p^{p+1} < \int_p^{p+1}\frac{1}{x}dx < \left[\frac{x}{p}\right]_p^{p+1}$

$\frac{1}{p+1} < \int_p^{p+1}\frac{1}{x}dx < \frac{1}{p}$ 1 mark

(iii) Sub. $p=2k+1$ into $\int_p^{p+1}\frac{dx}{x}<\frac{1}{p}$

We get $\int_{2k+1}^{2k+2}\frac{dx}{x}<\frac{1}{2k+1}$

Sub. $p=2k+2$ $\int_{2k+2}^{2k+3}\frac{dx}{x}<\frac{1}{2k+2}$

$\vdots$

Sub. $p=3k$ $\int_{3k}^{3k+1}\frac{dx}{x}<\frac{1}{3k}$

Adding $\int_{2k+1}^{3k+1}\frac{dx}{x}<\frac{1}{2k+1}+\cdots+\frac{1}{3k}$

$\therefore \int_{2k+1}^{3k+1}\frac{dx}{x}<V_k$ 1 mark

Sub. $p=2k$ into $\frac{1}{p+1}<\int_p^{p+1}\frac{dx}{x}$

We get $\frac{1}{2k+1}<\int_{2k}^{2k+1}\frac{dx}{x}$

Sub $p=2k+1$, $\frac{1}{2k+2}<\int_{2k+1}^{2k+2}\frac{dx}{x}$

$\vdots$

Sub. $p=3k-1$, $\frac{1}{3k}<\int_{3k-1}^{3k}\frac{dx}{x}$

Adding, $\frac{1}{2k+1}+\frac{1}{2k+2}+\cdots+\frac{1}{3k}<\int_{2k}^{3k}\frac{dx}{x}$

i.e. $V_k<\int_{2k}^{3k}\frac{dx}{x}$ 1 mark

$\therefore \int_{2k+1}^{3k+1}\frac{dx}{x}<V_k<\int_{2k}^{3k}\frac{dx}{x}$

(iv) $\therefore \ln\frac{3k+1}{2k+1}<V_k<\ln\frac{3k}{2k}$

$\ln\frac{3+\frac{1}{k}}{2+\frac{1}{k}}<V_k<\ln\frac{3}{2}$

as $k\to\infty$ $\ln\frac{3+\frac{1}{k}}{2+\frac{1}{k}}\to\ln\frac{3}{2}$

$\therefore \lim_{k\to\infty}V_k=\ln\frac{3}{2}$ 1 mark