**BAULKHAM HILLS HIGH SCHOOL**

**2014**

**YEAR 12**

**TERM 3 TRIAL ASSESSMENT TASK**

# Mathematics Extension 2

**General Instructions**

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All relevant mathematical reasoning and/or calculations must be shown

**Total marks – 100**

**Section I** (Pages 2-5)

**10 marks**

Attempt Questions 1-10

Allow about 15 minutes for this section

**Section II** (Pages 6-11)

**90 marks**

Attempt Questions 11-16

Allow about 2 hours 45 minutes for this section

## Section I

**10 marks**
**Attempt questions 1-10**
**Allow about 15 minutes for this section**

Use the multiple choice answer sheet for Questions 1-10.

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1. $z = a + ib$ , where $a \neq 0$ and $b \neq 0$ , which of the following statements is false

   (A) $z - \overline{z} = 2bi$
   (B) $|z|^2 = |z||\overline{z}|$
   (C) $|z| + |\overline{z}| = |z + \overline{z}|$
   (D) $\arg(z) + \arg(\overline{z}) = 0$

2. The number of points of intersection of the graphs $y = |x|$ and $y = \left|x^2 - 4\right|$ is

   (A) 0

   (B) 1

   (C) 2

   (D) 4

3. The equation $48x^3 - 64x^2 + 25x - 3 = 0$ has roots $\alpha, \beta, \gamma$ .
   If $\alpha = \beta\gamma$ , a possible value of $\alpha$ is

   (A) $-\frac{1}{2}$

   (B) $\frac{1}{4}$

   (C) $\frac{1}{2}$

   (D) $\frac{3}{8}$

4. The substitution $t = \tan\frac{\theta}{2}$ is used to find $\int \frac{d\theta}{\cos\theta}$.

Which of the following gives the correct expression for the required integral?

(A) $\int \frac{dt}{2(1-t^2)}$

(B) $\int \frac{2tdt}{(1-t^2)}$

(C) $\int \frac{2dt}{(1-t^2)}$

(D) $\int \frac{4tdt}{(1+t^2)^2}$

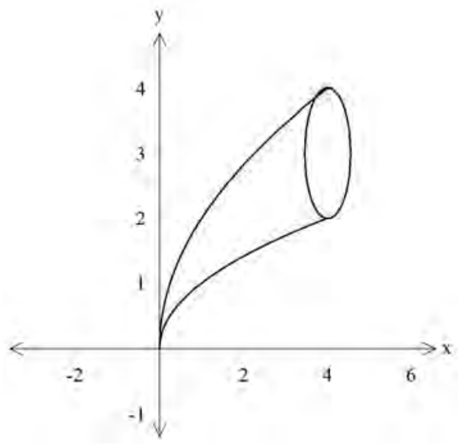
5. The cross section perpendicular to the $x$ axis between two curves $y = \sqrt{x}$ and $y = 2\sqrt{x}$ is a circle. If the curves are drawn between $x = 0$ and $x = 4$, the volume of the resulting horn is given by

(A) $\int_0^4 \sqrt{x}\ dx$

(B) $\int_0^4 \pi x\ dx$

(C) $\int_0^4 \frac{\pi x}{2}\ dx$

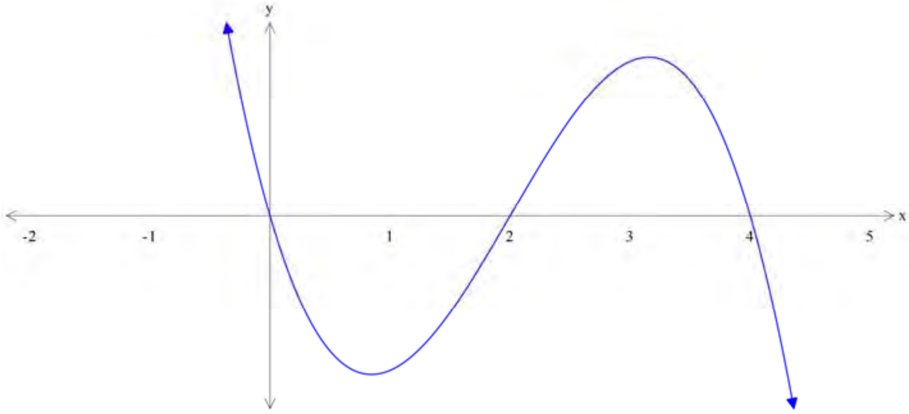
(D) $\int_0^4 \frac{\pi x}{4}\ dx$




6. An ellipse has the equation $\frac{x^2}{100} + \frac{y^2}{36} = 1$. The distance between the foci is:

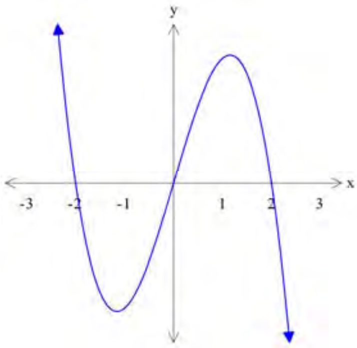
(A) 8
(B) 16
(C) 20
(D) 25

7. The graph of $y = f(x)$ is shown below.

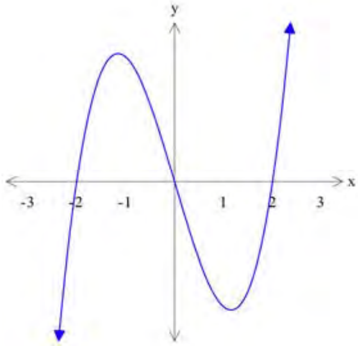



The graph of $y = f(2 - x)$ is:

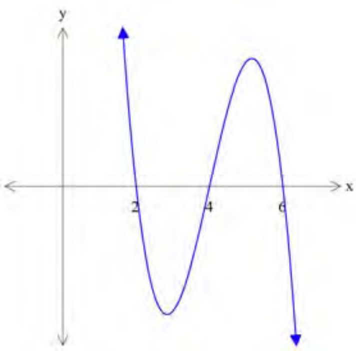
(A)



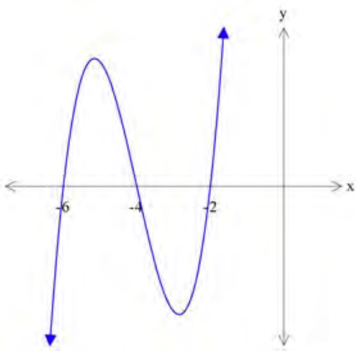

(B)




(C)




(D)




8. If a particle moves in a straight line so that its velocity at any particular time is given by $v = \sin^{-1} x$ , then the acceleration is given by

(A) $-\cos^{-1} x$

(B) $\cos^{-1} x$

(C) $-\dfrac{\sin^{-1} x}{\sqrt{1 - x^2}}$

(D) $\dfrac{\sin^{-1} x}{\sqrt{1 - x^2}}$

9. Given that $\frac{dy}{dx} = y^2 + 1$ and at $x = 0$, $y = 1$, then

(A) $y = y^2x + x + 1$

(B) $y = \tan\left(x + \frac{\pi}{4}\right)$

(C) $y = \tan\left(x - \frac{\pi}{4}\right)$

(D) $y = \log_e\left(\frac{y^2 + 1}{2}\right)$

10. From a set of $n$ objects of which two are white and the rest are black, four objects are to be chosen at random without replacement.
The probability that both white objects will be chosen is twice the probability that neither white will be chosen.

The total number of objects is:

(A) 5
(B) 7
(C) 8
(D) 9

**End of Section I**

**Section II**

**90 marks**
**Attempt questions 11-16**
**Allow about 2hours and 45 minutes for this section**

Answer each question on the appropriate pages of your answer booklet. Extra pages are available.
In questions 11-16, your responses should include relevant mathematical reasoning and/or calculations.

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**Question 11** (15 marks) Answer on the appropriate page in your answer booklet

(a) Let $z = 1 - 3i$ and $w = 2 + i$

(i) Express $zw$ in the form $a + ib$ **1**

(ii) Express $zw$ in modulus- argument form **2**

(iii) Hence find $x$ if $\dfrac{\sqrt{2}\,(\cos x - i\sin x)}{2 + i} = \dfrac{1 - 3i}{5}$ **2**

(b) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos^2 x}\,dx$ **3**

(c) Sketch the region on the Argand diagram defined by: **3**

$$-\frac{\pi}{2} \le \arg(z - 1 - i) \le \pi \text{ and } |z| \le \sqrt{2}$$

(d) (i) Find the constants $A$,$B$ and $C$ such that **2**

$$\frac{x^2 + x + 1}{(x + 2)\left(x^2 + 2x + 3\right)} = \frac{A}{x + 2} + \frac{Bx + C}{x^2 + 2x + 3}$$

(ii) Hence find $\displaystyle\int \frac{x^2 + x + 1}{(x + 2)\left(x^2 + 2x + 3\right)}$ **2**

**End of Question 11**

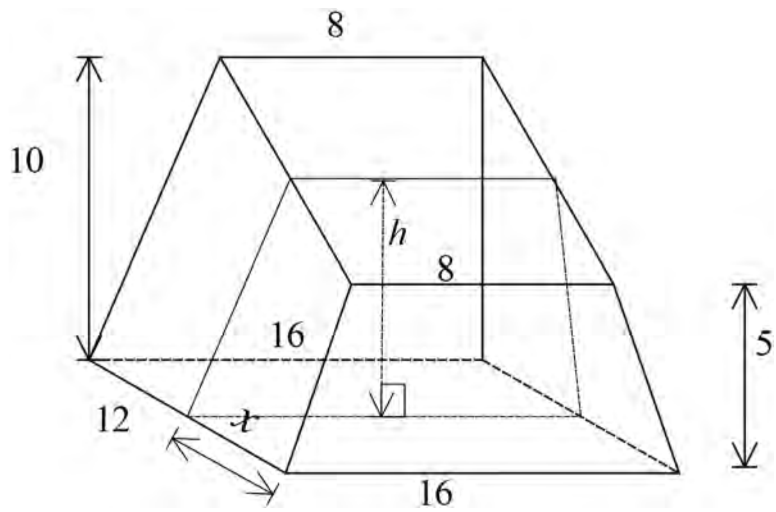
**Question 12** (15 marks) Answer on the appropriate page in your answer booklet

(a) Find $\int x(x+1)^{10}\,dx$ **3**

(b) Find $\int \dfrac{e^x + e^{2x}}{1 + e^{2x}}\,dx$ **3**

(c) Find the equations of the tangents to the curve $x^2 + y^2 = xy + 3$ when $x = 1$. **4**

(d) A stone monument has a rectangular base measuring 16 metres by 12 metres. The cross section formed by any slice perpendicular to the base is a trapezium with top edge 8 metres and bottom edge 16 metres. It is 5 metres high at the front and 10 metres high at the back.




(i) Show that the area of a trapezium at a distance of $x$ metres from the front of the stone monument is $A = 5x + 60$. **3**

(ii) Find the volume of the stone monument. **2**

**End of Question 12**

**Question 13** (15 marks) Answer on the appropriate page in your answer booklet

(a) Sketch the following curves on separate diagrams. There is no need to use calculus.

(i) $y = x^2(x + 3)$ **2**

(ii) $y = \left| x^2(x + 3) \right|$ **2**

(iii) $y = \dfrac{1}{x^2(x + 3)}$ **2**

(iv) $y = \sqrt{x^2(x + 3)}$ **2**

(v) $y = 2\ln|x| + \ln(x + 3)$ **2**

(b) Consider the polynomial $P(z) = z^3 + az^2 + bz + c$ where $a,b$ and $c$ are all real.

If $P(\alpha i) = 0$ where $\alpha$ is real and non zero:

(i) Explain why $P(-\alpha i) = 0$ **1**

(ii) Show that $P(z)$ has only one real zero. **1**

(iii) Hence show that $c=ab$ where $b>0$ **3**

**End of Question 13**

**Question 14** (15 marks) Answer on the appropriate page in your answer booklet

(a) If $\omega$ is a complex cube root of unity, prove that: **2**

$$(a+b)(a+\omega b)(a+\omega^2 b) = a^3 + b^3$$

(b) Factorise $P(x) = x^6 - 3x^2 + 2$ over the field of complex numbers given that it has two roots of multiplicity of at least two. **3**

(c) $P$, $Q$ and $R$ represent the complex numbers $z_1$, $z_2$ and $z_3$. If $z_1 - z_2 = i(z_3 - z_2)$ sketch a diagram and discuss the geometric properties of $\Delta PQR$, giving reasons for your answer. **2**

(d) (i) Prove that $\dfrac{x^2}{(x^2+1)^{n+1}} = \dfrac{1}{(x^2+1)^n} - \dfrac{1}{(x^2+1)^{n+1}}$ **1**

(ii) Given $I_n = \displaystyle\int_0^1 \frac{1}{(x^2+1)^n}\,dx$, prove that $2nI_{n+1} = 2^{-n} + (2n-1)I_n$ **2**

(iii) Hence evaluate $\displaystyle\int_0^1 \frac{1}{(x^2+1)^3}\,dx$ **2**

(e) Prove by mathematical induction $\displaystyle\sum_{r=1}^{n} r \log\frac{r+1}{r} = \log\frac{(n+1)^n}{n!}$ for $n \geq 1$. **3**

**End of Question 14**

**Question 15** (15 marks) Answer on the appropriate page in your answer booklet

(a) The locus of a point is defined by the equation $|z-2| = 2\text{Re}\left(z - \frac{1}{2}\right)$.

(i) If $z = x + iy$ explain why $x \geq \frac{1}{2}$. **1**

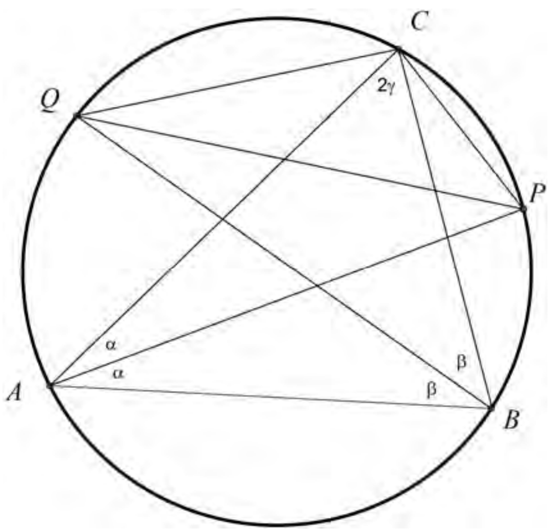
(ii) Show that the locus is a branch of the hyperbola $3x^2 - y^2 = 3$. **2**

(iii) Sketch the locus showing its asymptotes and vertex. **2**

(b) (i) Prove that the equation of the normal to the curve $xy = c^2$ at the point $P\left(cp, \frac{c}{p}\right)$ is given by $p^3x - py = c\left(p^4 - 1\right)$. **2**

(ii) The normal at $P$ meets the $x$ axis at $M$, and the tangent at $P$ meets the $y$ axis at $N$. Prove that the locus of the midpoint of $MN$ as $P$ varies is given by $2c^2xy = c^4 - y^4$. **3**

(c) $AB$ is a fixed chord of a circle and $C$ is a variable point on the major arc $AB$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the circle again at $P$ and $Q$ respectively.




Let $\angle CAB = 2\alpha$, $\angle ABC = 2\beta$, and $\angle BCA = 2\gamma$.

(i) Show that $\angle PCQ = \alpha + \beta + 2\gamma$ **1**

(ii) Hence explain why the length of $PQ$ is constant. **2**

(iii) Use the sine rule to show that $\frac{\text{AB}}{\text{PQ}} = 2\sin\gamma$ **2**

**End of Question 15**

**Question 16** (15 marks) Answer on the appropriate page in your answer booklet

(a) A food parcel is dropped vertically from a helicopter which is hovering 2000 metres above a group of stranded bushwalkers. After 10 seconds a parachute opens automatically. Air resistance is neglected for the first 10 seconds but then the effect of the open parachute is to supply a resistance of $2Mv$ newtons where $M$kg is the mass of the parcel plus parachute and $V$ ms$^{-1}$ is the velocity after $t$ seconds ( $t \geq 10$ seconds).

Take the position of the helicopter to be the origin, the downwards direction as positive and the value of $g$, the acceleration due to gravity, as 10 ms$^{-2}$.

(i) Use calculus to find the equations of motion in terms of $t$ for the parcel before the parachute opens and prove that the velocity at the end of 10 seconds is 100 $ms^{-1}$ and the distance fallen at the end of 10 seconds is 500 metres. **2**

(ii) Show that the velocity of the parcel after the parachute opens is given by **3**

$$v = 5 + 95e^{-2(t-10)} \quad \text{for} \quad t \geq 10$$

(iii) Find $x$, the distance fallen as a function of $t$ and calculate the height of the parcel above the bushwalkers 2 minutes after it leaves the helicopter. **2**

(iv) Calculate the terminal velocity of the parcel. **1**

(v) Draw a neat sketch of velocity against time from the instant the parcel leaves the helicopter. **2**

(b) Find $\prod_{r=0}^{\infty}\left(\cos\frac{\pi}{2^r} + i\sin\frac{\pi}{2^r}\right)$ given that $\prod_{j=1}^{10}$ means the product of terms from $j$=1 to $j$ =10. **2**

(c) Form a polynomial of the least possible positive degree with integer coefficients and one root of which is $\sqrt[3]{12} + \sqrt[3]{18}$ **3**

**END OF PAPER**

# EXTENSION 2 2014 TRIAL SOLUTIONS

1. (C) $|a+ib| + |a-ib|$

$= \sqrt{a^2+b^2} + \sqrt{a^2+b^2}$

$= 2\sqrt{a^2+b^2}$

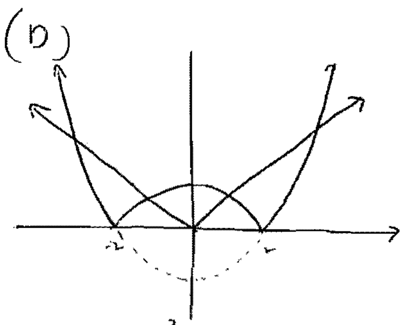
$|z+\bar{z}| = |a+ib+a-ib|$

$= |2a|$

$= 2a$

$\neq 2\sqrt{a^2+b^2}$

2. (D)




4

3. (B) $\alpha\beta\gamma = \frac{3}{48}$

$\alpha . \alpha = \frac{1}{16}$

$\alpha^2 = \frac{1}{16}$

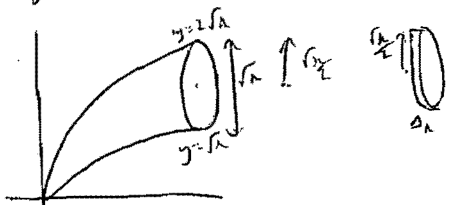
$\alpha = \pm\frac{1}{4}$

$\therefore B$

4. (C) $\displaystyle\int \frac{\frac{2dt}{1+t^2}}{\frac{1-t^2}{1+t^2}}$ $\qquad d\theta = \frac{2dt}{1+t^2}$

$= \displaystyle\int \frac{2dt}{1-t^2}$

5. (D)




$\Delta V = \pi\left(\frac{\sqrt{x}}{2}\right)^2 \Delta x$

$V = \pi \displaystyle\int_0^4 \frac{x}{4}\, dx$

6. (B) $b^2 = a^2(1-e^2)$

$36 = 100(1-e^2)$

$\frac{9}{25} = 1-e^2$

$e^2 = \frac{16}{25}$

$e = \pm\frac{4}{5}$

Foci $(\pm ae, 0)$

$(\frac{4}{5}\times 10, 0) = (\pm 8, 0)$ $\quad \therefore$ Distance $= 16$ units

7. (B) $f(2-x) = f(2\times 1 - x)$

$f(x)$ rotated about $x = 1$.

$\therefore (B)$

8. (D)

$V = \sin^{-1} x$

$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{1}{2}\frac{d}{dx}(\sin^{-1} x)^2$

$a = \dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$

$\therefore 0$

9. (B) $\frac{dy}{dx} = y^2+1$

$\frac{dx}{dy} = \frac{1}{y^2+1}$

$x = \tan^{-1} y + C$

when $x=0, y=1$

$0 = \tan^{-1} 1 + C$

$C = -\frac{\pi}{4}$

$\therefore x = \tan^{-1} y - \frac{\pi}{4}$

$x + \frac{\pi}{4} = \tan^{-1} y$

$y = \tan\left(x + \frac{\pi}{4}\right)$

$\therefore B$

10. (B) 2 white

$n-2$ black

$P(WW) = \dfrac{{}^2C_2 \times {}^{n-2}C_2}{{}^nC_4}$ $\qquad P(\text{no whites}) = \dfrac{{}^{n-2}C_4}{{}^nC_4}$

But $P(WW) = 2\,P(\text{no whites})$

$\dfrac{{}^{n-2}C_2}{{}^nC_4} = 2 \times \dfrac{{}^{n-2}C_4}{{}^nC_4}$

$\dfrac{(n-2)!}{(n-4)!\,2!} = \dfrac{2(n-2)!}{(n-6)!\,4!}$

$6\ 24\ (n-6)! = 4(n-4)(n-5)(n-6)!$

$n^2 - 9n + 20 - 6 = 0$

$n^2 - 9n + 14 = 0$

$(n-7)(n-2) = 0$

$n = 2, 7 \qquad n \geq 4$

$\therefore n = 7$ (B)

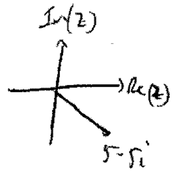
11 a) (i) $zw = (1-3i)(2+i)$

$= 2+3-5i$

$= 5-5i$ ✓

(ii) $5-5i = \sqrt{5^2+5^2}\ \tan^{-1}\left(\frac{-5}{5}\right)$

$= 5\sqrt{2}\ \text{cis}\left(-\frac{\pi}{4}\right)$ ✓ ✓




(iii) $\frac{\sqrt{2}(\cos x - i\sin x)}{2+i} = \frac{1-3i}{5}$

$\sqrt{2}\ \text{cis}(-x) = \frac{5-5i}{5}$

$\text{cis}(-x) = \frac{1-i}{\sqrt{2}}$ ✓

$\text{cis}(-x) = \frac{\sqrt{2}-\sqrt{2}i}{2}$

$\cos x = \frac{1}{\sqrt{2}}$ $\quad -i\sin x = \frac{-\sqrt{2}i}{2}$

$\therefore x$ acute

$x = \frac{\pi}{4}$ ✓

b) $-\int_0^{\frac{\pi}{2}} \frac{-\sin x\,dx}{1+\cos^2 x}$

let $u = \cos x$

$du = -\sin x\,dx$

when $x = \frac{\pi}{2}$, $u = \cos\frac{\pi}{2} = 0$

$x = 0$, $u = \cos 0 = 1$

$= -\int_1^0 \frac{du}{1+u^2}$ ✓

$= \int_0^1 \frac{du}{1+u^2}$ ✓

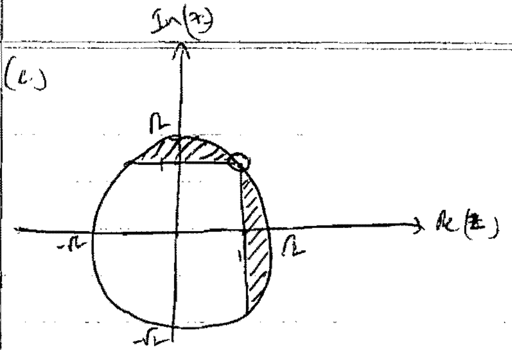
$= \left[\tan^{-1} u\right]_0^1$

$= \tan^{-1} 1 - \tan^{-1} 0$

$= \frac{\pi}{4} - 0$

$= \frac{\pi}{4}$ ✓

11 (c)




(3) correct
(2) circle & some progress
(1) circle radius √2 centre (0,0)

11 d(i) $\frac{x^2+x+1}{(x+2)(x^2+2x+3)} \equiv \frac{A}{x+2} + \frac{Bx+C}{x^2+2x+3}$

$x^2+x+1 = A(x^2+2x+3) + (x+2)(Bx+C)$

when $x=-2$ $\quad 3 = 3A$

$A = 1$

comparing coeff of $x^2$ $\quad 1 = A+B$

$1 = 1+B$

$B = 0$

comparing constant term: $\quad 1 = 3A+2C$

$1 = 3+2C$

$2C = -2$

$C = -1$

$\therefore A=1, B=0, C=-1$

(1) 1 value correct
(2) Both correct

11 d(ii) $\int \frac{x^2+x+1\,dx}{(x+2)(x^2+2x+3)} = \int \frac{1}{x+2} - \frac{1}{x^2+2x+3}\,dx$

$= \int \frac{1}{x+2} - \frac{1}{(x+1)^2+2}\,dx$

$= \ln|x+2| - \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right) + C$ ✓ ✓

12 a) $\int x(x+1)^{10}\,dx$

$= \int (x+1-1)(x+1)^{10}\,dx$ ✓

$= \int (x+1)^{11} - (x+1)^{10}\,dx$ ✓

$= \frac{(x+1)^{12}}{12} - \frac{(x+1)^{11}}{11} + C$ ✓

12 b) $\int \frac{e^x + e^{2x}}{1+e^{2x}}\,dx$

$= \int \frac{e^x(1+e^x)}{1+e^{2x}}\,dx$

let $u = e^x$, $du = e^x\,dx$.

$= \int \frac{(1+u)\,du}{1+u^2}$ ✓

$= \int \frac{1}{1+u^2} + \frac{1}{2}\frac{2u}{1+u^2}\,du$ ✓ ✓

$= \tan^{-1} u + \frac{1}{2}\ln(1+u^2) + C$

$= \tan^{-1} e^x + \frac{1}{2}\ln(1+e^{2x}) + C$

12 c) $x^2 + y^2 = xy + 3$

$2x + 2y\frac{dy}{dx} = y.1 + x.\frac{dy}{dx}$

$2x - y = \frac{dy}{dx}(x - 2y)$

$\frac{dy}{dx} = \frac{2x-y}{x-2y}$ ✓

when $x = 1$ $1 + y^2 = y + 3$

$y^2 - y - 2 = 0$

$(y-2)(y+1) = 0$

$\therefore y = -1, 2.$ ✓

At $(1,-1)$ $\frac{dy}{dx} = \frac{2+1}{1+2}$

$\frac{dy}{dx} = 1$

$y + 1 = 1(x-1)$

$\therefore y = x - 2$ ✓ &

At $(1,2)$ $\frac{dy}{dx} = \frac{2-2}{1-4}$

$\frac{dy}{dx} = 0$

$\therefore y = 2$

$y = 2$ ✓

12 d) (i) let $h = ax + b$

when $x = 0$, $h = 5$

$\therefore 5 = 0 + b$

$h = ax + 5$ ✓

when $x = 12$, $h = 10$

$10 = 12a + 5$

$5 = 12a$

$a = \frac{5}{12}$

$\therefore h = \frac{5x}{12} + 5$ ✓

Area $= \frac{1}{2}\left(\frac{16+8}{2}\right)\left(\frac{5x}{12} + 5\right)$

$= 12\left(\frac{5x}{12} + 5\right)$ ✓

$= 5x + 60$

12 d) (ii) Volume slice $= (5x + 60)\,\Delta x$

Total volume $= \lim_{\Delta x \to 0} \sum_{x=0}^{12} (5x + 60)\,\Delta x$

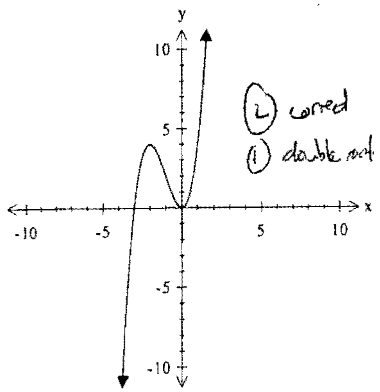
$V = \int_0^{12} 5x + 60\,dx$ ✓

$= \left[\frac{5x^2}{2} + 60x\right]_0^{12}$

$= (72 \times 5 + 720) - (0 + 0)$

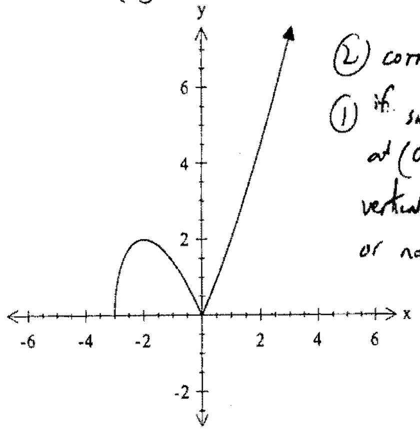
$= 1080\text{ cm}^3$ ✓

a(i)



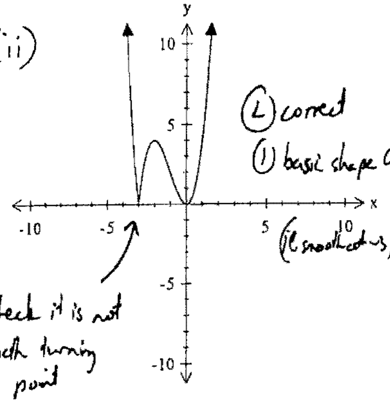
(2) correct
(1) double root

13 a(iv)



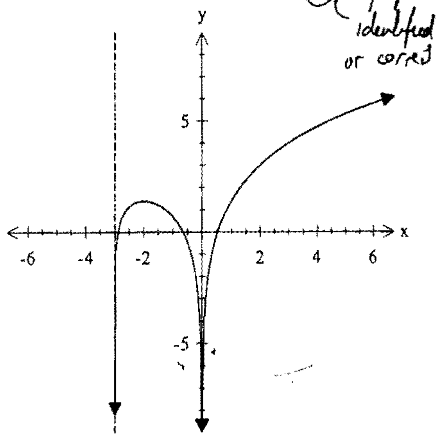
(2) correct
(1) if smooth t.p. at (0,0) or not vertical tangent at (−3,0) or not concave up

13 a(ii)



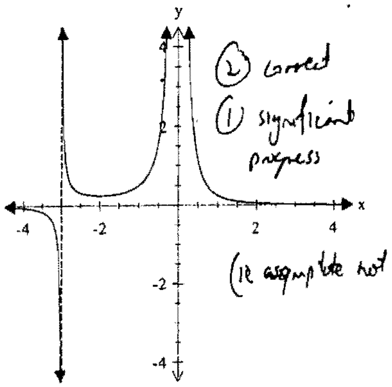
(2) correct
(1) basic shape OK
(ie smooth at −3,0)
check it is not smooth turning point

13 a(v)



(2) correct
(1) significant progress asymptote not identified at −3 or 0 or correct for x ≥ 0

13 a(iii)



(2) correct
(1) significant progress
(ie asymptote not identified)

13 (b) (i) since coefficients are all real, roots occur in conjugate pairs.
since $\alpha i$ is purely imaginary ($\alpha$ is real)
then the conjugate $-\alpha i$ is also a root. ✓

(ii) $\therefore$ Roots are $\alpha i, -\alpha i, \beta$

Sum of roots $= \alpha i - \alpha i + \beta = -a$

$\beta = -a$

$\therefore \beta$ is real since $a$ is real. ✓

$\therefore$ only one real root (2 are purely imaginary)

(iii) $P(\alpha i) = \alpha^3 i^3 + a\alpha^2 i^2 + b\alpha i + c = 0$ ✓

$P(\alpha i) = -\alpha^3 i - a\alpha^2 + b\alpha i + c = 0 + 0i$ ✓

$\therefore c - a\alpha^2 + (b\alpha - \alpha^3)i = 0 + 0i$

Equating real & imaginary parts ✓

$c - a\alpha^2 = 0 \qquad b\alpha - \alpha^3 = 0$

$c = a\alpha^2 \qquad \alpha(b - \alpha^2) = 0$

$\alpha^2 = \frac{c}{a} \qquad \alpha = 0$ or $\alpha^2 = b$

(but $\alpha$ is nonzero) ✓

$\therefore \frac{c}{a} = b$

$c = ab$

14 a) $1+w+w^2=0$ since $w$ is a complex root of unity

$LHS = (a+b)(a+wb)(a+w^2b)$

$= (a+b)(a^2+abw^2+abw+w^3b^2)$

$= (a+b)(a^2+ab(w+w^2)+b^2)$ since $w^3=1$

$= (a+b)(a^2-ab+b^2)$ $w+w^2=-1$

$= a^3+b^3$

$= RHS$ Q.E.D.

(2) correct
(1) uses $1+w+w^2=0$ to simplify

b) $P(x) = x^6-3x^2+2$ has roots of multiplicity at least 2

$P'(x) = 6x^5-6x$ has roots of multiplicity at least 1.

$P'(x) = 6x(x^4-1)$

When $P'(x)=0$

$x=0$ or $x^4-1=0$

but $P(0)=2$ $x^4=1$

$\neq 0$ $x^2=\pm 1$

$\therefore$ 0 is not a double root $x^2=1$ or $x^2=-1$

$x=\pm 1$ $x=\pm i$

$P(1) = 1-3+2$

$=0$

$\therefore x=1$ is a double root. ✓

$P(-1) = 1-3+2$

$=0$

$\therefore x=-1$ is also a double root. ✓

Sum of roots $2\times 1+2\times -1+\alpha+\beta=0$ (where $\alpha,\beta$ are also roots)

$\alpha+\beta=0$

$\beta=-\alpha$

Product of roots $1^2(-1)^2\cdot\alpha\cdot\beta = 2$ attempts to find other roots

$-\alpha^2=2$

$\alpha^2=2i^2$

$\alpha=\pm i\sqrt{2}$

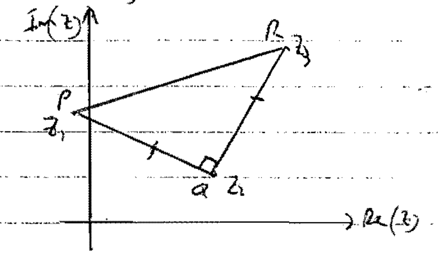
$\therefore$ Roots are $1,1,-1,-1,i\sqrt{2},-i\sqrt{2}$ ✓

$\therefore P(x) = (x-1)^2(x+1)^2(x-i\sqrt{2})(x+i\sqrt{2})$

14 (c) $z_1-z_2 = i(z_3-z_2)$

$\therefore z_1-z_2$ is same length as $z_3-z_2$ ✓

and is $z_3-z_2$ rotated $90^\circ$ anticlockwise ✓




(1) for diagram

(1) for right angled isosceles triangle

$\therefore PQR$ is a right angled isosceles triangle

14 d (i) $RHS = \dfrac{1}{(x^2+1)^n} - \dfrac{1}{(x^2+1)^{n+1}}$

$= \dfrac{x^2+1}{(x^2+1)^{n+1}} - \dfrac{1}{(x^2+1)^{n+1}}$

$= \dfrac{x^2}{(x^2+1)^{n+1}}$ ✓

$= LHS$ as req'd.

(ii) $I_n = \int_0^1 1.(x^2+1)^{-n}\,dx$ $u=(x^2+1)^{-n}$ $v'=1$

$u'=-2nx(x^2+1)^{-n-1}$ $v=x$

$= \left[\dfrac{x}{(x^2+1)^n}\right]_0^1 + \int_0^1 \dfrac{2nx^2}{(x^2+1)^{n+1}}\,dx$ ✓

$= \left(\dfrac{1}{2^n}-\dfrac{0}{1}\right) + 2n\int_0^1 \dfrac{x^2}{(x^2+1)^{n+1}}\,dx$ ✓

$= 2^{-n} + 2n\int \dfrac{1}{(x^2+1)^n} - \dfrac{1}{(x^2+1)^{n+1}}\,dx$ from (i)

Q14 (cont'd)

d(ii) (cont'd) $I_n = 2^{-n} + 2n\,I_n - 2n\,I_{n+1}$ ✓

$2n\,I_{n+1} = 2^{-n} + (2n-1)I_n$

4 d(iii) $I_3 = \int_0^1 \frac{1}{(x^2+1)^3}\,dx$

$I_{n+1} = \frac{2^{-n}}{2n} + \frac{2n-1}{2n} I_n$

$n=2 \Rightarrow I_3 = \frac{2^{-2}}{4} + \frac{3}{4} I_2$

$= \frac{1}{16} + \frac{3}{4} I_2$

$I_2 = \frac{2^{-1}}{2} + \frac{1}{2} I_1$

$= \frac{1}{4} + \frac{1}{2} I_1$

$I_1 = \int_0^1 \frac{dx}{x^2+1}$

$= \left[\tan^{-1} x\right]_0^1$

$= \frac{\pi}{4} - 0$

$= \frac{\pi}{4}$ ✓

$I_2 = \frac{1}{4} + \frac{\pi}{8}$

$\therefore I_3 = \frac{1}{16} + \frac{3}{4}\left(\frac{1}{4} + \frac{\pi}{8}\right)$

eg: (2) correct
(1) for significant progress finds $I_3$ in terms of $I_1$

4. (e) test $n=1$

LHS $= 1 \log \frac{2}{1}$
$= \log 2$

RHS $= \log \frac{(2)^1}{1!}$
$= \log 2$
$=$ LHS

$\therefore$ true for $n=1$ ✓

14(e) Assume true for $n=k$

$\sum_{r=1}^{k} r \log \frac{r+1}{r} = \log \frac{(k+1)^k}{k!}$

For $n=k+1$ we wish to prove

$\sum_{r=1}^{k+1} r \log \frac{r+1}{r} = \log \frac{(k+2)^{k+1}}{(k+1)!}$

LHS $= \sum_{r=1}^{k+1} r \log \frac{r+1}{r}$

$= \sum_{r=1}^{k} r \log \frac{r+1}{r} + (k+1)\log \frac{k+2}{k+1}$

$= \log \frac{(k+1)^k}{k!} + \log \left(\frac{k+2}{k+1}\right)^{k+1}$ ✓

$= \log \left( \frac{(k+1)^k}{k!} \cdot \frac{(k+2)^{k+1}}{(k+1)^{k+1}} \right)$

$= \log \frac{(k+2)^{k+1}}{(k+1)\,k!}$ ✓

$= \log \frac{(k+2)^{k+1}}{(k+1)!}$ as req'd. since $(k+1)! = (k+1)\,k!$

So if true for $n=k$, its true for $n=k+1$. But is true for $n=1$, $\therefore$ true for $n=2$, $n=3$ and so on for all $n$.

$\therefore \sum_{r=1}^{n} r \log \frac{r+1}{r} = \log \frac{(n+1)^n}{n!}$ for $n \geq 1$ by mathematical induction

15 a(i) if $|z-2| = 2\,\text{Re}(x+iy-\frac{1}{2})$

$|z-2| = 2\,\text{Re}(x-\frac{1}{2}+iy)$

Since $|z-2| \geq 0$

$2\,\text{Re}(x-\frac{1}{2}+iy) \geq 0$ ✓

$2(x-\frac{1}{2}) \geq 0$

$x-\frac{1}{2} \geq 0$

$x \geq \frac{1}{2}$

15 a(ii) $|x-2+iy| = 2\left(x-\frac{1}{2}\right)$

$\sqrt{(x-2)^2+y^2} = 2\left(x-\frac{1}{2}\right)$

$(x-2)^2+y^2 = 4\left(x^2-x+\frac{1}{4}\right)$ ✓

$x^2-4x+4+y^2 = 4x^2-4x+1$

$3 = 3x^2-y^2$

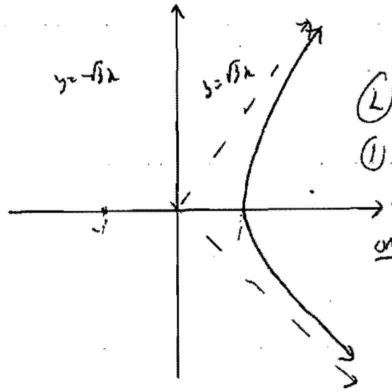
$3x^2-y^2=3$ as reqd ✓

But as $x \geq \frac{1}{2}$ only the right hand branch is possible

a(iii) $x^2 - \frac{y^2}{3} = 1$

asymptotes : $y=\pm\sqrt{3}x$

vertex $(1,0)$




(2) correct
(1) asymptotes or x intercepts wrong
or asymptotes, x intercept correct but wrong graph.

(NB if two branches max is (1))

15b (i) $xy=c^2$

$y=\frac{c^2}{x}$

$\frac{dy}{dx} = \frac{-c^2}{x^2}$

when $x=cp$

$\frac{dy}{dx} = \frac{-c^2}{c^2p^2}$

$= -\frac{1}{p^2}$ ✓

$\therefore m_{normal} = p^2$

$\therefore y-\frac{c}{p} = p^2(x-cp)$

$py - c = p^3x - cp^4$

$p^3x - py = cp^4 - c$

$p^3x - py = c(p^4-1)$ ✓

(ii) At M, $y=0$

$xp^3 = c(p^4-1)$

$x = \frac{c(p^4-1)}{p^3}$

$\therefore$ M is $\left(\frac{c(p^4-1)}{p^3}, 0\right)$

(3) correct
(2) substantial progress
(1) finds M or N

Tangent $y-\frac{c}{p} = \frac{-1}{p^2}(x-cp)$

when $x=0$, $y-\frac{c}{p} = \frac{c}{p}$

$y = \frac{2c}{p}$ ✓

$\therefore$ N is $\left(0, \frac{2c}{p}\right)$

Midpoint MN $= \left(\frac{0+cp^4-c}{2p^3}, \frac{0+2c}{2p}\right)$ ✓

$x = \frac{c(p^4-1)}{2p^3}$ $y=\frac{c}{p}$

$\therefore p = \frac{c}{y}$

Sub in x

$2xp^3 = c(p^4-1)$

$2x\frac{c^3}{y^3} = c\left(\frac{c^4}{y^4}-1\right)$

$2xc^3y = c^5 - cy^4$ ✓

$2xc^2y = c^4 - y^4$

15 c) (i) $\angle BAP = \angle BCP = \alpha$ (angles at circumference standing on arc BP)

$\angle ABQ = \angle ACQ = \beta$ (angles at circumference standing on arc QA) ✓

$\therefore \angle PCQ = \alpha + \beta + 2\gamma$ (adjacent ∠'s at point C)

(ii) $\angle CAB + \angle ABC + \angle BCA = 180°$ (∠ sum of $\Delta ABC$)

$2\alpha + 2\beta + 2\gamma = 180°$

$\alpha + \beta + \gamma = 90°$ ✓

$\therefore$ In $\Delta QCP$,

$\angle QCP = \alpha + \beta + 2\gamma$

$\angle QCP = 90 + \gamma$

Since $\gamma$ is a constant as C varies (∠ at circumference standing on the chord PQ)

then PQ is a constant. (= ∠'s at the circumference subtend equal chords) ✓

(iii) $\frac{PQ}{\sin \angle PCQ} = \frac{QC}{\sin \angle QPC}$ (sine rule $\Delta QCP$)

But $\angle QPC = \angle QBC = \beta$ (∠'s at circumference standing on arc QC)

$\therefore \frac{PQ}{\sin(90+\gamma)} = \frac{QC}{\sin\beta}$

But in $\Delta QCB$ $\angle PQB = \angle PAB = \alpha$ (∠'s at circumference standing on arc PB)

$\frac{QC}{\sin\beta} = \frac{BC}{\sin 2\alpha}$ (sine rule $\Delta QCB$)

But in $\Delta ACB$, using the sine rule,

$\frac{BC}{\sin 2\alpha} = \frac{AB}{\sin 2\gamma}$

$\therefore \frac{PQ}{\sin(90+\gamma)} = \frac{QC}{\sin\beta} = \frac{BC}{\sin 2\alpha} = \frac{AB}{\sin 2\gamma}$

$\therefore \frac{PQ}{\cos(90-(90+\gamma))} = \frac{AB}{2\sin\gamma\cos\gamma}$

$\frac{PQ}{\cos(-\gamma)} = \frac{AB}{2\sin\gamma\cos\gamma}$

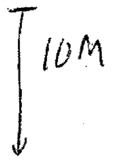
$\therefore \frac{AB}{PQ} = \frac{2\sin\gamma\cos\gamma}{\cos\gamma}$ ($\cos(-\theta) = \cos\theta$) $= 2\sin\gamma$.

② correct

① significant progress using sine rule

OR ① correct without reasons

16 a (i)




$M\ddot{x} = 10M$

$\ddot{x} = 10$

$\int_0^v \frac{dv}{dt} = \int_0^t 10\,dt$

$[v]_0^v = [10t]_0^t$

$v - 0 = 10t - 0$

$v = 10t$

After 10 seconds,

$V = 10 \times 10$

$= 100\,ms^{-1}$

Also $\frac{dx}{dt} = 10t$

$\int_0^x dx = \int_0^{10} 10t\,dt$

$[x]_0^x = [5t^2]_0^{10}$

$x - 0 = 500 - 0$

$\therefore x = 500$

$\therefore$ After 10 seconds it is fallen 500m and is travelling $100\,ms^{-1}$.

② correct solution

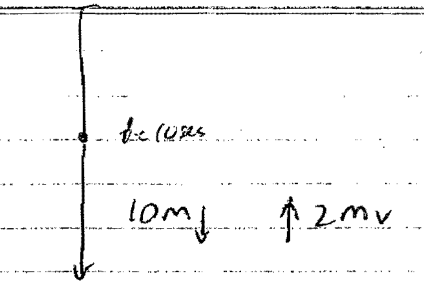
① substantial progress

eg (finds 100 or 500)

OR finds expressions for x and v but not values.

OR correctly finds x given incorrect v.

16 a(ii)




$m\ddot{x} = 10m - 2mv$

$\frac{dv}{dt} = 10 - 2v$

$\int_{100}^{v} \frac{dv}{10-2v} = \int_{10}^{t} dt$ ✓

$[t]_{10}^{t} = -\frac{1}{2}\int_{100}^{v} \frac{dv}{v-5}$

$t - 10 = -\frac{1}{2}\left[\ln(v-5)\right]_{100}^{v}$

$-2(t-10) = \ln(v-5) - \ln(100-5)$ ✓

$-2(t-10) = \ln\left(\frac{v-5}{100-5}\right)$

$e^{-2(t-10)} = \frac{v-5}{95}$

$v - 5 = 95e^{-2(t-10)}$ ✓

$v = 5 + 95e^{-2(t-10)}$

a (iii) $\frac{dx}{dt} = 5 + 95e^{-2(t-10)}$

$[x]_{500}^{x} = \left[5t - \frac{95}{2}e^{-2(t-10)}\right]_{10}^{t}$

$x - 500 = 5t - \frac{95}{2}e^{-2(t-10)} - \left(50 - \frac{95}{2}e^{0}\right)$ ✓

$\therefore x = 5t - \frac{95}{2}e^{-2(t-10)} + \frac{995}{2}$

when $t=120$ $x = 600 - \frac{95}{2}e^{-220} + \frac{995}{2}$

$x = 1097.5$

$\therefore$ Height above $= 2000 - 1097.5$

$= 902.5\text{m}$ ✓

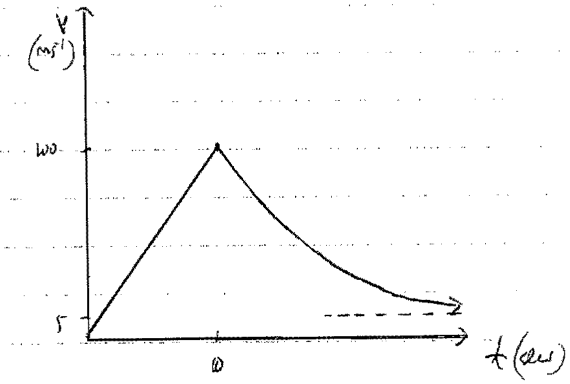
16 a(iv) terminal velocity:

$\ddot{x} = 0$

$10 - 2v = 0$

$v = 5\ \text{ms}^{-1}$ ✓

a(v)




② correct
① some progress (only 1 of the graphs)
or ① both graphs no scale/W

16 (b) $\prod_{r=0}^{\infty}\left(\cos\frac{\pi}{2^r} + i\sin\frac{\pi}{2^r}\right)$

$= \left(\cos\frac{\pi}{1} + i\sin\frac{\pi}{1}\right)\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) + \ldots$

$(\text{cis}\,\pi)\left(\text{cis}\,\frac{\pi}{2}\right)\left(\text{cis}\,\frac{\pi}{4}\right)\left(\text{cis}\,\frac{\pi}{8}\right)\ldots$

$= \text{cis}\left(\pi + \frac{\pi}{2} + \frac{\pi}{4} + \frac{\pi}{8} + \ldots\right)$ ✓

GP $a = \pi$ $r = \frac{1}{2}$

$S_\infty = \frac{\pi}{1 - \frac{1}{2}}$

$= 2\pi$

$= \text{cis}(2\pi)$

$= \cos 2\pi + i\sin 2\pi$

$= +1 + i0$ ✓

$= 1$

6 (c) let $x = \sqrt[3]{12} + \sqrt[3]{18}$

$x^3 = 12 + {}^3C_1 (12)^{2/3}(18)^{1/3} + {}^3C_2 (12)^{1/3}(18)^{2/3} + 18$ ✓

$x^3 = 30 + 3\left[(12^{2/3})(18^{1/3}) + (12)^{1/3}(18^{2/3})\right]$

$x^3 = 30 + 3(12^{1/3})(18^{1/3})\left[12^{1/3} + 18^{1/3}\right]$

$x^3 = 30 + 3\left(2^{2/3} 3^{1/3}\right)\left(2^{1/3} 3^{2/3}\right)\left[x\right]$ ✓

$x^3 = 30 + 3 \times 2 \times 3\,x$ ✓

$\therefore x^3 - 18x - 30 = 0$ is the req'd eqn.