BAULKHAM HILLS HIGH SCHOOL

2015

YEAR 12 TRIAL
HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations
- Marks may be deducted for careless or badly arranged work

Total marks – 100

Section I Pages 2 – 6

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II Pages 7 – 15

90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours 45 minutes for this section

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10

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1 What is the eccentricity of the hyperbola $\frac{y^2}{4}-\frac{x^2}{9}=1$?

(A) $\frac{\sqrt{5}}{3}$

(B) $\frac{\sqrt{5}}{2}$

(C) $\frac{\sqrt{13}}{3}$

(D) $\frac{\sqrt{13}}{2}$

2 The substitution of $x = \sin\theta$ in the integral $\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}}\, dx$ results in

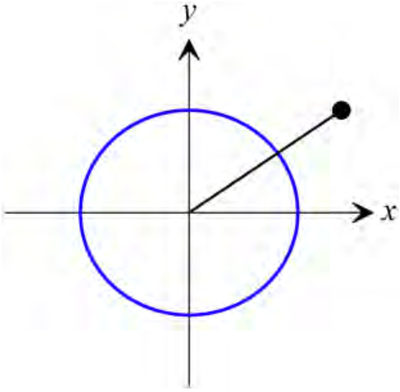
(A) $\int_0^{\frac{1}{2}} \frac{\sin^2\theta}{\cos\theta}\, d\theta$

(B) $\int_0^{\frac{1}{2}} \sin^2\theta\ \ d\theta$

(C) $\int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{\cos\theta}\, d\theta$

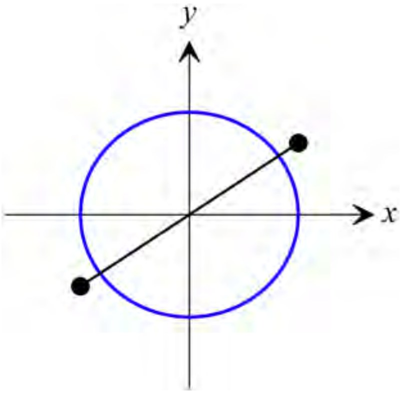
(D) $\int_0^{\frac{\pi}{6}} \sin^2\theta\ \ d\theta$

3 The Argand diagram below shows the complex number z, represented as a vector, along with the unit circle.

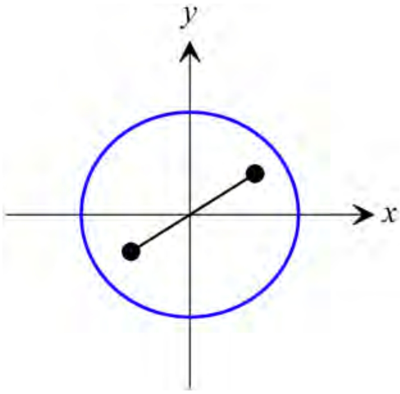


Which diagram best illustrates the vectors representing $\sqrt{z}$?

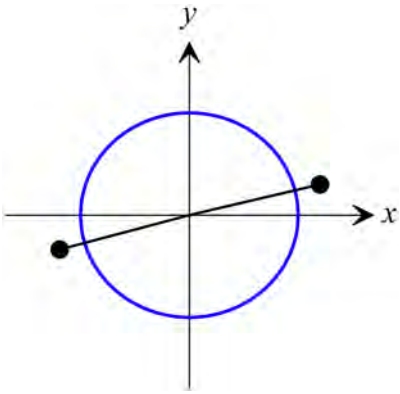
(A)



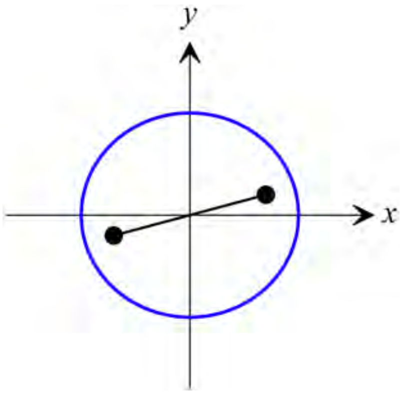
(B)



(C)



(D)



4 The circle $|z - 3 - 2i| = 2$ is intersected exactly twice by the line given by

(A) $|z - 3 - 2i| = |z - 5|$

(B) $|z - i| = |z + 1|$

(C) $Re(z) = 5$

(D) $Im(z) = 0$

5 If $\frac{dy}{dx} = \sqrt{2x^6 + 1}$ and $y = 5$ when $x = 1$, then the value of y when $x = 4$ is given by

(A) $\int_1^4 \left(\sqrt{2x^6 + 1} + 5\right) dx$

(B) $\int_1^4 \sqrt{2x^6 + 1}\, dx + 5$

(C) $\int_1^4 \left(\sqrt{2x^6 + 1} - 5\right) dx$

(D) $\int_1^4 \sqrt{2x^6 + 1}\, dx - 5$

6 The graph of $y = \frac{1}{ax^2 + bx + c}$ has asymptotes at $x = -5$ and $x = 3$.

Given that the graph has one stationary point with a y value of $-\frac{1}{8}$, it follows that

(A) $a = \frac{1}{2}, b = 1 , c = -\frac{15}{2}$

(B) $a = \frac{1}{2}, b = -1 , c = -\frac{15}{2}$

(C) $a = 1 , b = 2 , c = -15$

(D) $a = 1 , b = -2 , c = -15$

7 The polynomial equation $x^3 - 3x^2 + 2 = 0$ has roots α , β and γ .

What is the value of $\alpha^3 + \beta^3 + \gamma^3$?

(A) 9

(B) 13

(C) 21

(D) 25

8 The base of a solid is the circle $x^2 + y^2 = 4$. Every cross section of the solid taken perpendicular to the x axis is a right-angled, isosceles triangle with its hypotenuse lying in the base of the solid.

Which of the following is an expression for the volume of the solid?

(A) $\frac{1}{4}\int_{-2}^{2} (4 - x^2)\, dx$

(B) $\int_{-2}^{2} (4 - x^2)\, dx$

(C) $2\int_{-2}^{2} (4 - x^2)\, dx$

(D) $4\int_{-2}^{2} (4 - x^2)\, dx$

9 A particle of mass 1 kg is projected vertically upwards from ground level with a velocity of u m/s.

The particle is subject to a constant gravitational force and a resistance which is proportional to twice the square of its velocity v m/s, (with k being the constant of proportionality).

Let x be the displacement in metres from the ground after t seconds and let g be the acceleration due to gravity.

Which of the following expressions gives the maximum height reached by the particle?

(A) $\int_{u}^{0} \frac{v}{g + 2kv^2}\, dv$

(B) $\int_{u}^{0} \frac{v}{g - 2kv^2}\, dv$

(C) $\int_{0}^{u} \frac{v}{g + 2kv^2}\, dv$

(D) $\int_{0}^{u} \frac{v}{g - 2kv^2}\, dv$

10 Marudan has 10 jellybeans left in a jar, 5 black, 3 red and 2 yellow. He chooses 2 jellybeans at random and puts them in his pocket.

Later he takes one jellybean out of his pocket and sees that it is black. What is the probability that the jellybean that is left in his pocket is also black?

(A) $\frac{2}{9}$

(B) $\frac{2}{7}$

(C) $\frac{4}{9}$

(D) $\frac{1}{2}$

END OF SECTION I

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours 45 minutes for this section

Answer each question on the appropriate answer sheet. Each answer sheet must show your BOS#. Extra paper is available.

In Questions 11 to 16, your responses should include relevant mathematical reasoning and/or calculations.

Marks

Question 11 ***(15 marks)*** Use a *separate* answer sheet

(a) Evaluate $\displaystyle\int_e^{e^2} \frac{dx}{x\ln x}$ *2*

(b) $z = p + 2i$, where p is a real number, and $w = 1 - 2i$ represent two complex numbers

(i) Find $\dfrac{z}{w}$ in the form $a + ib$ where a and b are real. *1*

(ii) Given that $\left|\dfrac{z}{w}\right| = 13$, find the possible values of p. *2*

(c) The roots of the equation $2z^3 - 3z^2 + 8z + 5 = 0$ are α , β and γ . *2*
Given that $\alpha = 1 + 2i$, find β and γ .

(d) (i) On an Argand diagram sketch the locus of z represented by $|z - 3| = 3$ *2*

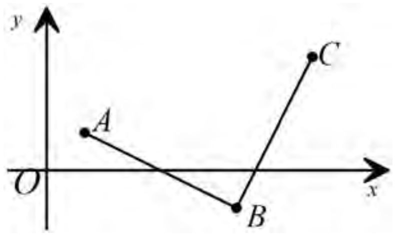
(ii) Explain why $\arg(z - 3) = 2\arg z$ *1*

Question 11 continues on page 8

Marks

<u>Question 11</u> (continued)

(e) *2*



The points A and C represent the complex numbers $1 + i$ and $7 + 3i$ respectively.

Find the complex number ω , represented by B such that ΔABC is isosceles and right angled at B.

(f) (i) Express $(5 - i)^2(1 + i)$ in the form $a + ib$ where a and b are real. *1*

(ii) Hence, prove that $\tan^{-1}\frac{7}{17} + 2\tan^{-1}\frac{1}{5} = \frac{\pi}{4}$ *2*

End of Question 11

Marks

<u>Question 12</u> ***(15 marks)*** Use a *separate* answer sheet

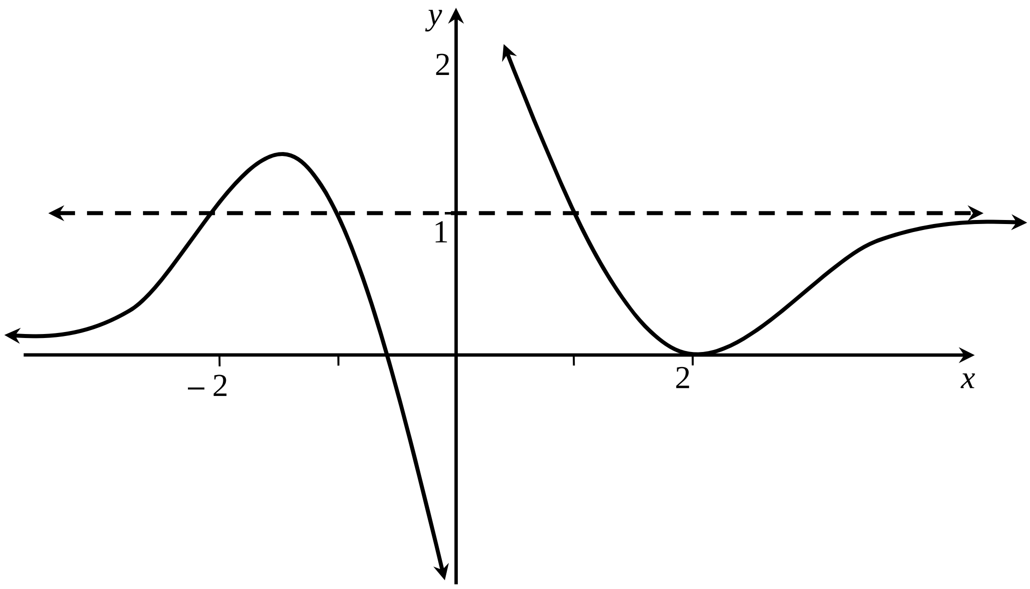
(a) Using the substitution $t = \tan\frac{x}{2}$, or otherwise, evaluate *3*

$$\int_0^{\frac{\pi}{3}} \frac{dx}{1 - \sin x}$$

(b) Find $\int \frac{x+1}{\sqrt{x^2 + 4x - 3}}\, dx$ *3*

(c) The diagram below is a sketch of the function $y = f(x)$.

The lines $x = 0$, $y = 0$ and $y = 1$ are asymptotes.



Draw separate one-third page sketches of the following curves, clearly indicating any important features such as turning points or asymptotes.

(i) $y = f(4 - x)$ *2*

(ii) $y = \sqrt{f(x}$ *2*

(iii) $y = e^{f(x)}$ *2*

(d) (i) Show that for $a > 0$ and $n \neq 0$, $\log_{a^n} x = \frac{1}{n}\log_a x$ *1*

(ii) Hence evaluate $\log_2 3 + \log_4 3 + \log_{16} 3 + \log_{256} 3 + \ldots$ *2*

Marks

<u>Question 13</u> ***(15 marks)*** Use a *separate* answer sheet

(a) The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a > b > 0$, has eccentricity $e=\frac{1}{2}$. The point $P(2,3)$ lies on the ellipse.

(i) Find the values of a and b. *3*

(ii) Find the foci and the directrices of the ellipse. *1*

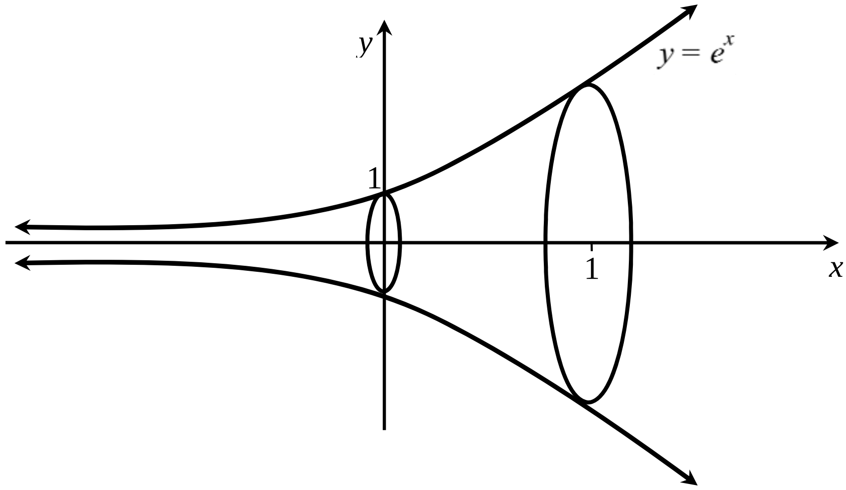
(b) Let α , β and γ be the roots of the cubic equation $x^3+Ax^2+Bx+8=0$, where A and B are real. Furthermore $\alpha^2+\beta^2=0$ and $\beta^2+\gamma^2=0$.

(i) Explain why β is real and both α and γ are not real. *1*

(ii) Show that α and γ are purely imaginary. *1*

(iii) Find A and B. *2*

(c) The arc defined by $y=e^x$, $0 \le x \le 1$, is rotated about the x-axis to form a curved bowl.



(i) Using the method of cylindrical shells, show that the volume ,V, of the solid that makes the bowl is given by *3*

$$V=\pi e^2-2\pi\int_1^e y\ln y\ dy$$

(ii) Find the volume, leaving your answer in exact form. *3*

(iii) Use the result in part (ii) to evaluate $\int_0^1 e^{2x}\ dx$ *1*

Marks

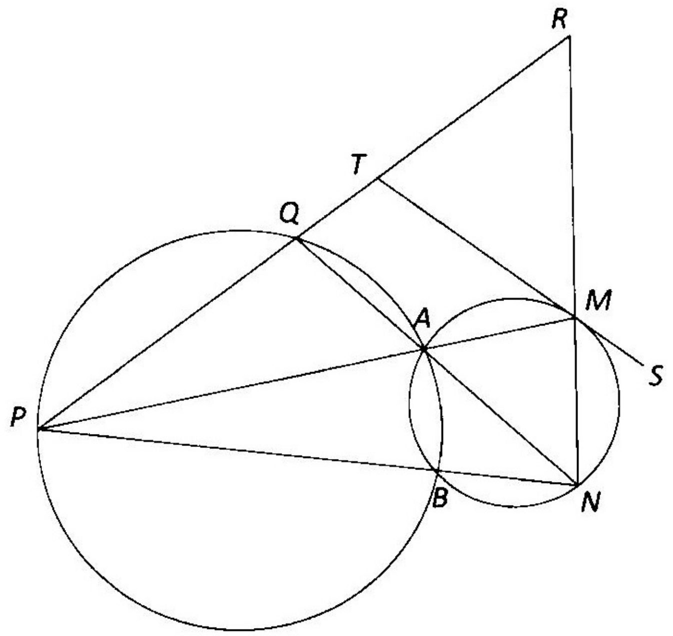
Question 14 ***(15 marks)*** Use a *separate* answer sheet

(a) (i) Use De Moivre's Theorem to prove that if $z = \cos\theta + i\sin\theta$, then 1

$$2\cos n\theta = z^n + \frac{1}{z^n}$$

(ii) Hence , or otherwise solve the equation $5x^4 - 11x^3 + 16x^2 - 11x + 5 = 0$ 3

(b) In the diagram, the two circles intersect at *A* and *B*. *P* is a point on one circle. *PA* and *PB* produced meet the other circle at *M* and *N* respectively. *NA* produced meets the first circle at *Q*. *PQ* and *NM* produced meet at *R*. The tangent to the second circle meets *PR* at *T*.



Copy or trace the diagram into your answer booklet.

(i) Show that *QAMR* is a cyclic quadrilateral. 2

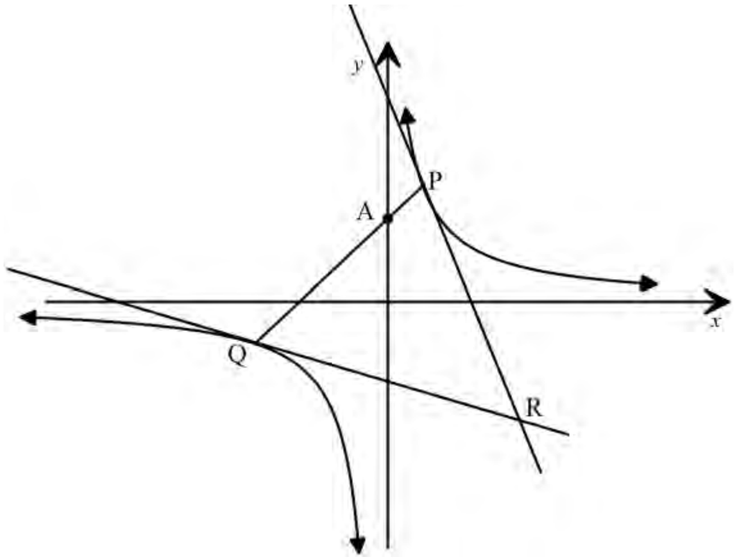
(ii) Show that *TM* = *TR* 3

Question 14 continues on page 12

Marks

<u>Question 14</u> (continued)

(c) The tangents to the curve $xy = c^2$ at the points $P\left(cp, \dfrac{c}{p}\right)$ and $Q\left(cq, \dfrac{c}{q}\right)$ intersect at R. The chord PQ passes through the point $A(0, 2c^2)$



(i) Show that $2cpq = p + q$ *1*

(ii) Show that the tangent at P has the equation $x + p^2 y = 2cp$ *2*

(iii) Hence show that R has coordinates $\left(\dfrac{2cpq}{p+q}, \dfrac{2c}{p+q}\right)$ *2*

(iv) Hence, or otherwise, find the equation of the locus of R. *1*

End of Question 14

Marks

Question 15 ***(15 marks)*** Use a *separate* answer sheet

(a)

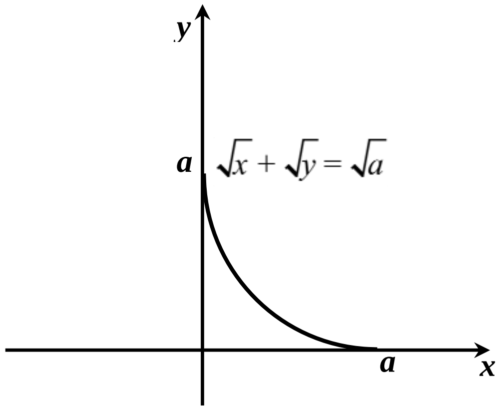


Diagram 1

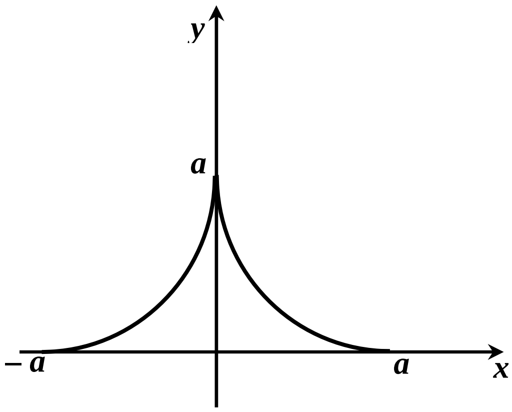


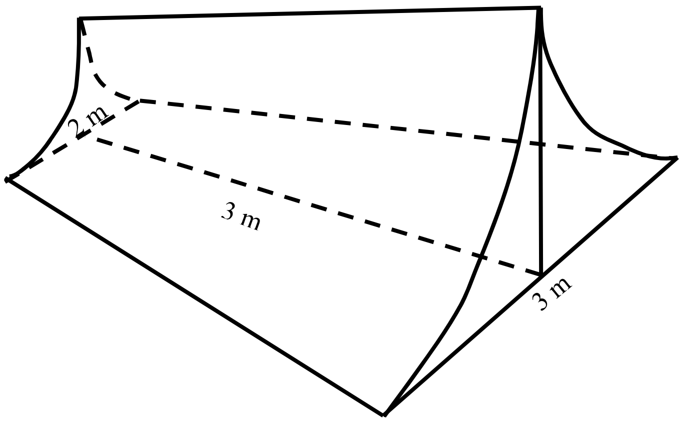
Diagram 2

Diagram 1 shows the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$. Diagram 2 shows the same curve and its reflection in the y-axis.

(i) What is the equation of the function illustrated in Diagram 2? *1*

(ii) Show that the area enclosed by the function and the x-axis, as illustrated in Diagram 2, is given by $\frac{a^2}{3}$ units^2 . *2*

(b) The a tent is formed by draping material over a pole. The tent ends up being 3 metres in length and the base of the tent is a trapezium with parallel sides of 3 metres and 2 metres.



Slices are taken perpendicular to the axis of the tent, and it is noted that each slice is similar to the region illustrated in Diagram 2 of part a). *3*

Calculate the volume of the tent.

Question 15 continues on page 14

Marks

<u>Question 15</u> (continued)

(c) In an aerobatics display, Cynthia and Rebel jump from a great height and go through a period of free fall before opening their parachutes. While in free fall at speed v m/s, Cynthia experiences air resistance kv per unit mass, but Rebel, who spread eagles, experiences air resistance $kv + \frac{2k^2}{g}v^2$ per unit mass.

(i) Using a force diagram, show that Cynthia's terminal velocity is $\frac{g}{k}$ m/s *2*

(ii) Find Rebel's terminal velocity. *2*

(iii) Cynthia opens her parachute when her speed is $\frac{g}{3k}$ m/s. Find the time she has been in free fall. *2*

(d) Prove by mathematical induction that for all integers $n \geq 0$ *3*

$$\cos\alpha\cos 2\alpha\cos 4\alpha\ldots\cos 2^n\alpha = \frac{\sin\left(2^{n+1}\alpha\right)}{2^{n+1}\sin\alpha}$$

End of Question 15

Marks

Question 16 ***(15 marks)*** Use a *separate* answer sheet

(a) (i) How many five digit integers use the digits 1 and 2, and no other? *2*
(e.g. 11 221 , 21 212 but **not** 11 111 or 22 222)

(ii) How many five digit integers use exactly two different digits? *2*

(b) Let $I_n = \int_0^1 \sqrt{x}\,(1-x)^n\,dx$, where n is an integer and $n \geq 0$

(i) Show that $I_n = \dfrac{2n}{2n+3} I_{n-1}$ *3*

(ii) Hence evaluate $\int_0^1 \sqrt{x}\ (1-x)^3\,dx$ *1*

(c) The positive integers x, y and z, where $x < y$, satisfy

$$x^3 + y^3 = kz^3$$

where k is a given positive integer.

(i) In the case $x + y = k$, show that $z^3 = k^2 - 3kx + 3x^2$ *2*

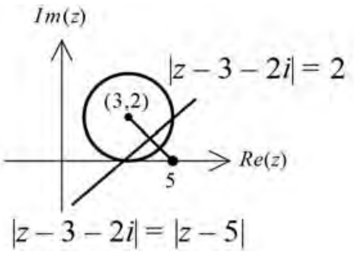
(ii) Deduce that $\dfrac{4z^3 - k^2}{3}$ is a perfect square *1*

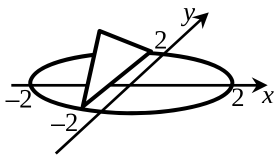
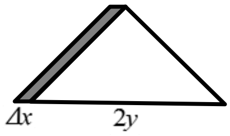
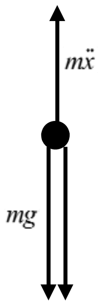
(iii) Hence, or otherwise, deduce that $\dfrac{1}{4}k^2 \leq z^3 < k^2$ *2*

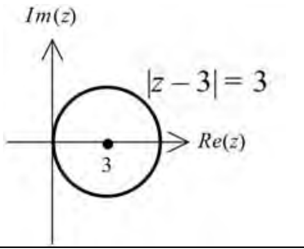
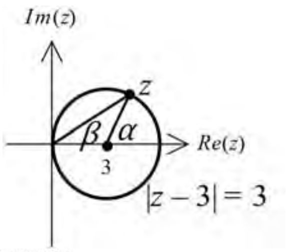
(iv) Use these results to find a solution of $x^3 + y^3 = 20z^3$ *2*

End of paper

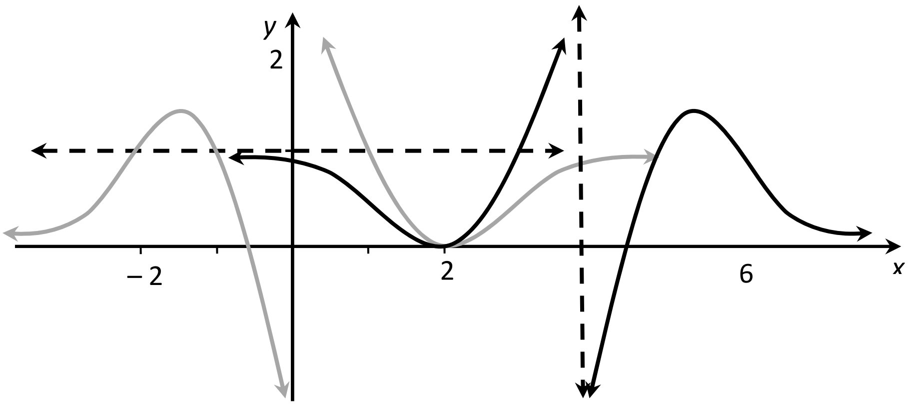
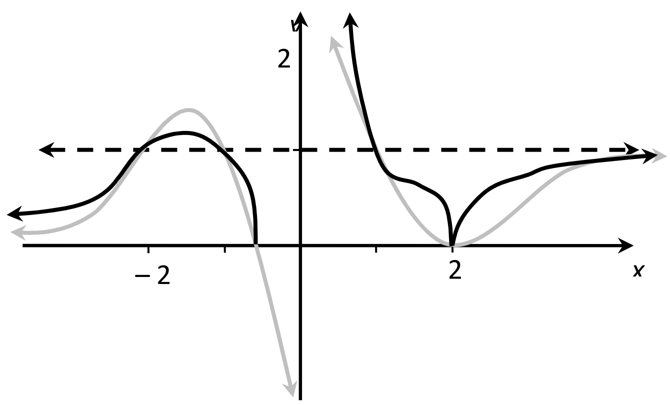
BAULKHAM HILLS HIGH SCHOOL
YEAR 12 EXTENSION 2 TRIAL HSC 2015 SOLUTIONS

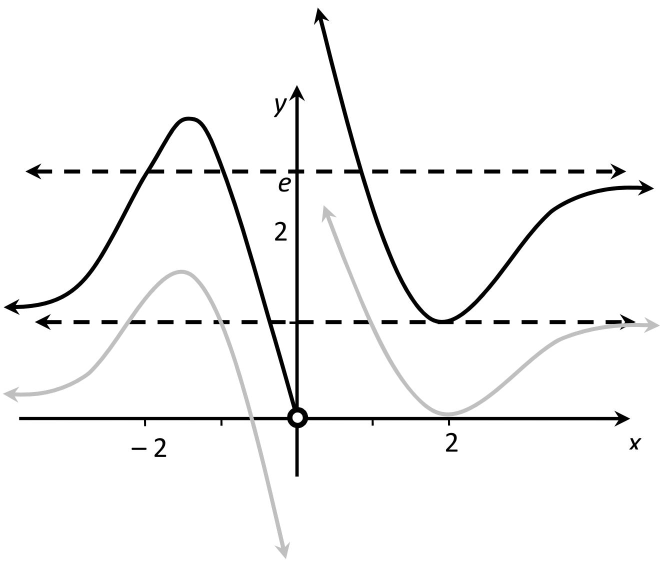
Solution	Marks	Comments
SECTION I		
1. D – as $y^2 > 0$, focii are located on the y axis, thus $e^2 = \frac{a^2 + b^2}{b^2}$ $= \frac{9+4}{4}$ $= \frac{13}{4}$ $e = \frac{\sqrt{13}}{2}$	**1**	
2. D - $\int_0^{\frac{1}{2}} \frac{x^2}{\sqrt{1-x^2}}\,dx = \int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \times \cos\theta\,d\theta$ $= \int_0^{\frac{\pi}{6}} \sin^2\theta\,d\theta$ $x = \sin\theta$ $dx = \cos\theta\,d\theta$ when $x = 0$, $\theta = 0$ $x = \frac{1}{2}$, $\theta = \frac{\pi}{6}$	**1**	
3. C – $z = r\text{cis}\,\theta$ $\sqrt{z} = \sqrt{r}\text{cis}\frac{\theta}{2}$ Thus the argument would be halved (eliminating A and B), as $r > 1$ then $\sqrt{r} > 1$ so **C**	**1**	
4. A –  *Note:* • $\lvert z-3-2i\rvert = \lvert z-5\rvert$ is the perpendicular bisector of (3,2) and (5,0) • Both *Re(z)* = 5 and *Im(z)*= 0 are tangents to the circle • $\lvert z - i\rvert = \lvert z+1\rvert$ would miss the circle all together (perpendicular bisector of (- 1,0) and (0,1))	**1**	
5. B - $\int_5^y dy = \int_1^4 \sqrt{2x^6+1}\,dx$ $y - 5 = \int_1^4 \sqrt{2x^6+1}\,dx$ $y = \int_1^4 \sqrt{2x^6+1}\,dx + 5$	**1**	
6. A – $ax^2 + bx + c = a(x+5)(x-3)$ $= a\left(x^2 + 2x - 15\right)$ This parabola will have a stationary point at $(-1, -8)$ *Noting that AOS is the average of the roots* $\therefore -8 = a\{(-1)^2 + 2(-1) - 15\}$ $= -16a$ $a = \frac{1}{2}$ $a = \frac{1}{2}, b = 1, c = -\frac{15}{2}$	**1**	
7. C – $\Sigma\alpha^3 - 3\Sigma\alpha^2 + 6 = 0$ $\Sigma\alpha^3 - 3(9) + 6 = 0$ $\Sigma\alpha^3 = 21$ $\Sigma\alpha = 3$ $\Sigma\alpha\beta = 0$ $\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$ $= (-3)^2 - 2(-0)$ $= 9$	**1**	

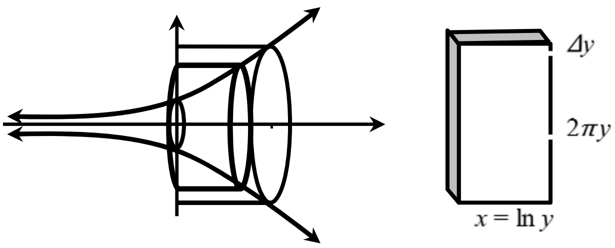
Solution	Marks	Comments
8. B –   $A = \frac{1}{2}\left(\frac{1}{2} \times 2y \times 2y\right)$ $= y^2$ $\Delta V = \left(4 - x^2\right)\Delta x$ $V = \lim_{\Delta x \to 0} \sum_{x=-2}^{2} \left(4 - x^2\right) \Delta x$ $= \int_{-2}^{2} \left(4 - x^2\right) dx$	1	
9. C -  $m\ddot{x} = -mg - 2mkv^2$ $\ddot{x} = -g - kv^2$ $v\frac{dv}{dx} = -\left(g + kv^2\right)$ $\int_0^H dx = -\int_u^0 \frac{v}{g + kv^2} dv$ $H = \int_0^u \frac{v}{g + kv^2} dv$	1	
10. B – As we know that one jellybean is definitely black, then we only need to consider outcomes involving black jellybeans $P(BB \text{ given that one jellybean is definitely } B) = \frac{P(BB)}{1 - P(RR)}$ $= \frac{\frac{2}{9}}{1 - \frac{2}{9}}$ $= \frac{2}{7}$	1	

Solution	Marks	Comments
SECTION II		
QUESTION 11		
11(a) $\int_e^{e^2} \frac{dx}{x\ln x} = \int_1^2 \frac{du}{u}$ $u = \ln x$, $du = \frac{dx}{x}$ when $x = e$, $u = 1$; $x = e^2$, $u = 2$ $= \left[\ln u\right]_1^2$ $= \ln 2$	**2**	**2 marks** • Correct solution **1 mark** • Finds the correct primitive function i.e. $\ln(\ln x)$ • Changes the definite integral via a correct substitution
11(b) (i) $\frac{z}{w} = \frac{p+2i}{1-2i} \times \frac{1+2i}{1+2i}$ $= \frac{p + 2p\,i + 2i - 4}{1+4}$ $= \frac{p-4}{5} + \frac{2(p+1)}{5}i$	**1**	**1 mark** • Correct answer
11(b) (ii) $\left\lvert\frac{z}{w}\right\rvert = 13$ $\frac{\lvert z\rvert}{\lvert w\rvert} = 13$ $\frac{p^2+4}{5} = 169$ $p^2 = 841$ $p = \pm\sqrt{841}$	**2**	**2 marks** • Correct solution **1 mark** • Correctly finds the modulus of z, w or $\frac{z}{w}$
11(c) As the coefficients are real, then complex solutions appear in conjugate pairs i.e. $1 - 2i$ is also a root. $\alpha\beta\gamma = -\frac{5}{2}$ $(1+2i)(1-2i)\gamma = -\frac{5}{2}$ $5\gamma = -\frac{5}{2}$ $\gamma = -\frac{1}{2}$ $\therefore \beta = 1 - 2i$ and $\gamma = -\frac{1}{2}$	**2**	**2 marks** • Correct answers **1 mark** • Uses conjugate root theorem in an attempt to identify the second root • Correctly identifies one of the other two roots
11(d) (i)  	**2**	**2 marks** • Correct solution **1 mark** • Identifies the locus as a circle
11(d) (ii)  $\alpha = \arg(z-3)$ and $\beta = \arg z$ $\alpha = 2\beta$ ($\angle$ at centre = $\angle$ at circumference on same "arc") $\therefore \arg(z-3) = 2\arg z$	**1**	**1 mark** • Correct explanation
11(e) $\overrightarrow{BA} = i\,\overrightarrow{BC}$ $(1+i) - B = i[(7+3i) - B]$ $= 7i - 3 - iB$ $(1-i)B = 1 + i + 3 - 7i$ $B = \frac{4-6i}{1-i} \times \frac{1+i}{1+i}$ $= \frac{10-2i}{2}$ $= 5 - i$	**2**	**2 marks** • Correct solution **1 mark** • Recognises that the rotation of a vector 90° anticlockwise represents the multiplication by i

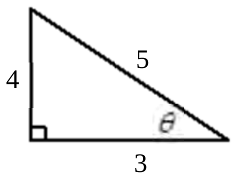
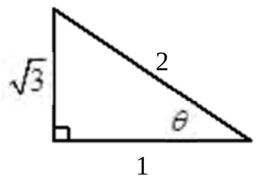
Solution	Marks	Comments
11(f) (i) $(5-i)^2(1+i) = (25-10i-1)(1+i)$ $= (24-10i)(1+i)$ $= 24-10i+24i+10$ $= 34+14i$	**1**	**1 mark** • Correct answer
11(f) (ii) $\arg\left[(5-1)^2(1+i)\right] = \arg(34+14i)$ $2\arg(5-i)+\arg(1+i) = \arg(34+14i)$ $2\tan^{-1}\left(-\frac{1}{5}\right)+\frac{\pi}{4} = \tan^{-1}\frac{14}{34}$ $-2\tan^{-1}\frac{1}{5}+\frac{\pi}{4} = \tan^{-1}\frac{7}{17}$ $\tan^{-1}\frac{7}{17}+2\tan^{-1}\frac{1}{5} = \frac{\pi}{4}$	**2**	**2 marks** • Correct solution **1 mark** • Uses the properties of arguments in a valid attempt to prove the result

Solution	Marks	Comments
QUESTION 12		
12(a) $\displaystyle\int_0^{\frac{\pi}{3}} \frac{dx}{1-\sin x} = \int_0^{\frac{1}{\sqrt{3}}} \frac{\frac{2dt}{1+t^2}}{1-\frac{2t}{1+t^2}}$ $t = \tan\frac{x}{2}$ $dt = \frac{2dt}{1+t^2}$ when $x = 0$, $t = 0$ $x = \frac{\pi}{3}$, $t = \frac{1}{\sqrt{3}}$ $\displaystyle = \int_0^{\frac{1}{\sqrt{3}}} \frac{2dt}{1+t^2-2t}$ $\displaystyle = \int_0^{\frac{1}{\sqrt{3}}} \frac{2dt}{(t-1)^2}$ $\displaystyle = 2\left[\frac{1}{1-t}\right]_0^{\frac{1}{\sqrt{3}}}$ $\displaystyle = 2\left(\frac{\sqrt{3}}{\sqrt{3}-1} - 1\right)$ $\displaystyle = \frac{2}{\sqrt{3}-1}$	3	**3 marks** • Correct solution **2 marks** • Obtains the correct primitive in terms of the substituted variable **1 mark** • Obtains the correct integrand in terms of the substituted variable.
12(b) $\displaystyle\int \frac{x+1}{\sqrt{x^2+4x-3}}\,dx = \frac{1}{2}\int \frac{2x+4}{\sqrt{x^2+4x-3}}\,dx - \int \frac{dx}{\sqrt{(x+2)^2-7}}$ $\displaystyle = \sqrt{x^2+4x-3} - \ln\left|x+2+\sqrt{x^2+4x-3}\right| + c$	3	**3 marks** • Correct solution **2 marks** • Successfully splits the original integral into two new integrals **1 mark** • Completes the square in the denominator • Creates $\frac{du}{\sqrt{u}}$ • Correctly uses standard integral for their denominator, after completing the square
12(c) (i)  	2	**2 marks** • Correct graph **1 mark** • Basic shape correct with most of key features **Key Features** • Reflection in the line $x = 2$ • Vertical asymptote moves to $x = 4$ • Maximum turning point between $x= 5$ and $x= 6$ • y-values of key features are unchanged
12(c) (ii)  	2	**2 marks** • Correct graph **1 mark** • Basic shape correct with most of key features **Key Features** • x-intercepts become critical points. • Horizontal asymptote remains at $y= 1$ • turning point stays with same x-value • double root becomes cusp • single root becomes vertical tangent

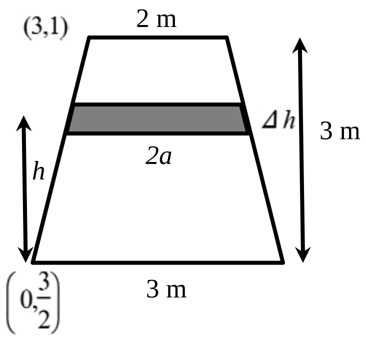
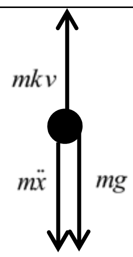
Solution	Marks	Comments
12(c) (iii) 	**2**	**2 marks** • Correct graph **1 mark** • Basic shape correct with most of key features **Key Features** • Horizontal asymptotes: LHS becomes y = 1 , RHS becomes y = e • Point discontinuity at the origin • Vertical asymptote on positive side of x= 0. • Minimum turning point has $y$value of 1 • Maximimum turning point has y value > e
12(d) (i) $y = \log_{a^n} x \;\Rightarrow\; x = (a^n)^y$ $= a^{ny}$ $ny = \log_a x$ $y = \frac{1}{n}\log_n x$ $\therefore \log_{a^n} x = \frac{1}{n}\log_a x$	**1**	**1 mark** • Shows result correctly
12(d) (ii) $\log_2 3 + \log_4 3 + \log_{16} 3 + \log_{256} 3 + ... = \log_2 3 + \log_{2^2} 3 + \log_{2^4} 3 + \log_{2^8} 3 + ...$ $= \log_2 3 + \frac{1}{2}\log_2 3 + \frac{1}{4}\log_2 3 + \frac{1}{8}\log_2 3 + ...$ $= \log_2 3\left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...\right)$ $= \log_2 3\left\{\frac{1}{1-\frac{1}{2}}\right\}$ $= 2\log_2 3$	**2**	**2 marks** • Correct solution **1 mark** • Uses part (i) to simplify all of the terms in the given expression.

Solution	Marks	Comments
QUESTION 13		
13(a) (i) $e^2=\frac{a^2-b^2}{a^2}$ $\frac{1}{4}=\frac{a^2-b^2}{a^2}$ $a^2=4a^2-4b^2$ $b^2=\frac{3}{4}a^2$ $\frac{4}{a^2}+\frac{9}{b^2}=1$ $\frac{4}{a^2}+\frac{12}{a^2}=1$ $a^2=16$ $\therefore \quad a=4\,,\ b=2\sqrt{3}$	**3**	**3 marks** • Correct solution **2 marks** • Finds the value of *a* or *b* **1 mark** • Creates simultaneous equations that could be used to find values
13(a) (ii) focii: $(\pm ae,0)=\left(4\times\frac{1}{2},0\right)$ $=(\pm 2,0)$ dircetrices: $x=\pm\frac{a}{e}$ $x=\pm 4\times 2$ $x=\pm 8$	**1**	**1 mark** • Correct answer
13(b) (i) $\alpha^2+2\beta^2+\gamma^2=0$ **,** as $x=0$ is not a root, then all of the roots cannot be real other wise this sum would have to be >0. As the coefficients are real, there must be an even number of complex roots, so the only possibility is two complex roots, which would be conjugates. However; $\alpha^2=-\beta^2$ and $\gamma^2=-\beta^2 \ \Rightarrow\ \alpha^2=\gamma^2$ $\alpha=\pm\gamma$, meaning if α is real, then so is β , yet there is only one real root Thus β is real and both α and γ are not real	**1**	**1 mark** • Valid explanation
13(b) (ii) $\alpha^2=\gamma^2$ $\alpha=\pm\gamma$ As they are conjugates, this is only possible if the real part is zero. i.e. α and γ are purely imaginary	**1**	**1 mark** • Valid explanation
13(b) (iii) Let $\alpha=ki\,,\ \beta=m\,,\ \gamma=-ki$ $ki+m-ki=-A$ $m=-A$ $kim-kim-k^2i^2=B$ $B=k^2$ $-k^2mi^2=-8$ $k^2m=-8$ $\alpha^2+\beta^2=0$ $k^2i^2+m^2=0$ $m^2=k^2$ $\therefore \quad m^3=-8$ $m=-2\,,\ k=\pm 2$ Thus $A=\ 2$ and $B=\ 4$	**2**	**2 marks** • Correct solution **1 mark** • Evaluates *A* or *B* • Identifies the roots of the equation
13(c) (i)  Note: if using cylindrical shells for this problem, it is easier to calculate the "outside" volume and subtract from the surrounding cylinder. $A(y)=2\pi y\ln y$ $\Delta V=2\pi y\ln y\ \Delta y$ outside volume $=\lim_{\Delta y\to 0}\sum_{y=1}^{e}2\pi y\ln\ y\ \Delta y$ $=2\pi\int_1^e y\ln y\ \ dy$ $\therefore V=\pi\left(e^2\right)(1)-$ outside volume $V=\pi e^2-2\pi\int_1^e y\ln y\ \ dy$	**3**	**3 marks** • Correct solution **2 marks** • Uses the method of cylindrical shells correctly **1 mark** • Correctly establishes the area of the cross-sectional face or equivalent. • Finds the volume of the surrounding cylinder

Solution	Marks	Comments
13(c) (ii) $V = \pi e^2 - 2\pi\left\{\left[\frac{1}{2}y^2\ln y\right]_1^e - \frac{1}{2}\int_1^e y\ dy\right\}$ $\quad u = \ln y \quad v = \frac{1}{2}y^2 \quad du = \frac{dy}{y} \quad dv = y\,dy$ $= \pi e^2 - 2\pi\left\{\frac{1}{2}e^2 - \frac{1}{4}\left[y^2\right]_1^e\right\}$ $= \pi e^2 - 2\pi\left(\frac{1}{2}e^2 - \frac{1}{4}e^2 + \frac{1}{4}\right)$ $= \pi e^2 - \pi e^2 + \frac{\pi}{2}e^2 - \frac{\pi}{2}$ $= \frac{\pi}{2}\left(e^2 - 1\right)\ units^3$	**3**	**3 marks** • Correct solution **2 marks** • Uses integration by parts to find the value of the integral, or equivalent **1 mark** • Attempts to use integration by parts, or equivalent merit
13(c) (iii) Had we sliced perpendicular to the rotation instead of parallel $V = \pi\int_0^1 e^{2x}\,dx$ $\pi\int_0^1 e^{2x}\ dx = \frac{\pi}{2}\left(e^2 - 1\right)$ $\therefore \int_0^1 e^{2x}\ dx = \frac{1}{2}\left(e^2 - 1\right)$	**1**	**1 mark** • Establishes result using their answer to part (ii)

Solution	Marks	Comments
QUESTION 14		
14(a) (i) $z^n + \frac{1}{z^n} = (\text{cis}\theta)^n + (\text{cis}\theta)^{-n}$ $= \text{cis}(n\theta) + \text{cis}(-n\theta)$ $= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$ $= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$ $= 2\cos n\theta$	**1**	**1 mark** • Correct solution
14(a) (ii) $5x^4 - 11x^3 + 16x^2 - 11x + 5 = 0$ $5x^2 - 11x + 16 - \frac{11}{x} + \frac{5}{x^2} = 0$ $5\left(x^2 + \frac{1}{x^2}\right) - 11\left(x + \frac{1}{x}\right) + 16 = 0$ $10\cos 2\theta - 22\cos\theta + 16 = 0$ $5\cos 2\theta - 11\cos\theta + 8 = 0$ $10\cos^2\theta - 5 - 11\cos\theta + 8 = 0$ $10\cos^2\theta - 11\cos\theta + 3 = 0$ $(5\cos\theta - 3)(2\cos\theta - 1) = 0$ $\cos\theta = \frac{3}{5}$ or $\cos\theta = \frac{1}{2}$   $\therefore x = \frac{3}{5} \pm \frac{4}{5}i$ or $x = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$	**3**	**3 marks** • Correct solution **2 marks** • Correctly finding values for $\cos\theta$ • Correct solution for incorrect values of $\cos\theta$ **1 mark** • Correctly finding the equation $10\cos 2\theta - 22\cos\theta + 16 = 0$ or equivalent
14(b) (i) $\angle RMA = \angle ABN$ (exterior ∠ cyclic quadrilateral = opposite interior ∠) Similarly $\angle AQP = \angle ABN$ (in cyclic quadrilateral *ABPQ*) $\therefore \angle AQP = \angle RMA$ Thus exterior $\angle AQP$ is equal to the opposite interior $\angle RMA$ i.e. *QAMR* is a cyclic quadrilateral	**2**	**2 marks** • Correct solution **1 mark** • Significant progress towards correct solution
14(b) (ii) $\angle TMR = \angle SMN$ (vertically opposite ∠ 's =) $\angle SMN = \angle MAN$ (alternate segment theorem) $\angle MAN = \angle PAQ$ (vertically opposite ∠ 's =) $\angle PAQ = \angle TRM$ (exterior ∠ cyclic quadrilateral = opposite interior ∠) $\therefore \angle TRM = \angle TMR$ $\Delta\, TRM$ is isosceles (2 = ∠'s) $TM = TR$ (= sides in isosceles Δ)	**3**	**3 marks** • Correct solution **2 marks** • Correct solution with poor reasoning • Significant progress towards a correct solution **1 mark** • Progress towards a correct solution involving some relevant logic
14(c) (i) $m_{PQ} = m_{PA}$ $\dfrac{\frac{c}{p} - \frac{c}{q}}{cp - cq} = \dfrac{\frac{c}{p} - 2c^2}{cp - 0}$ $\frac{q - p}{pq(p - q)} = \frac{1 - 2cp}{p^2}$ $-\frac{1}{pq} = \frac{1 - 2cp}{p^2}$ $-p^2 = pq - 2cp^2q$ $-p = q - 2cpq$ $2cpq = p + q$	**1**	**1 mark** • Correct solution

Solution	Marks	Comments
14(c) (ii) $y=\frac{c^2}{x}$ $\frac{dy}{dx}=-\frac{c^2}{x^2}$ when $x=2p$, $\frac{dy}{dx}=-\frac{c^2}{c^2p^2}$ $=-\frac{1}{p^2}$ $\therefore$ required slope $=-\frac{1}{p^2}$ $y-\frac{c}{p}=-\frac{1}{p^2}(x-cp)$ $p^2y-cp=-x+cp$ $x+p^2y=2cp$	**2**	**2 marks** • Substitutes into point-slope formula and arrives at the required result **1 mark** • Finds the required slope
14(c) (iii) $x+p^2y=2cp$ $x+q^2y=2cq$ $(p^2-q^2)y=2c(p-q)$ $y=\frac{2c(p-q)}{(p+q)(p-q)}$ $y=\frac{2c}{p+q}$ $\Rightarrow$ $x=2cp-\frac{2cp^2}{p+q}$ $x=2c\left(\frac{p(p+q)-p^2}{p+q}\right)$ $x=\frac{2c(p^2+pq-p^2)}{(p+q)}$ $x=\frac{2cpq}{p+q}$ $\therefore R\left(\frac{2cpq}{p+q},\frac{2c}{p+q}\right)$ ***OR*** $x+p^2y=\frac{2cpq}{p+q}+\frac{2cp^2}{p+q}$ $=\frac{2cp(q+p)}{p+q}$ $=2cp$ $x+q^2y=\frac{2cpq}{p+q}+\frac{2cq^2}{p+q}$ $=\frac{2cq(p+q)}{p+q}$ $=2cq$ $\therefore$ as R lies on both tangents, it must be the point of intersection	**2**	**2 marks** • Correct solution **1 mark** • Successfully finds the x or y coordinate using a valid method. • Successfully substitutes R into one of the tangents. • Attempts to substitute R into both tangents
14(c) (iv) $x=\frac{2cpq}{p+q}$ $x=\frac{p+q}{p+q}$ $x=1$ $\therefore$ locus of R is $x=1$, however as tangents cannot meet inside the hyperbola or on the x axis, the locus is restricted to $y<c^2$, $y\neq 0$	**1**	**1 mark** • Correct answer *Note: no penalty for ignoring restriction on locus.*

Solution	Marks	Comments
QUESTION 15		
15(a) (i) A reflection in the y-axis would be represented by $y=f(\lvert x \rvert)$ So the function in Diagram 2 is $\sqrt{\lvert x \rvert}+\sqrt{y}=\sqrt{a}$	1	**1 mark** • Correct answer
15(a) (ii) $A=2\int_0^a (\sqrt{a}-\sqrt{x})^2\,dx$ $=2\int_0^a (a-2\sqrt{a}\sqrt{x}+x)\,dx$ $=2\left[ax-\frac{4}{3}\sqrt{a}\,x\sqrt{x}+\frac{1}{2}x^2\right]_0^a$ $=2\left(a^2-\frac{4}{3}a^2+\frac{1}{2}a^2\right)$ $=\frac{a^2}{3}$	2	**2 marks** • Correct solution **1 mark** • Establishes the correct integral required to find the desired result.
15(b)  $m=\frac{1-\frac{3}{2}}{3-0}$ $=-\frac{1}{6}$ $a-1=-\frac{1}{6}(h-3)$ $a=-\frac{h}{6}+\frac{3}{2}$ $a=\frac{9-h}{6}$ $A(h)=\frac{(9-h)^2}{6\times 36}$ $=\frac{(9-h)^2}{108}$ $\Delta V=\frac{1}{108}(9-h)^2\,\Delta h$ $V=\lim_{\Delta h\to 0}\sum_{h=0}^{3}\frac{1}{108}(9-h)^2\,\Delta h$ $=\frac{1}{108}\int_0^3 (9-h)^2\,dh$ $=-\frac{1}{324}\left[(9-h)^3\right]_0^3$ $=-\frac{1}{324}\left(6^3-9^3\right)$ $=\frac{19}{12}\,units^3$	3	**3 marks** • Correct solution **2 marks** • Establishes an integral that would find the correct volume • Finds a volume from an integral that used the formula from part (a) **1 mark** • Finds the area of the cross-sectional face • Establishes an integral using the formula from part (a)
15(c) (i) $m\ddot{x}=mg-mkv$ $\ddot{x}=g-kv$ Terminal velocity occurs when $\ddot{x}=0$ $0=g-kv$ $v=\frac{g}{k}$  	2	**2 marks** Correct solution **1 mark** • Acknowledges the condition for terminal velocity • Correct force diagram *Note: force diagram does not have to have resultant force drawn*

Solution	Marks	Comments
15(c) (ii) $m\ddot{x} = \text{mg} - m\left(kv + \frac{2k^2}{g}v^2\right)$ $\ddot{x} = g - kv - \frac{2k^2}{g}v^2$ Terminal velocity occurs when $\ddot{x} = 0$ $\frac{2k^2}{g}v^2 + kv - g = 0$ $2k^2v^2 + gkv - g^2 = 0$ $(2kv - g)(kv + g) = 0$ $v = \frac{g}{2k}$ or $v = -\frac{g}{k}$ But $v > 0$, $\therefore v = \frac{g}{2k}$	**2**	**2 marks** Correct solution **1 mark** • Establishes a quadratic that will lead to a correct answer.
15(c) (iii) $\frac{dv}{dt} = g - kv$ $\int_0^T dt = \int_0^{\frac{g}{3k}} \frac{dv}{g - kv}$ $T = -\frac{1}{k}\left[\ln(g - kv)\right]_0^{\frac{g}{3k}}$ $T = \frac{1}{k}\ln\frac{3}{2}$	**2**	**2 marks** Correct solution **1 mark** • Correct integrand in terms of v
15(d) When $n = 0$; LHS = LHS $= \cos 2^0\alpha$ $= \cos\alpha$ RHS $= \frac{\sin 2\alpha}{2\sin\alpha}$ $= \frac{2\sin\alpha\cos\alpha}{2\sin\alpha}$ $= \cos\alpha$ $\therefore$ LHS=RHS Hence the result is true for $n = 1$ Assume the result is true for $n = k$ where k is an integer i.e. $\cos\alpha\cos 2\alpha\cos 4\alpha\ldots\cos 2^k\alpha = \frac{\sin\left(2^{k+1}\alpha\right)}{2^{k+1}\sin\alpha}$ Prove the result is true for $n = k + 1$ i.e. $\cos\alpha\cos 2\alpha\cos 4\alpha\ldots\cos 2^{k+1}\alpha = \frac{\sin\left(2^{k+2}\alpha\right)}{2^{k+2}\sin\alpha}$ ***PROOF:*** $\cos\alpha\cos 2\alpha\cos 4\alpha\ldots\cos 2^{k+1}\alpha = \cos\alpha\cos 2\alpha\cos 4\alpha\ldots\cos 2^k\alpha\cos 2^{k+1}\alpha$ $= \frac{\sin\left(2^{k+1}\alpha\right)}{2^{k+1}\sin\alpha} \times \cos 2^{k+1}\alpha$ $= \frac{\frac{1}{2} \times 2\sin 2^{k=1}\alpha\cos 2^{k=1}\alpha}{2^{k+1}\sin\alpha}$ $= \frac{\sin\left(2 \times 2^{k=1}\alpha\right)}{2 \times 2^{k+1}\sin\alpha}$ $= \frac{\sin 2^{k+2}\alpha}{2^{k+2}\sin\alpha}$ Hence the result is true for $n = k + 1$, if it is true for $n = k$ Since the result is true for $n = 1$, then it is true for all positive integers by induction.	**3**	There are 4 key parts of the induction; 1. Proving the result true for $n = 0$ 2. Clearly stating the assumption and what is to be proven 3. Using the assumption in the proof 4. Correctly proving the required statement **3 marks** • Successfully does all of the 4 key parts **2 marks** • Successfully does 3 of the 4 key parts **1 mark** • Successfully does 2 of the 4 key parts

Solution	Marks	Comments
QUESTION 16		
16(a) (i) all numbers between 10 000 and 100 000 contain 5 digits, so we could have Four 1’s and a 2 $= {}^5\mathbf{C}_4$ Three 1’s and two 2’s $= {}^5\mathbf{C}_3$ Two 1’s and three 2’s $= {}^5\mathbf{C}_2$ A 1 and four 2’s $= {}^5\mathbf{C}_1$ Total numbers $= {}^5\mathbf{C}_1 + {}^5\mathbf{C}_2 + {}^5\mathbf{C}_3 + {}^5\mathbf{C}_4$ $= 2^5 - 2$ $= 30$	2	**2 marks** • Correct answer **1 mark** • Breaks the problem into logical cases • Finds the amount of numbers for at least one case
16(a) (ii) There are ${}^9\mathbf{C}_2$ pairings of two digits, not including 0 Total numbers (not using 0) $= {}^9\mathbf{C}_2 \times 30$ $= 1080$ If 0 is included, it cannot occupy the first spot Total numbers (using 0) $= 9 \times ({}^4\mathbf{C}_0 + {}^4\mathbf{C}_1 + {}^4\mathbf{C}_2 + {}^4\mathbf{C}_3)$ $= 9 \times \left(2^4 - 1\right)$ $= 135$ Total numbers $= 1080 + 135$ $= 1215$	2	**2 marks** • Correct answer **1 mark** • Splits the problem into valid cases, and correctly finds one of these cases.
16(b) (i) $I_n = \int_0^1 \sqrt{x}\,(1-x)^n\,dx$ $= \left[\frac{2}{3}x\sqrt{x}\,(1-x)^n\right]_0^1 + \frac{2n}{3}\int_0^1 x\sqrt{x}\,(1-x)^{n-1}\,dx$ $= -\frac{2n}{3}\int_0^1 (1-x)\sqrt{x}\,(1-x)^{n-1}\,dx + \frac{2n}{3}\int_0^1 \sqrt{x}\,(1-x)^{n-1}\,dx$ $= -\frac{2n}{3}I_n + \frac{2n}{3}I_{n-1}$ $3I_n = -2nI_n + 2nI_{n-1}$ $(2n+3)I_n = 2nI_{n-1}$ $I_n = \frac{2n}{2n+3}I_{n-1}$	3	**3 marks** • Correct solution **2 marks** • Substantial progress towards a solution using logical techniques **1 mark** • Attempts to create the reduction formula by using integration by parts, or similar merit.
16(b) (ii) $I_3 = \frac{6}{9}I_2$ $= \frac{2}{3} \times \frac{4}{7} \times I_1$ $= \frac{2}{3} \times \frac{4}{7} \times \frac{2}{5}I_0$ $= \frac{16}{105}\int_0^1 \sqrt{x}\,dx$ $= \frac{16}{105} \times \frac{2}{3}\left[x\sqrt{x}\right]_0^1$ $= \frac{32}{315}$	1	**1 mark** • Correct uses formula to reduce down to $\frac{16}{105}\int_0^1 \sqrt{x}\,dx$

Solution	Marks	Comments
16(c) (i) $x^3 + y^3 = kz^3$ $(x+y)(x^2 - xy + y^2) = (x+y)z^3$ $z^3 = x^2 - xy + y^2$ $= x^2 - x(k-x) + (k-x)^2$ $= x^2 - kx + x^2 + k^2 - 2kx + x^2$ $= k^2 - 3kx + 3x^2$	**2**	**2 marks** • Correct solution **1 mark** • Uses $x + y = k$ and the sum of two cubes in an attempt to show the given result, or equivalent merit.
16(c) (ii) $\dfrac{4z^3 - k^2}{3} = \dfrac{3k^2 - 12kx + 12x^2}{3}$ $= k^2 - 4kx + 4x^2$ $= (k-2x)^2$ which is a perfect square	**1**	**1 mark** • Correct solution
16(c) (iii) $\dfrac{4z^3 - k^2}{3} \geq 0$ $4z^3 - k^2 \geq 0$ $4z^3 \geq k^2$ $z^3 \geq \dfrac{k^2}{4}$ $z^3 = k^2 - 3x(k-x)$ as $k > x$ then $(k-x) > 0$ $\therefore k^2 - 3x(k-x) > 0$ i.e. $z^3 < k^2$ thus $\dfrac{k^2}{4} \leq z^3 < k^2$	**2**	**2 marks** • Correct solution **1 mark** • shows $z^3 \geq \dfrac{k^2}{4}$ • shows $z^3 < k^2$
16(c) (iv) If $k = 20$, $100 \leq z^3 < 400$ $\therefore z$ is either 5 , 6 or 7 However $\dfrac{4z^3 - 400}{3}$ is a perfect square thus $z = 7$ $\left(\dfrac{4z^3 - 400}{3} = 324 = 18^2\right)$ Further $20 - 2x = 18$ $2x = 2$ $x = 1$ $\therefore x = 1, y = 19, z = 7$	**2**	**2 marks** • Correct solution **1 mark** • Finds z