BAULKHAM HILLS HIGH SCHOOL

2016

YEAR 12 TRIAL
HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Black pen is preferred
- Board-approved calculators may be used
- A reference sheet is provided at the back of this paper
- All relevant mathematical reasoning and/or calculations must be shown

Total marks – 100

Section I (Pages 2-6)
10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Section II (Pages 7-16)
90 marks
Attempt Questions 11-16
Allow about 2 hours 45 minutes for this section

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 to 10.

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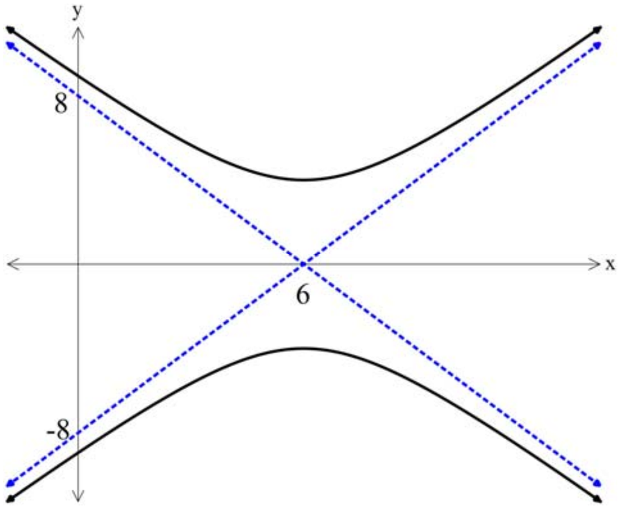
1 The graph shown below could have the equation



(A) $\dfrac{(x-6)^2}{16} - \dfrac{y^2}{9} = -1$

(B) $\dfrac{(x-6)^2}{9} - \dfrac{y^2}{16} = -1$

(C) $\dfrac{(x-6)^2}{4} - \dfrac{y^2}{3} = -1$

(D) $\dfrac{(x-6)^2}{3} - \dfrac{y^2}{4} = -1$

2 On an Argand diagram, the points *A* and *B* represent the complex numbers $z_1 = -2i$ and $z_2 = 1 - \sqrt{3}\,i$. Which of the following statements is true?

(A) $\arg(z_2)^2 = \arg(z_1)$

(B) $\arg\left(\dfrac{z_1}{z_2}\right) = \dfrac{\pi}{6}$

(C) $\arg(z_1\,z_2) = -\dfrac{5\pi}{6}$

(D) $\arg(z_1 - z_2) = \dfrac{3}{2}$

3 Let α, β and γ be the roots of the equation $x^3 + x^2 - 1 = 0$. The polynomial equation with roots $2\alpha, 2\beta$ and 2γ is:

(A) $x^3 + 2x^2 - 8 = 0$

(B) $x^3 + 2x^2 + 8 = 0$

(C) $8x^3 - 4x^2 + 1 = 0$

(D) $8x^3 + 4x^2 - 1 = 0$

4 Given that $w^3 = 1$ and that w is complex, the value of $(1 + w - w^2)^3$ is:

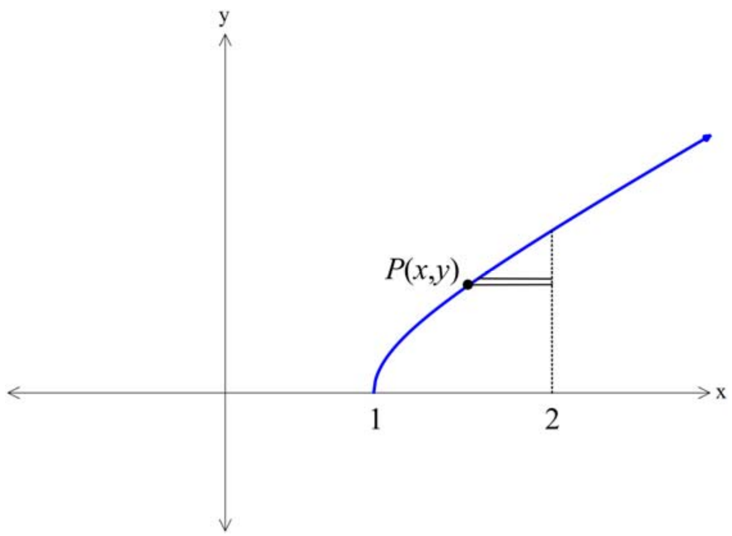
(A) -8

(B) -1

(C) 1

(D) 8

5 The area enclosed by $y = \sqrt{x^2 - 1}$ and the line $x = 2$ and the x axis is rotated about the y axis.



The slice at $P(x,y)$ on the curve is perpendicular to the axis of rotation.
The volume δV on the slice of the annulus is

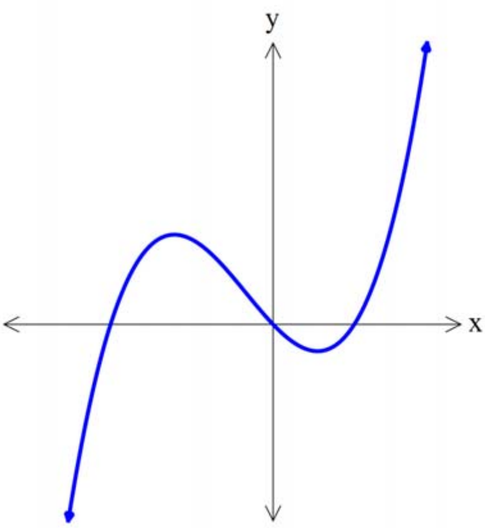
(A) $\pi\left(4 - \sqrt{y^2 + 1}\right)\delta y$

(B) $\pi\left(2 - \sqrt{y^2 + 1}\right)\delta y$

(C) $\pi(1 - y^2)\delta y$

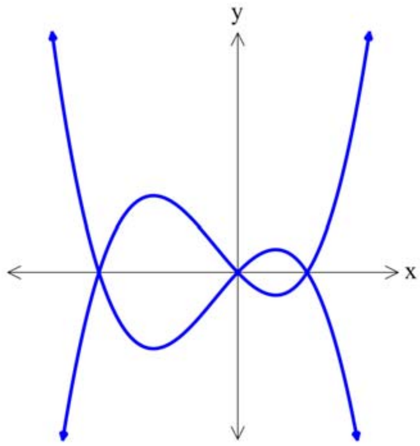
(D) $\pi(3 - y^2)\delta y$

6 The graph of $y = f(x)$ is shown below.

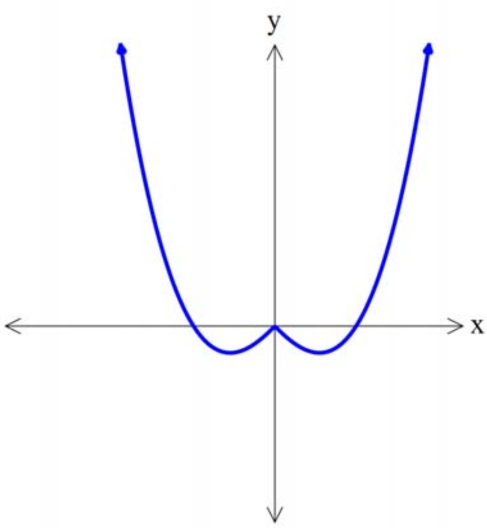


Which is the correct graph of $|y| = f(x)$

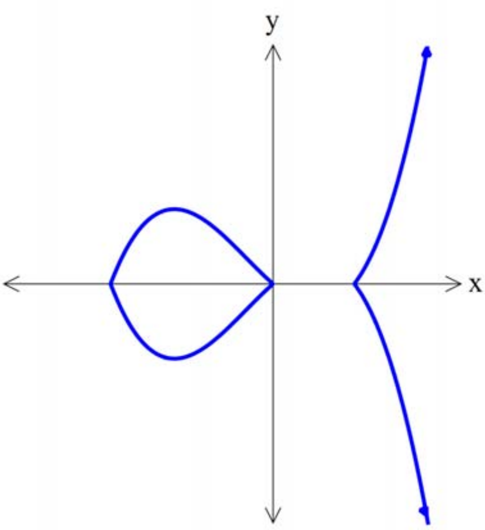
(A)



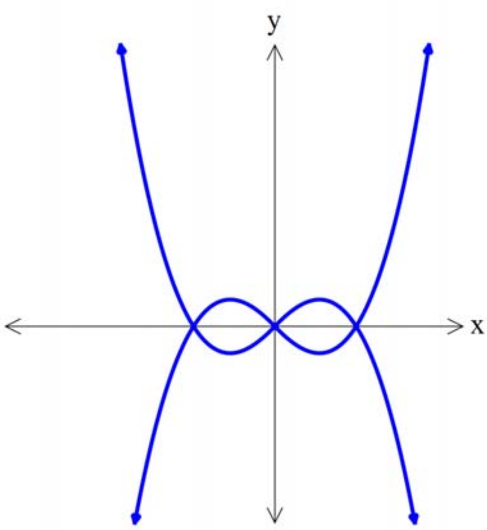
(B)



(C)



(D)



7 Find $\int \frac{x^3 dx}{x^2 + x + 1}$

(A) $\frac{x^2}{2} - x + \tan^{-1} \frac{2x + 1}{\sqrt{3}} + c$

(B) $\frac{x^2}{2} - x + \tan^{-1} \frac{4x + 2}{3} + c$

(C) $\frac{x^2}{2} - x + \frac{4}{3} \tan^{-1} \frac{4x + 2}{3} + c$

(D) $\frac{x^2}{2} - x + \frac{2}{\sqrt{3}} \tan^{-1} \frac{2x + 1}{\sqrt{3}} + c$

8 If $e^x + e^y = 1$ then $\frac{dy}{dx}$ is:

(A) $-e^{x-y}$

(B) e^{x-y}

(C) e^{y-x}

(D) $-e^{y-x}$

9 A particle of mass M kg is projected vertically upwards, from rest, with velocity V ms^{-1} . The resistive force is kv^2 Newtons, where k is a positive constant.
The equation of motion which will enable determination of the maximum height reached is:

(A) $-Mg - Mkv^2 = Mv\frac{dv}{dx}$

(B) $-Mg - kv^2 = Mv\frac{dv}{dx}$

(C) $Mg - Mkv^2 = -Mv\frac{dv}{dx}$

(D) $Mg + kv^2 = M\frac{dv}{dt}$

10 How many ways are there of choosing 3 different numbers in increasing order from the numbers 1, 2, 3, 4, ..., 10 so that no two of the numbers are consecutive?

(A) 20

(B) 48

(C) 56

(D) 72

End of Section I

Section II

90 marks
Attempt questions 11 -16
Allow about 2 hours 45 minutes for this section

Answer each question on the appropriate page of your answer booklet
In Questions 11–16, your responses should include relevant mathematical reasoning and/ or calculations.

Question 11 (15 marks) Answer on the appropriate page

a) Find:

i) $\displaystyle\int \frac{e^{\tan x}}{\cos^2 x}\, dx$ **1**

ii) $\displaystyle\int \frac{1}{\sqrt{x}\,(1-\sqrt{x})}\, dx$ **2**

b) If $z = 2 + i$ and $w = 3 - i$ find $\dfrac{z}{w}$ in the form $a + ib$. **2**

c) i) Show that $z = 1 + i$ is a root of the equation $z^2 - (3 - 2i)z + (5 - i) = 0$. **2**

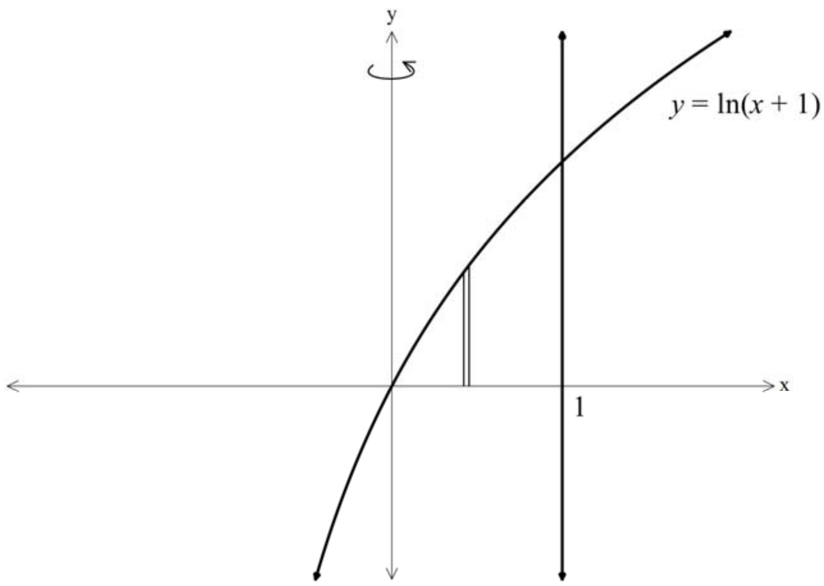
ii) Find the other root of the equation. **1**

d) Shade on an Argand diagram the region represented by the complex number z where $\dfrac{\pi}{4} \le \arg z \le \pi$, $1 \le Im(z) \le 3$ and $|z| \le 4$. **3**

Question 11 continues on next page

Question 11 (continued)

e) The area between the curve $y = \ln(x + 1)$, the x axis and the line $x = 1$ is rotated about the y axis.



Find the volume of the solid of revolution formed using the method of cylindrical shells. **4**

End of Question 11

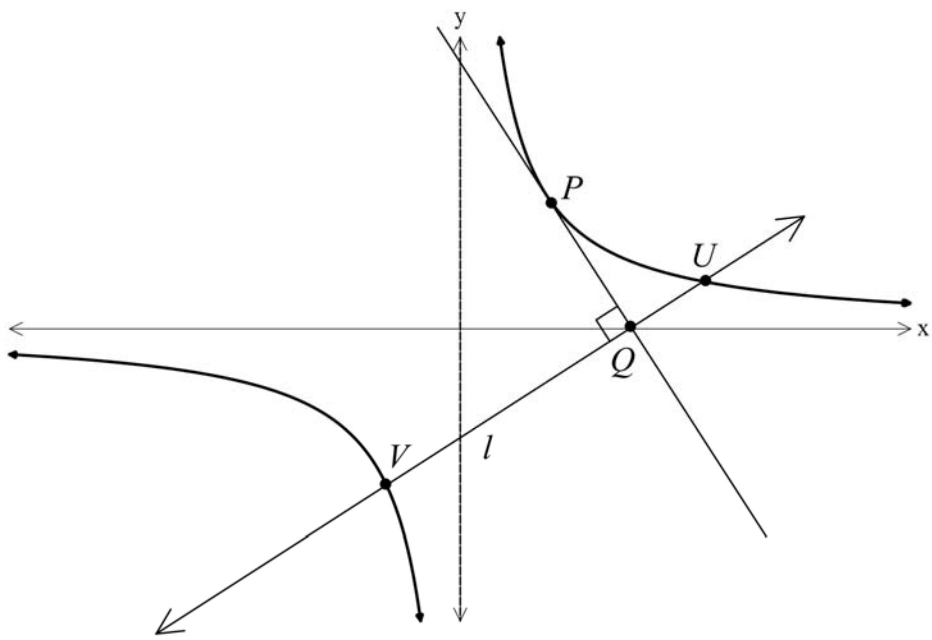
Question 12 (15 marks) Answer on the appropriate page

(a) If α, β and γ are the roots of the cubic equation $x^3 + px + q = 0$, find in terms of p and q, the values of

(i) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$ **2**

(ii) $\alpha^3 + \beta^3 + \gamma^3$ **2**

(b) A tangent is drawn at any point $P\left(ct, \dfrac{c}{t}\right)$ on the hyperbola $xy = c^2$. This tangent meets the x axis at Q. Through Q a straight line l is drawn perpendicular to the tangent. The line l cuts the hyperbola in the two points U and V.



(i) Show the equation of the tangent is $x + t^2y = 2ct$ **2**

(ii) Find the coordinates of Q. **1**

(iii) Find the equation of the line l. **1**

(iv) If M is the midpoint of the interval UV, show that the coordinates of M are $(ct, -ct^3)$. **3**

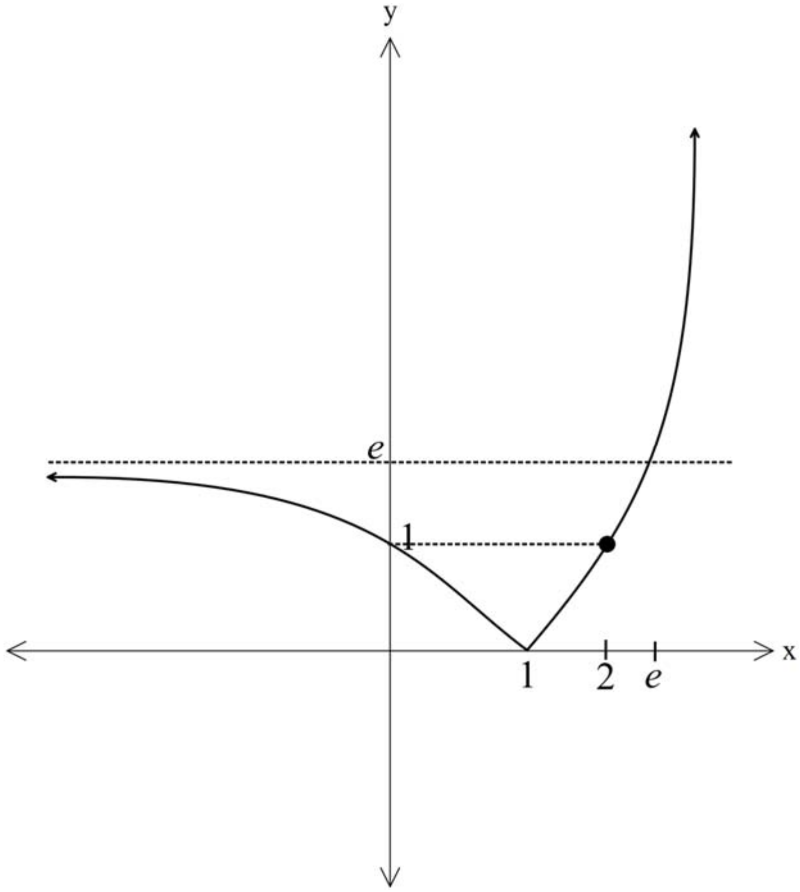
(v) Hence find the locus of M as the point P varies. **2**

(c) Find $\displaystyle\lim_{x \to -5} \frac{\sqrt{20 - x} - 5}{5 + x}$ without the aid of a calculator. **2**

End of Question 12

Question 13 (15 marks) Answer on the appropriate page

(a) The diagram is a sketch of $y = f(x)$.



Draw separate one third page sketches of the graphs of the following:

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = f(|x|)$ **2**

(iii) $y = \sqrt{f(x)}$ **2**

(iv) $y = \ln(f(x))$ **2**

Question 13 continues on the next page

Question 13 (continued)

(b) Use the result $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ to evaluate $\int_0^{\frac{\pi}{4}} \frac{1 - \tan x}{1 + \tan x} dx$. **3**

(c) Given that $I_n = \int_0^{\frac{\pi}{2}} \cos^n x dx$

(i) Show that $I_n = \frac{n-1}{n} I_{n-2}$ **2**

(ii) If $I_n = \frac{105\pi}{768}$ explain, without finding the value of *n*, why *n* must be even. **1**

(iii) Hence find the value of *n* if $I_n = \frac{105\pi}{768}$. **1**

End of Question 13

Question 14 (15 marks) Answer on the appropriate page

(a) Find $\displaystyle\int \frac{(x+1)dx}{(2x-1)(1-x)}$ **3**

(b) (i) If $t = \tan x$ prove that $\tan 4x = \dfrac{4t(1-t^2)}{t^4 - 6t^2 + 1}$ **2**

(ii) If $\tan x \tan 4x = 1$ deduce that $5t^4 - 10t^2 + 1 = 0$ **1**

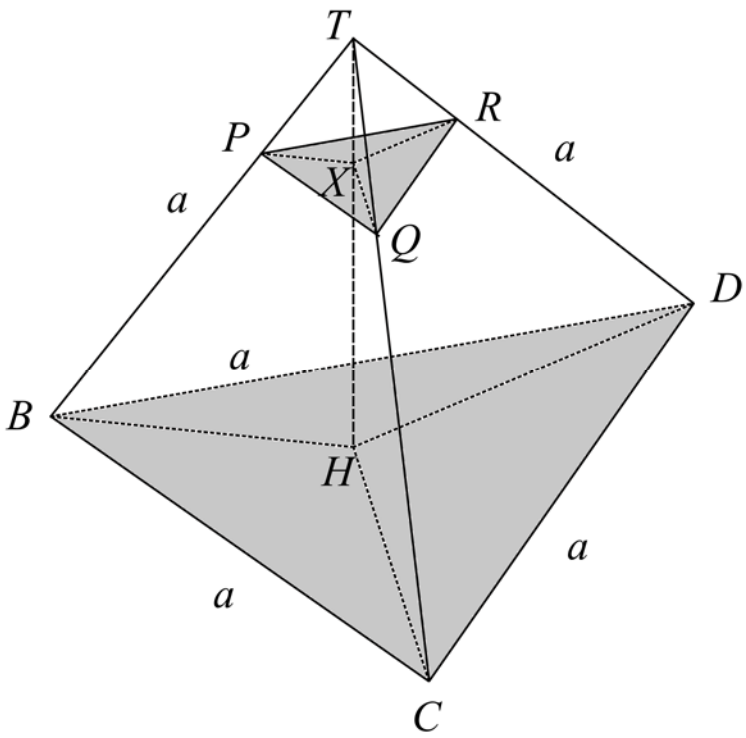
(iii) It is known that both $x = 18°$ and $x = 54°$ satisfy the equation $\tan x \tan 4x = 1$.

Find the exact value of $\tan 54°$. **3**

Question 14 continues on the following page

Question 14 (continued)

(c) A regular tetrahedron *TBCD* has six sides each of length *a* units. The point *H* marks the centre of equilateral triangle *BDC.* The line *TXH* is perpendicular to the plane *BDC.* The plane *PQR* is parallel to the plane *BDC.*
TX is taken *x* units from *T* such that $0 < x \leq TH$.



(i) Show that $TH^2 = \dfrac{2a^2}{3}$ **2**

(ii) Show that the cross sectional area of the slice of ΔPQR is $\dfrac{3\sqrt{3}}{8}x^2$ square units. **2**

(iii) Hence, by considering the typical slice ΔPQR of thickness Δx units, **2**

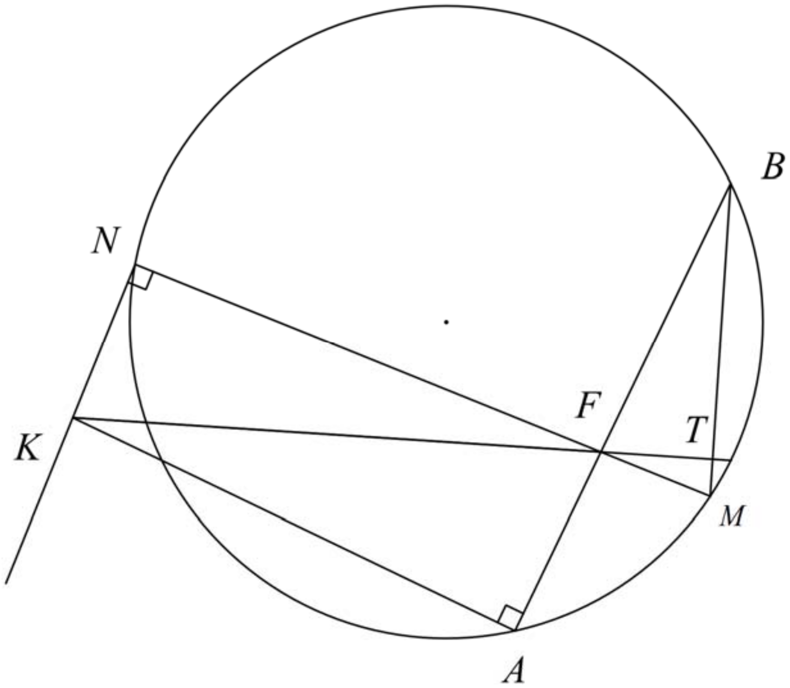
show that the volume of the tetrahedron *TBCD* is $\dfrac{a^3\sqrt{2}}{12}$ cubic units.

End of Question 14

Question 15 (15 marks) Answer on the appropriate page

(a) The letters *A, B, C, D, E, F, I* and *O* are arranged in a circle. In how many ways can this be done if at least two of the vowels are together? **3**

(b) A circle has two chords *AB* and *MN* intersecting at *F*. Perpendiculars are drawn to these chords at *A* and at *N* intersecting at *K*. *KF* produced meets *MB* at *T*.



(i) Copy or trace into your answer booklet

(ii) Explain why *AKNF* is a cyclic quadrilateral. **1**

(iii) Prove that *KT* is perpendicular to *MB*. **3**

Question 15 continues on the following page

Question 15 continued

(c) A plane of mass M kilograms on landing experiences a variable resistance force (due to air resistance) of magnitude Bv^2 Newtons, where v is the speed of the plane.

After the brakes are applied the plane experiences a constant resistive force A Newtons (due to the brakes) as well as the variable resistive force Bv^2.

(i) Show that the distance travelled, D_1, in slowing from speed V to speed U under the effect of air resistance is given by $D_1 = \frac{M}{B}\ln\left(\frac{V}{U}\right)$. **3**

(ii) After the brakes are applied with the plane travelling at speed, U, show that the distance, D_2, required to come to rest is given by $D_2 = \frac{M}{2B}\ln\left(1 + \frac{B}{A}U^2\right)$. **3**

(iii) Use the above information to estimate the stopping distance for a 100 tonne plane if it slows from $90ms^{-1}$ to $60ms^{-1}$ under a resistive force of magnitude $125v^2$ Newtons and is finally brought to rest with the assistance of constant braking force of magnitude 75000 Newtons. **2**

End of Question 15

Question 16 (15 marks) Answer on the appropriate page

(a) If $x = \cot\theta$ and $y = \sin^2\theta$

(i) Show that $\frac{dy}{dx} = -2\sin^3\theta\cos\theta$ **1**

(ii) Prove, by mathematical induction, $\frac{d^ny}{dx^n} = (-1)^n\, n!\, \sin^{n+1}\theta\, \sin(n+1)\theta$ **3**

for all positive integral values of n.

(b) The points A, B and C, represented by the **non zero** complex numbers z, w and t respectively, are the vertices of a right angled triangle ABC on an Argand diagram.

If AC is the hypotenuse and AB is 3 times the length of BC show that **3**
$2w(z + 9t) = z^2 + 9t^2 + 10w^2$.

(c) A point $P(a,b)$ lies on the circle $x^2 + y^2 - 10x - 14y + 73 = 0$. Prove that **3**

$$\frac{3}{4} < \frac{3a + 2b}{4a + b} < \frac{17}{11}$$

(d) (i) Show that $\tan^{-1}(n + 1) - \tan^{-1}(n - 1) = \tan^{-1}\frac{2}{n^2}$ where n is a positive integer. **2**

(ii) Hence or otherwise show that for $n \geq 1$, **2**

$$\sum_{r=1}^{n} \tan^{-1}\frac{2}{r^2} = \frac{3\pi}{4} + \tan^{-1}\frac{2n + 1}{1 - n - n^2}$$

(iii) Hence write down $\sum_{r=1}^{\infty} \tan^{-1}\frac{2}{r^2}$ **1**

End of paper

EXTENSION 2 2016 TRIAL SOLNS BHHS

SECTION 1 MULTIPLE CHOICE

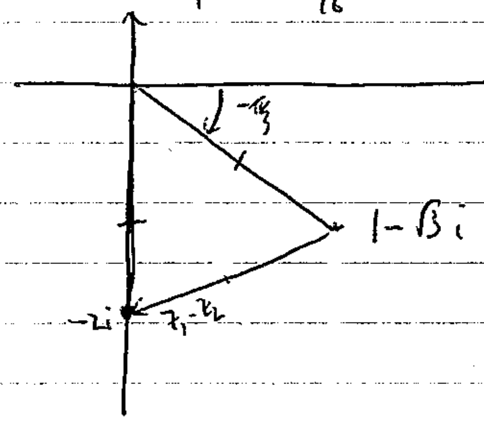
① Asymptotes have gradient $\pm\frac{b}{a}$ ie $\pm\frac{8}{6}$

$\pm\frac{4}{3}$

$a=3,\ b=4$

$\therefore \frac{(x-6)^2}{9} - \frac{y^2}{16} = -1$ is equation $\therefore B$

2.



$\arg(z_2^2) = 2\times -\frac{\pi}{3}$

$= -\frac{2\pi}{3}$ not A

$\arg(z_1/z_2) = -\frac{\pi}{2} - -\frac{\pi}{3}$

$= -\frac{\pi}{6}$ not B

$\arg(z_1 z_2) = -\frac{\pi}{2} + -\frac{\pi}{3}$

$= -\frac{5\pi}{6} \therefore C$

$\arg(z_1 + z_2) < 0$ not D

3. sub $y = \frac{x}{2}$ into equation

$\left(\frac{x}{2}\right)^3 + \left(\frac{x}{2}\right)^2 - 1 = 0$

$\frac{x^3}{8} + \frac{x^2}{4} - 1 = 0$

$x^3 + 2x^2 - 8 = 0 \quad \therefore A$

4.

$1 + w + w^2 = -1$

$(1 + w - w^2)^3 = (-w^2 - w^2)^3$

$= (-2w^2)^3$

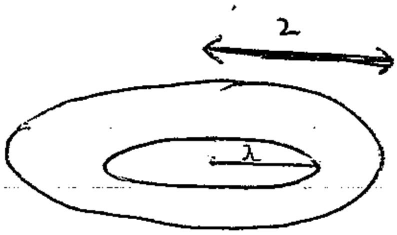
$= -8w^6$

$= -8 \times (w^3)^2$

$= -8 \times (-1)^2$

$= -8$

$\therefore A$

5. 

$\delta V = \pi(2^2 - x^2)\delta y$

$= \pi(4-(y^2+1))\delta y$

$= \pi(3-y^2)\delta y$

$\therefore D$

$y = \sqrt{x^2-1}$

$y^2 = x^2-1$

$x^2 = y^2+1$

6. $|y| = f(x)$

ie $y = f(x)$ for $y \geq 0$

$y = -f(x)$ for $y < 0$

$\therefore C$

7. $$\int \frac{x^3-1+1}{x^2+x+1}\,dx = \int x-1 + \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx$$

$$= \frac{x^2}{2} - x + \frac{2}{\sqrt{3}}\tan^{-1}\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}} + c$$

$$= \frac{x^2}{2} - x + \frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + c$$

$\therefore D$

8. $e^x + e^y = 1$

Differentiating wrt x

$$e^x + \frac{d}{dy}e^y\frac{dy}{dx} = 0$$

$$e^x + e^y\frac{dy}{dx} = 0$$

$$e^y\frac{dy}{dx} = -e^x$$

$$\frac{dy}{dx} = \frac{-e^x}{e^y}$$

$$= -e^{x-y}$$

$\therefore A$

9 $F = ma$

$mv\frac{dv}{dx} = -Mg - kv^2$

$\therefore B$

10. No of ways = total ways − 2 consecutives together + 3 consecutives together

$= {}^{10}C_3 - {}^{9}C_1 \times 8 + 8$

$= 120 - 72 + 8$

$= 56$

$\therefore C$

11

a) i) $\int \frac{e^{\tan x}}{\cos^2 x}\,dx = \int \sec^2 x\, e^{\tan x}\,dx$ (1) correct answer

$= e^{\tan x} + C$

ii) $\int \frac{2 \times 1}{2\sqrt{x}(1-\sqrt{x})}\,dx$ let $u = \sqrt{x}$ (2) correct answer

$du = \frac{1}{2}x^{-\frac{1}{2}}\,dx$ (1) changes to expression using suitable substitution

$du = \frac{1}{2\sqrt{x}}\,dx$

$= \int \frac{2}{1-u}\,du$

$= -2\ln(1-u) + C$

$= -2\ln(1-\sqrt{x}) + C$

NB OR $= \ln \frac{1}{(1-\sqrt{x})^2} + C$

b) (2) correct answer
(1) correctly multiplies by conjugate to make denominator real.

$\frac{z}{w}$

$= \frac{2+i}{3-i}$

$= \frac{2+i}{3-i} \cdot \frac{3+i}{3+i}$

$= \frac{6-1+5i}{9+1}$

$= \frac{5}{10} + \frac{5}{10}i$

$= \frac{1}{2} + \frac{i}{2}$

11 c i) $f(1+i) = (1+i)^2 - (3-2i)(1+i) + 5 - i$

$= 2i - (3+2+i) + 5 - i$

$= 5 + i - 5 - i$

$= 0$

$\therefore$ $1+i$ is a root.

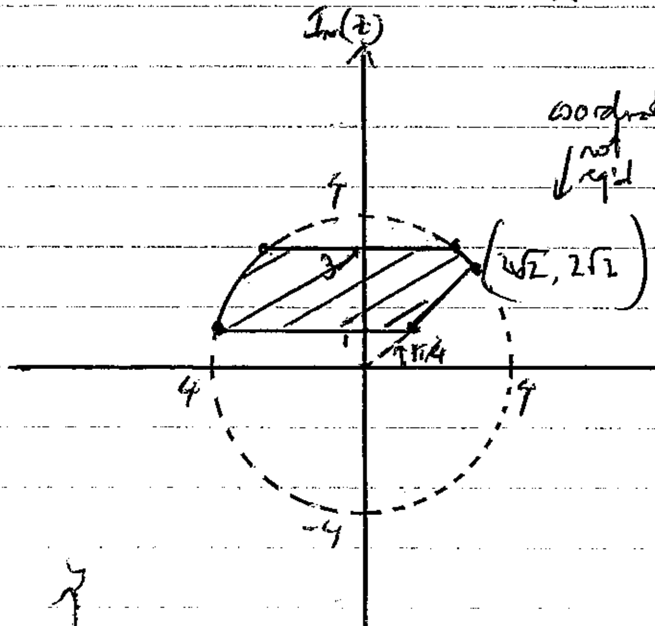
(2) correct solution
(1) substitutes $1+i$ in $f(z)$

(ii) sum of roots $1+i+\alpha = 3-2i$

$\alpha = 2-3i$

(1) correct answer

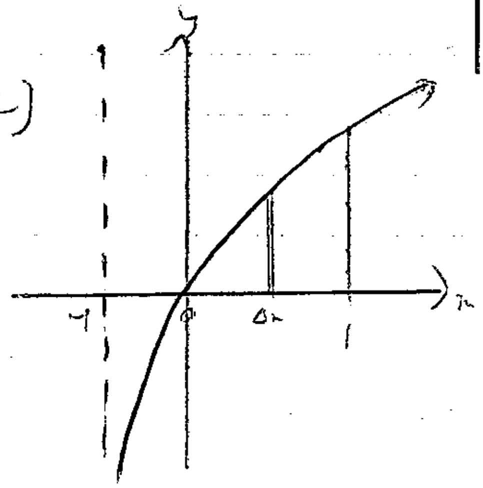
d)

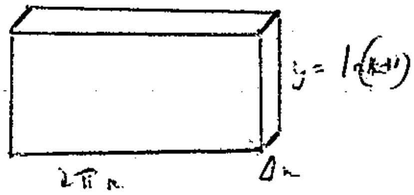


coordinates not req'd

(3) correct solution
(2) 2 correct regions
· all 3 graphs drawn with no incorrect region
(1) · 2 correct graphs
· 1 correct region

e)





$\Delta V = 2\pi x \ln x \, \Delta x$

Volume $= \lim_{\Delta x \to 0} \sum_{x=0}^{1} 2\pi x \ln(x+1) \Delta x$

$= 2\pi \int_0^1 x \ln(x+1) dx$ ✓

$= 2\pi \left[\left[\frac{x^2}{2} \ln(x+1) \right]_0^1 - \int_0^1 \frac{x^2}{2} \frac{1}{x+1} dx \right]$ ✓

$= 2\pi \left[\left(\frac{1}{2} \ln 2 - 0 \right) - \frac{1}{2} \int_0^1 \frac{x^2+1-1}{x+1} dx \right]$

(4) correct solution
(3) substantial progress ie correctly applies int by parts but not evaluate
(2) correctly attempts to integrate $x \ln(x+1)$ by parts (but does not find $\int \frac{x^2}{1+x}$)
(1) Expresses as $2\pi \int_0^1 x \ln(x+1) dx$
(1) Integrates by parts incorrect integral or
(1) some limited progress

11 e

$$= \pi\left(\ln 2 - \int_0^1 \frac{x^2-1}{x+1} + \frac{1}{x+1}\, dx\right)$$

$$= \pi\left(\ln 2 - \int_0^1 x-1 + \frac{1}{x+1}\, dx\right)$$

$$= \pi\left(\ln 2 - \left[\frac{x^2}{2} - x + \ln(x+1)\right]_0^1\right) ✓$$

$$= \pi \ln 2 - \pi\left[\left(\frac{1}{2} - 1 + \ln 2\right) - \left(0 - 0 + \ln 1\right)\right]$$

$$= \frac{\pi}{2} \text{ units}^3 \quad ✓$$

12 a) (i) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$

$= \frac{\beta\gamma}{\alpha\beta\gamma}+\frac{\alpha\gamma}{\alpha\beta\gamma}+\frac{\alpha\beta}{\alpha\beta\gamma}$

$= \frac{\alpha\beta+\beta\gamma+\alpha\gamma}{\alpha\beta\gamma}$

$= \frac{-p}{q}$ ✓

(2) correct answer
(1) finds sum in pairs, product
(1) expresses as $\frac{\text{sum in pairs}}{\text{product}}$

ii) $\left.\begin{aligned} \alpha^3+p\alpha+q &= 0 \\ \beta^3+p\beta+q &= 0 \\ \gamma^3+p\gamma+q &= 0 \end{aligned}\right\}$ since α, β, γ are roots

$\alpha^3+\beta^3+\gamma^3+p(\alpha+\beta+\gamma)+3q=0$ ✓

$\alpha^3+\beta^3+\gamma^3 = -p\times 0-3q$

$\alpha^3+\beta^3+\gamma^3 = -3q$ ✓

(2) correct answer
(1) substitutes α, β, γ
(1) correctly expands $(\alpha+\beta+\gamma)^3$

b) i) $x=ct \quad y=\frac{c}{t}$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$= \frac{\frac{-c}{t^2}}{c}$

$\frac{dy}{dx} = \frac{-1}{t^2}$ ✓

At P $\frac{dy}{dx}=\frac{-1}{t^2}$, $y-\frac{c}{t}=\frac{-1}{t^2}(x-ct)$

$t^2y-ct=-x+ct$

$x+t^2y=2ct$ ✓

(2) correct solution
(1) calculates gradient

ii) when $y=0 \quad x=2ct$

$\therefore Q$ is $(2ct, 0)$ ✓

(1) correct answer

12 b (iii) $m_l = t^2$

Eqn of l $\quad y - 0 = t^2(x - 2ct)$

$y = t^2 x - 2ct^3$

① correct solution

(iv) $y = t^2 x - 2ct^3 \quad ①$

$xy = c^2 \quad ②$

① $\times x$ $\quad xy = t^2 x^2 - 2ct^3 x$

sub in ② $\quad c^2 = t^2 x^2 - 2ct^3 x$

$t^2 x^2 - 2ct^3 x - c^2 = 0$

$\therefore t^2 x^2 - 2ct^3 x - c^2$ has roots x_1 and x_2 ✓

③ correct solution
② finds x or y value
① finds quadratic eqn

midpoint M of U, V $\left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

Sum of roots $x_1 + x_2 = \dfrac{2ct^3}{t^2}$

$= 2ct$ ✓

$\therefore$ midpoint M, $x = \dfrac{2ct}{2}$

$= ct$

sub in ① $\quad y = t^2(ct) - 2ct^3$

$= -ct^3$

$\therefore M$ is $(ct, -ct^3)$. ✓

v) $x = ct \Rightarrow t = \frac{x}{c}$

$y = -ct^3 \Rightarrow y = -c\dfrac{x^3}{c^3}$

$y = \dfrac{-x^3}{c^2}$

② correct locus
① attempts to eliminate parameter

c) $\displaystyle\lim_{x \to -5} \frac{\sqrt{20-x} - 5}{5+x} = \lim_{x \to -5} \frac{(\sqrt{20-x} - 5)(\sqrt{20-x} + 5)}{(\sqrt{20-x} + 5)(5+x)}$

$\displaystyle = \lim_{x \to -5} \frac{20 - x - 25}{(\sqrt{20-x}+5)(5+x)}$

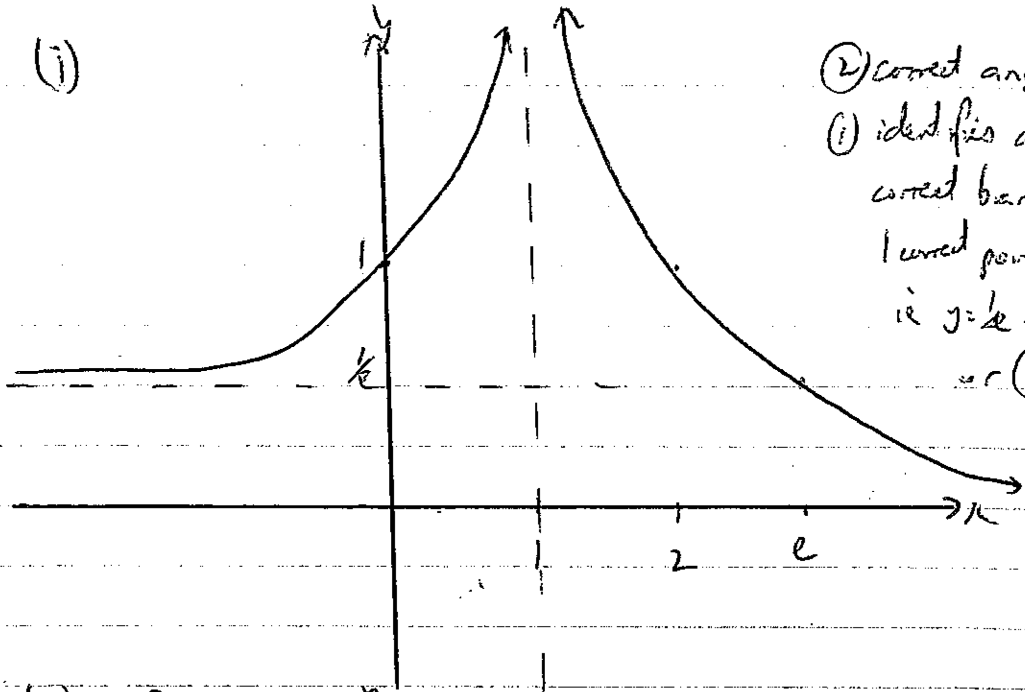
$\displaystyle = \lim_{x \to -5} \frac{-\cancel{(5+x)}^{1}}{(\sqrt{20-x}+5)\cancel{(5+x)}}$

$= \dfrac{-1}{10}$

② correct answer
① rationalises numerator

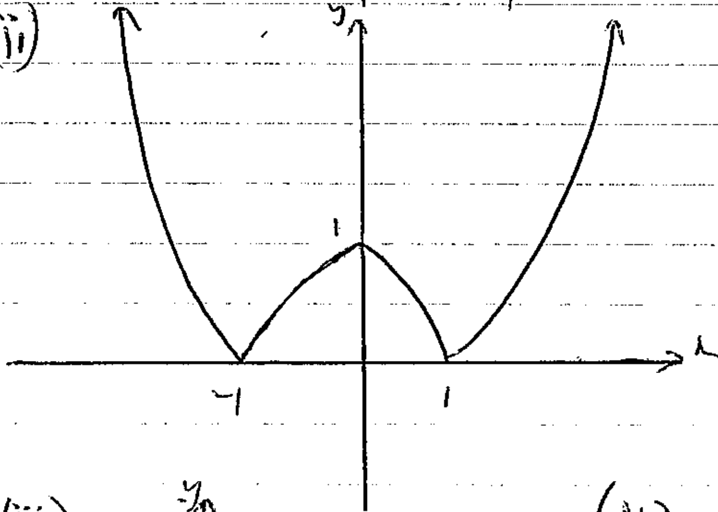
B

a) (i)



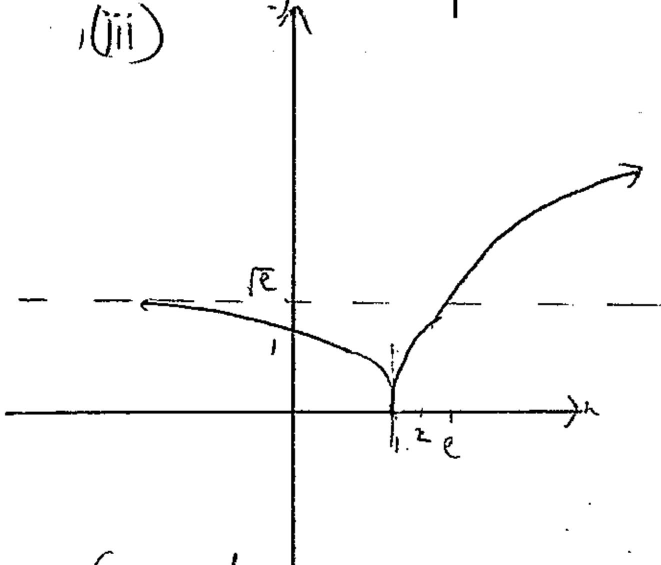
② correct answer
① identifies asymptote and 1 correct branch with at least 1 correct point/horizontal asymptote ie $y = 1/e$ or (0,1) or (2,1) or (e,e)

(ii)



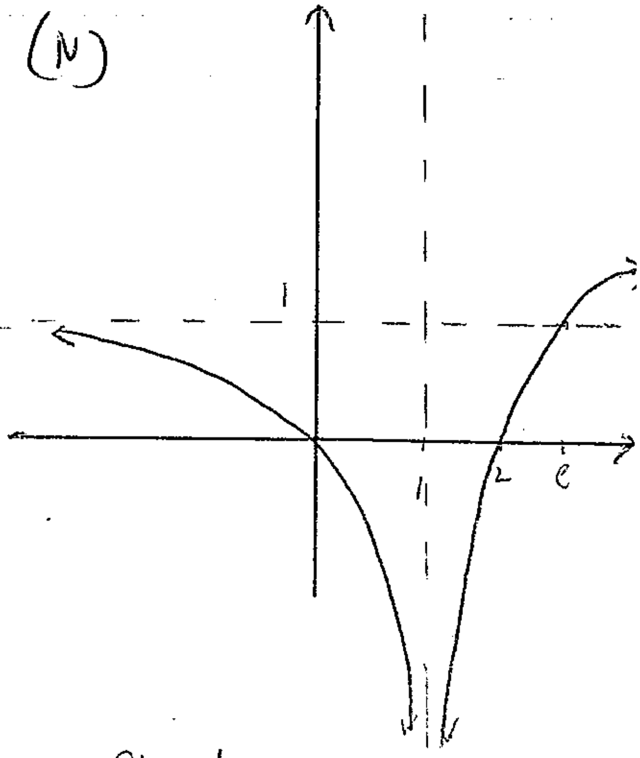
② correct answer
① basic shape without intercepts

(iii)



② correct answer
① 1 correct branch with vertical tangent
correct shape without vertical tangent

(iv)



② correct answer
① one correct branch with (2,0) and (e,1
or (0,0) and one asymptote

13 b)

$$\int_0^{\frac{\pi}{4}} \frac{1-\tan x}{1+\tan x}\,dx = \int_0^{\frac{\pi}{4}} \frac{1-\tan\left(\frac{\pi}{4}-x\right)}{1+\tan\left(\frac{\pi}{4}-x\right)}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1-\frac{\left(\tan\frac{\pi}{4}-\tan x\right)}{1+\tan\frac{\pi}{4}\tan x}}{1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1+\tan x-(1-\tan x)}{1+\tan x+1-\tan x}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{\cancel{2}\tan x}{\cancel{2}}\,dx$$

$$= -\Big[\ln(\cos x)\Big]_0^{\frac{\pi}{4}}$$

$$= -\left(\ln\frac{1}{\sqrt{2}} - \ln\cos 0\right)$$

$$= \ln\sqrt{2} + \ln 1$$

$$= \ln\sqrt{2} \text{ or } \frac{\ln 2}{2}$$

③ correct answer
② correctly integrates $\tan x$
① applies given rule and tan expansion

c) (i)

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$$

$$= \int_0^{\frac{\pi}{2}} \cos x \cos^{n-1} x\,dx$$

$$= \Big[\sin x\cos^{n-1}x\Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x.(n-1)\cos^{n-2}x.-\sin x\,dx$$

$$= \left(\sin\frac{\pi}{2}\cos^{n-1}\frac{\pi}{2} - \sin 0\cos^{n-1}0\right) + (n-1)\int_0^{\frac{\pi}{2}} (1-\cos^2 x)\cos^{n-2}x\,dx$$

$$I_n = 0-0+(n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2}x - \cos^n x\,dx$$

$$I_n = (n-1)I_{n-2} - (n-1)I_n$$

$$I_n(1+n-1) = (n-1)I_{n-2}$$

$$I_n = \frac{n-1}{n}I_{n-2}$$

② correct proof
① correctly applies integration by parts

13 c(ii) $I_0 = \int_0^{\pi/2} (\cos x)^0 dx$

$= \int_0^{\pi/2} 1\, dx$

$= \left[x\right]_0^{\pi/2}$

$= \frac{\pi}{2} - 0$

$= \frac{\pi}{2}$

$I_1 = \int_0^{\pi/2} \cos x\, dx$

$= \left[\sin x\right]_0^{\pi/2}$

$= 1 - 0$

$= 1$

① correct justification (finds I_0, I_1 and links I_n to I_{n-2})

Since $I_n = \frac{n-1}{n} I_{n-2}$ for all which differ by a multiple of 2 will contain I_0 (or $\frac{\pi}{2}$). ie for all even positive n.
(Values of n which differ by multiples of 2 from I_1 (eg $I_3, I_5, \ldots$) will contain I_1 or 1 and not $\frac{\pi}{2}$)

13 c(iii) $I_2 = \frac{1}{2} I_0$

$= \frac{\pi}{4}$

$I_4 = \frac{3}{4}\frac{\pi}{4}$

$= \frac{3\pi}{16}$

$I_6 = \frac{5}{6}\frac{3\pi}{16}$

$= \frac{15\pi}{96}$

$I_8 = \frac{7}{8}\frac{15\pi}{96}$

$= \frac{105\pi}{768}$

$\therefore n = 8$

14 a) let $\frac{x+1}{(2x-1)(1-x)} \equiv \frac{a}{2x-1} + \frac{b}{1-x}$

$x+1 \equiv a(1-x) + b(2x-1)$

let $x=1$ $\quad 2 = b$

let $x=\frac{1}{2}$ $\quad \frac{3}{2} = \frac{a}{2}$

$a=3$

$$\int \frac{x+1\,dx}{(2x-1)(1-x)} \equiv \int \frac{3}{2x-1} + \frac{2}{1-x}\,dx$$

$$= \frac{3}{2}\ln|2x-1| - 2\ln|1-x| + C$$

(3) correct answer

(2) finds correct values of a and b and attempts to integrate using logs

(2) finds incorrect values of a and b and correctly integrate using these values found

(1) finds a and b

(ignore absolute values)

b) i) $\tan 4x = \frac{2\tan 2x}{1-\tan^2 2x}$

$$= \frac{2\,\frac{2t}{1-t^2}}{1-\left(\frac{2t}{1-t^2}\right)^2} \quad \text{where } t = \tan x$$

$$= \frac{4t(1-t^2)}{(1-t^2)^2 - 4t^2}$$

$$= \frac{4t(1-t^2)}{1-2t^2+t^4-4t^2}$$

$$= \frac{4t(1-t^2)}{t^4-6t^2+1}$$

(2) correct solution

(1) uses double angle to find expression in t

(1) finds expression for $\sin 4\theta$ and $\cos 4\theta$ and makes some progress

b ii) $t\tan 4x = \frac{4t^2(1-t^2)}{t^4-6t^2+1}$

$$\tan x \tan 4x = \frac{4t^2(1-t^2)}{t^4-6t^2+1}$$

$$1 = \frac{4t^2(1-t^2)}{t^4-6t^2+1} \quad (\text{since } \tan x \tan 4x = 1)$$

$t^4-6t^2+1 = 4t^2-4t^4$

$5t^4-10t^2+1=0$

(1) correct solution

b iii) $t^2 = \dfrac{10 \pm \sqrt{100-4\times 5}}{10}$ ✓

$= \dfrac{10 \pm \sqrt{80}}{10}$

$= \dfrac{10 \pm 4\sqrt{5}}{10}$

$t^2 = \dfrac{5 \pm 2\sqrt{5}}{5}$

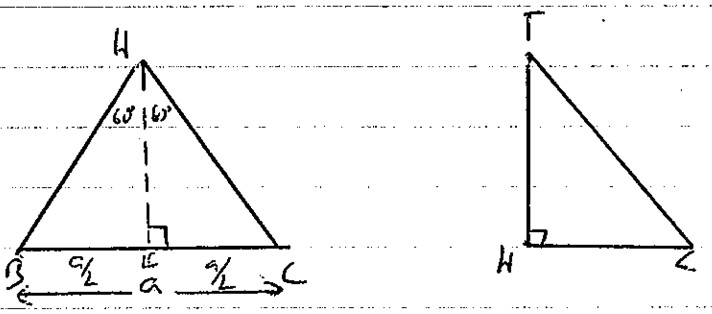
$t = \sqrt{\dfrac{5 \pm 2\sqrt{5}}{5}}$ ✓ (since $\begin{cases}\tan 54° > 0 \Rightarrow t > 0 \\ \tan 18° > 0\end{cases}$)

But $\tan 54° > \tan 18°$

$\therefore \tan 54° = \sqrt{\dfrac{5+2\sqrt{5}}{5}}$ ✓

(3) correct solution
(2) finds correct expression for t
(1) finds expression for t^2

c (i)



(2) correct solution
(1) progress towards eg. expression for CH

In ΔHEC $\sin 60° = \dfrac{\frac{a}{2}}{CH}$

$CH = \dfrac{a}{2} \cdot \dfrac{2}{\sqrt{3}}$

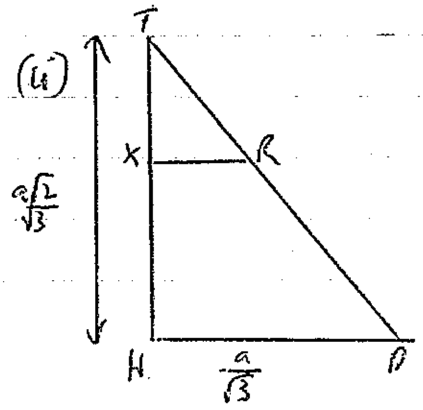
$= \dfrac{a}{\sqrt{3}}$

In ΔTHC $CT^2 = TH^2 + HC^2$

$a^2 = TH^2 + \dfrac{a^2}{3}$

$TH^2 = \dfrac{2a^2}{3}$

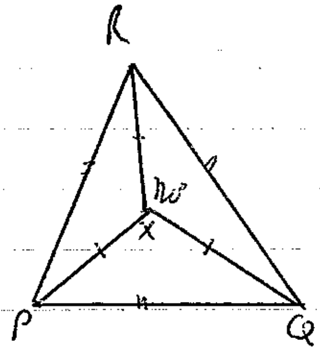
(ii)



$\dfrac{x}{\frac{a\sqrt{2}}{\sqrt{3}}} = \dfrac{XR}{\frac{a}{\sqrt{3}}}$ (matching sides in same ratio, $\Delta TXR ||| \Delta THD$)

$XR = \dfrac{x}{\sqrt{2}}$

(2) correct solution
(1) obtains expression for XR or equivalent.
(NB reason not required)



Area $\Delta PRQ = 3 \times A_{\Delta XRQ}$

$$= 3 \times \frac{1}{2}\left(\frac{x}{\sqrt{2}}\right)^2 \sin 120^\circ$$

$$= \frac{3x^2}{4} \times \frac{\sqrt{3}}{2}$$

$$= \frac{3\sqrt{3}}{8} x^2$$

iii) Area slice $\dfrac{3\sqrt{3}x^2}{8}$

(2) correct solution

(1) obtains correct integral.

Volume of slice $\Delta V = \dfrac{3\sqrt{3}x^2}{8} \Delta x$

$$\text{Volume} = \lim_{\Delta x \to 0} \sum_{x=0}^{\frac{a\sqrt{2}}{\sqrt{3}}} \frac{3\sqrt{3}}{8} x^2 \Delta x$$

$$= \frac{3\sqrt{3}}{8} \int_0^{\frac{a\sqrt{2}}{\sqrt{3}}} x^2 \, dx$$

$$= \frac{3\sqrt{3}}{8} \left[\frac{x^3}{3}\right]_0^{\frac{a\sqrt{2}}{\sqrt{3}}}$$

$$= \frac{\cancel{3}\cancel{\sqrt{3}}}{4\cancel{8}} \left(\frac{a^3 \cancel{2}\sqrt{2}}{3 \times \cancel{3}\cancel{\sqrt{3}}} - 0\right)$$

Volume $= \dfrac{a^3\sqrt{2}}{12}$ units^3.

15 a) AEIO BCDF

No. of ways = total arrangements − vowels all separated

$= 7! - 3! \times 4!$

$= 4896$

(2) correct answer
(1) calculates vowels separated
OR (1) significant progress
(0) finds total arrangements

b) i) opposite angles in KNFA are supplementary. (1) correct reason

ii) $\angle AKF = \angle ANF$ (angles standing on same chord = in KNFA) ✓ (3) correct solution

$\angle ANF = \angle ABM$ (angles standing on same chord AM = in ANBM) (2) proves ΔKF is ||| to BFT

$\angle KFA = \angle BFT$ (vertically opposite ∠'s) ✓ (1) uses ∠ on same arc/chord for relevant ∠'s

$\Delta KFA \ ||| \ \Delta BFT$ (AA)

$\therefore \angle KAF = \angle BTF$ (matching ∠'s in similar Δ's KAF, BTF) ✓

$\therefore BT \perp KT$

c i) $F = ma$

$m\ddot{x} = -Bv^2$

$\ddot{x} = -\frac{B}{m}v^2$

$v\frac{dv}{dx} = -\frac{B}{m}v^2$ ✓

$\frac{dv}{dx} = -\frac{B}{m}v$

$\frac{dx}{dv} = -\frac{m}{B}\frac{1}{v}$

$\int_0^{D_1} dx = -\frac{m}{B}\int_V^u \frac{1}{v}\,dv$ ✓

$[x]_0^{D_1} = -\frac{m}{B}[\ln v]_V^u$

$D_1 - 0 = \frac{m}{B}[\ln v]_u^V$

$D_1 = \frac{m}{B}(\ln V - \ln u)$

$\therefore D_1 = \frac{m}{B}\ln\frac{V}{u}$ ✓

(3) correct solution
(2) expresses as correct integral
(1) finds $v\frac{dv}{dx}$ in terms of v

c ii) $v\frac{dv}{dx} = -\frac{1}{m}(A+Bv^2)$

$\frac{dv}{dx} = -\frac{1}{m}\left(\frac{A+Bv^2}{v}\right)$

$\frac{dx}{dv} = -m\left(\frac{v}{A+Bv^2}\right)$ ✓

$\int_0^{D_2} dx = -m\int_u^0 \frac{v\,dv}{A+Bv^2}$

$[x]_0^{D_2} = \frac{-m}{2B}\left[\ln(A+Bv^2)\right]_u^0$ ✓

$D_2 - 0 = -\frac{m}{2B}\left(\ln A - \ln(A+Bu^2)\right)$

$D_2 = \frac{m}{2B}\ln\left(\frac{A+Bu^2}{A}\right)$

$= \frac{m}{2B}\ln\left(1+\frac{Bu^2}{A}\right)$ ✓

(3) correct solution
(2) finds $\frac{dx}{dV}$ and correctly integrates
(1) expresses $\frac{dx}{dv}$ in terms of v

iii) distance $= D_1 + D_2$

$= \frac{10^5}{125}\ln\left(\frac{90}{60}\right) + \frac{10^5}{250}\ln\left(1+\frac{125}{75\times10^3}\times60^2\right)$ ✓

$= 1163$ metres nearest metre

(2) correct answer
✓ (1) identifies answer as sum of two distances and attempts to evaluate
or (1) correctly determines one distance.

16 a) i) $x = \cot\theta \quad y = \sin^2\theta$

① correct

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$= \frac{2\sin\theta\cos\theta}{-\operatorname{cosec}^2\theta}$$ ✓

$$= -2\sin^3\theta\cos\theta$$

ii) test $n=1$

③ correct solution

② proves true for $n=1$ uses chain rule when differentiating assumption

① proves true for $n=1$

① uses chain rule when differentiating assumption

RHS $= (-1)^1 \, 1! \sin^2\theta \sin 2\theta$

$= -2\sin\theta\cos\theta \sin^2\theta$

$= -2\sin^3\theta\cos\theta$

$\therefore \quad = LHS$

$\therefore$ True for $n=1$ ✓

Assume true for $n=k$

$$\frac{d^k y}{dx^k} = (-1)^k k! \sin^{k+1}\theta \sin(k+1)\theta$$

For $n=k+1$ we wish to prove

$$\frac{d^{k+1}y}{dx^{k+1}} = (-1)^{k+1}(k+1)! \sin^{k+2}\theta \sin(k+2)\theta$$

$$\frac{d^{k+1}y}{dx^{k+1}} = \frac{d\,(-1)^k (k)! \sin^{k+1}\theta \sin(k+1)\theta}{d\theta} \times \frac{d\theta}{dx}$$ ✓

$$= (-1)^k k! \left(\sin(k+1)\theta\,(k+1)\sin^k\theta\cos\theta + \sin^{k+1}\theta\,(k+1)\cos(k+1)\theta\right)(-\sin^2\theta)$$

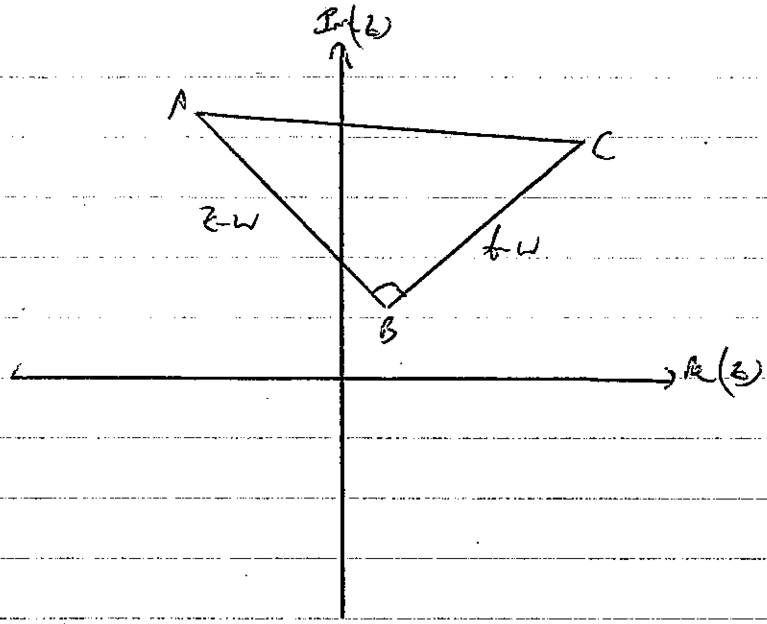
$$= (-1)^k k!(k+1)\sin^k\theta \left(\sin(k+1)\theta\cos\theta + \sin\theta\cos(k+1)\theta\right)(-1)\sin^2\theta$$

$$= (-1)^{k+1}(k+1)!\sin^{k+2}\theta \sin((k\theta+\theta)+\theta)$$

$$= (-1)^{k+1}(k+1)!\sin^{k+2}\theta \sin(k+2)\theta$$ as req'd. ✓

$\therefore$ If true for $n=1$, its true for $n=k+1$. But is true for $n=1$, $\therefore$ true for $n=2, 3, 4$ and so on for all $n \geq 1$.

16 b



③ correct solution
② significant progress
① expresses $\vec{AB}$ in terms of $\vec{BC}$ (or vice versa)

$\vec{BC} = t-w$

$\vec{AB} = 3i(t-w) = z-w$ ✓

$|3i(t-w)| = |z-w|$

$9i^2(t^2-2wt+w^2) = z^2-2wz+w^2$ ✓

$-9t^2+18wt-9w^2 = z^2-2wz+w^2$

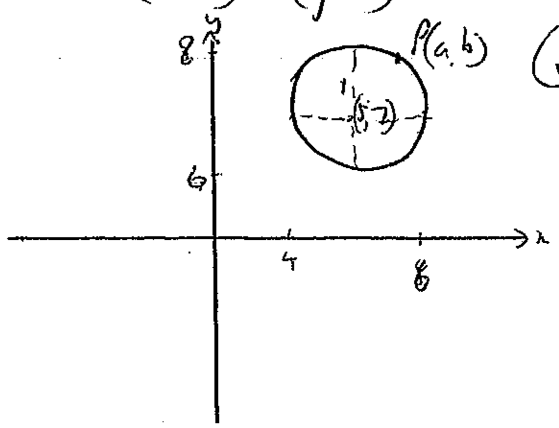
$2wz+18wt = z^2+9t^2+10w^2$ ✓

$2w(z+9t) = z^2+9t^2+10w^2$

16 c

$x^2-10x+y^2-14y+25+49 = 74-73$

$(x-5)^2+(y-7)^2 = 1$



✓

② correct solution
① identifies max & min values for numerator or denominator

$4 \le a \le 6$ $\quad 2 \le 3a+2b \le 34$ ①

$6 \le b \le 8$ $\quad 22 \le 4a+b \le 32$ ② ✓

$$\frac{\text{smallest ①}}{\text{largest ②}} \le \frac{3a+2b}{4a+b} \le \frac{\text{largest ①}}{\text{smallest ②}}$$

$$\frac{24}{32} \le \frac{3a+2b}{4a+b} \le \frac{34}{22}$$

$$\frac{3}{4} \le \frac{3a+2b}{4a+b} \le \frac{17}{11}$$ ✓

16d (i) let $\tan^{-1}(n+1) = \alpha$ $\quad \tan\alpha = n+1$ ② correct soln

$\tan^{-1}(n-1) = \beta$ $\quad \tan\beta = n-1$ ① progress towards soln

[uses $\tan(\alpha-\beta)$]

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$$

$$= \frac{(n+1)-(n-1)}{1+(n+1)(n-1)}$$

$$= \frac{2}{1+n^2-1}$$

$$\tan(\alpha-\beta) = \frac{2}{n^2}$$

$$\alpha - \beta = \tan^{-1}\frac{2}{n^2}$$

$$\therefore \tan^{-1}(n+1) - \tan^{-1}(n-1) = \frac{2}{n^2}$$

(ii) $\sum_{r=1}^{n} \tan^{-1}\frac{2}{r^2}$

$$= [\tan^{-1}2 - \tan^{-1}0] + [\tan^{-1}3 - \tan^{-1}1] + [\tan^{-1}4 - \tan^{-1}2]$$
$$+ \dots + [\tan^{-1}(n-2) - \tan^{-1}(n-4)] + [\tan^{-1}(n-1) - \tan^{-1}(n-3)]$$
$$+ [\tan^{-1}n - \tan^{-1}(n-2)] + [\tan^{-1}(n+1) - \tan^{-1}(n-1)]$$

$\boxed{\text{① correct Soln} \quad \text{① determined 4 terms of sum}}$

$$= -\tan^{-1}0 - \tan^{-1}1 + \tan^{-1}n + \tan^{-1}(n+1)$$

now $\tan^{-1}n + \tan^{-1}(n+1) = \tan^{-1}\left(\frac{n+n+1}{1-n(n+1)}\right)$

$$= \tan^{-1}\left(\frac{2n+1}{1-n-n^2}\right) < 0$$

as $n > 1$

$$\frac{\pi}{4} < \tan^{-1}n < \frac{\pi}{2}$$

Thus $\frac{\pi}{2} < \tan^{-1}n + \tan^{-1}(n+1) < \pi$

ie $\tan^{-1}n + \tan^{-1}(n+1) = \pi + \tan^{-1}\left(\frac{2n+1}{1-n-n^2}\right)$

Hence $\sum_{r=1}^{n} \tan^{-1}\frac{2}{r^2} = 0 - \frac{\pi}{4} + \pi \tan^{-1}\left(\frac{2n+1}{1-n-n^2}\right)$

$$= \frac{3\pi}{4} + \tan^{-1}\left(\frac{2n+1}{1-n-n^2}\right)$$

(iii) $\sum_{r=1}^{\infty} \frac{2}{r^2} = \frac{3\pi}{4} + \lim_{n\to\infty} \tan^{-1}\left(\frac{2n+1}{1-n-n^2}\right)$ ① correct answer.

$$= \frac{3\pi}{4} + \tan^{-1}0$$

$$= \frac{3\pi}{4}$$