BAULKHAM HILLS HIGH SCHOOL

2017

YEAR 12 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A reference sheet is provided at the back of this paper
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations
- Marks may be deducted for careless or badly arranged work

Total marks: 100

Section I – 10 marks (pages 2 – 6)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 7 – 15)

- Attempt Questions 11 – 16
- Allow about 2 hours 45 minutes for this section

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10

---

1 If ω is an imaginary cube root of unity, then $(1 + \omega - \omega^2)^{2017}$ is equal to

(A) $-2^{2017}\omega$

(B) $2^{2017}\omega$

(C) $-2^{2017}\omega^2$

(D) $2^{2017}\omega^2$

2

$$I(a) = \int_0^1 (x^2 - a)^2 dx$$

The smallest value of $I(a)$, as a varies is

(A) $\frac{3}{20}$

(B) $\frac{4}{45}$

(C) $\frac{1}{5}$

(D) $\frac{7}{13}$

3 A particle moves with a constant acceleration in a straight line so that at time t seconds its velocity is v m/s and its displacement from a fixed point on the line is x metres.

Which of the following could **NOT** be true?

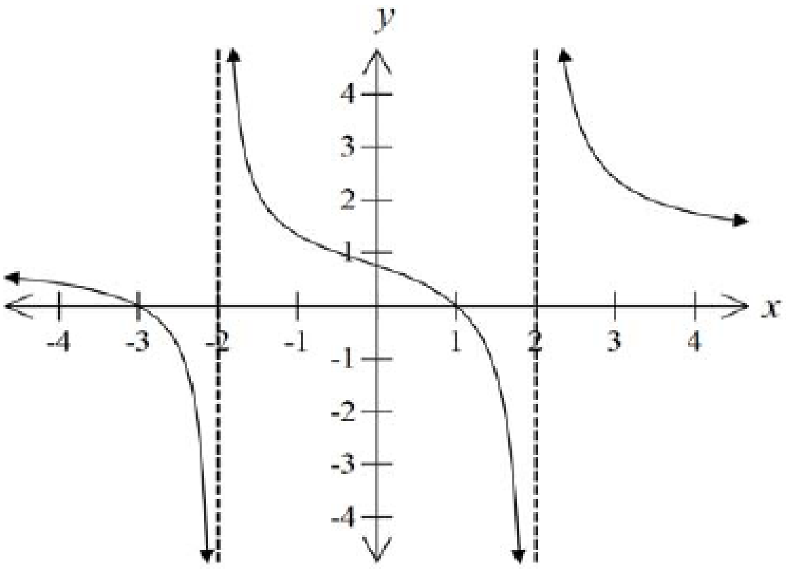
(A) $x = t^2 + t + 4$

(B) $2x + 4 = v^2$

(C) $4t = v - 9$

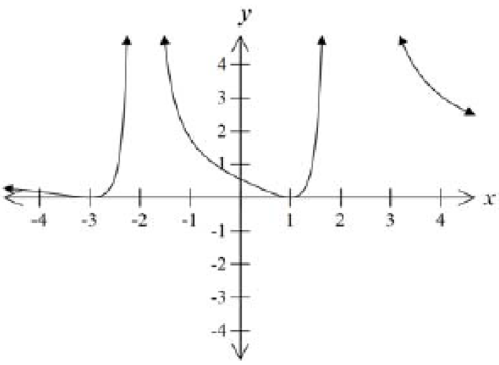
(D) $t^3 = x - 1$

4 The diagram shows the graph of the function $y = f(x)$

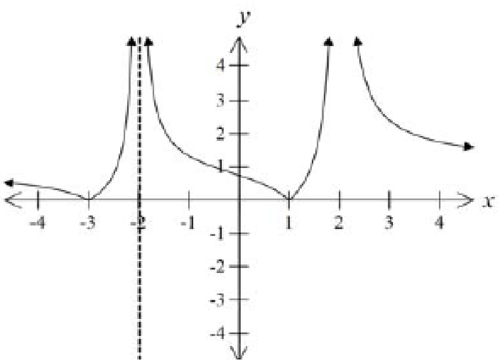


Which of the following is the graph of $y = [f(x)]^2$?

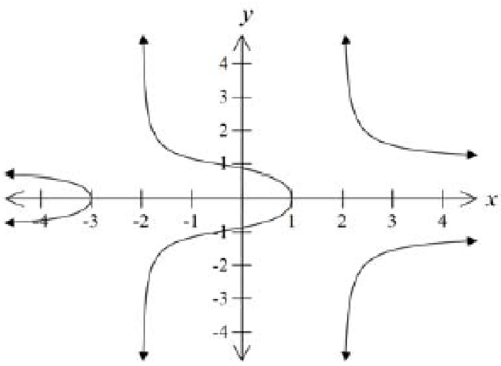
(A)



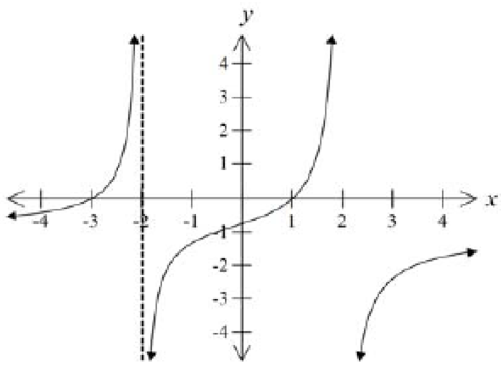
(B)



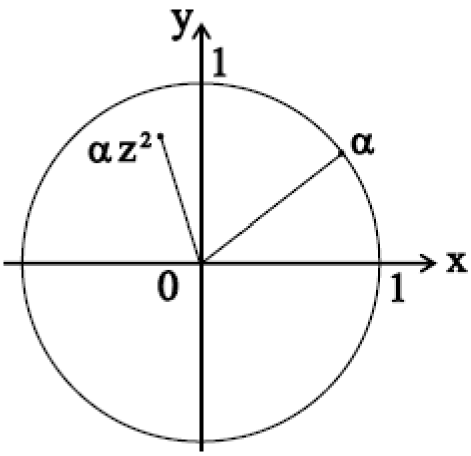
(C)



(D)

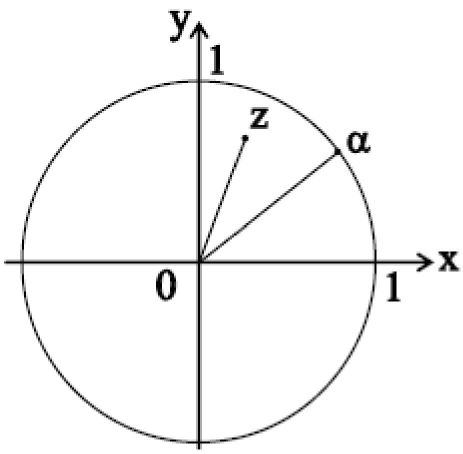


5 The Argand diagram below shows the complex numbers α and αz^2

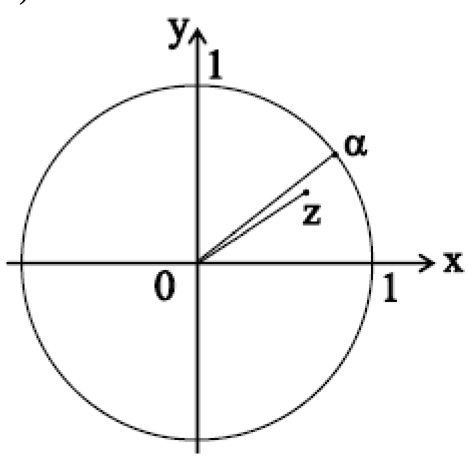


Which of the following best represents the positions of z and α ?

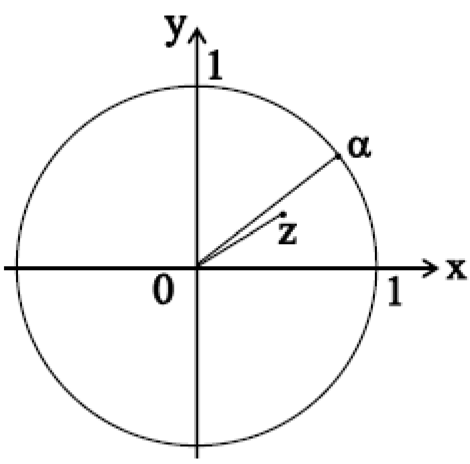
(A)



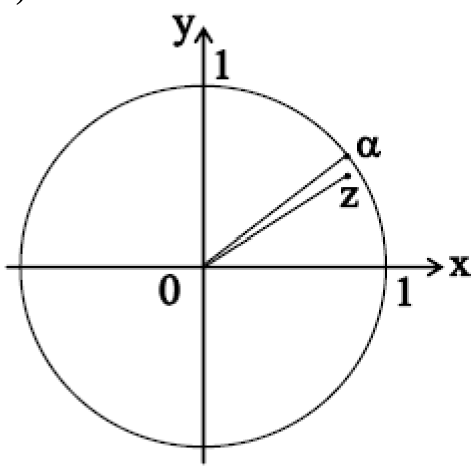
(B)



(C)



(D)



6 The fraction of the interval $0 \leq x \leq 2\pi$, for which one (or both) of the inequalities

$$\sin x \geq \frac{1}{2} \quad \text{and} \quad \sin 2x \geq \frac{1}{2}$$

is true, equals

(A) $\frac{1}{3}$

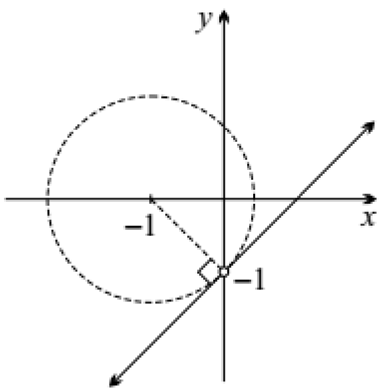
(B) $\frac{13}{24}$

(C) $\frac{7}{12}$

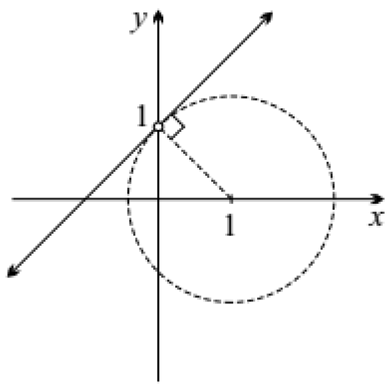
(D) $\frac{5}{8}$

7 If $\omega = \dfrac{z+1}{z+i}$ and ω is imaginary, what is the locus of z ?

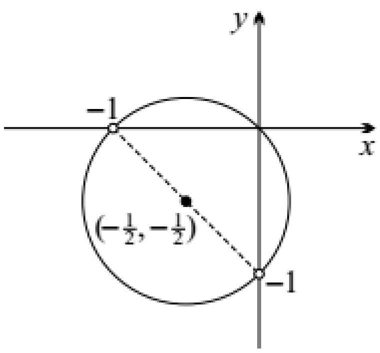
(A)



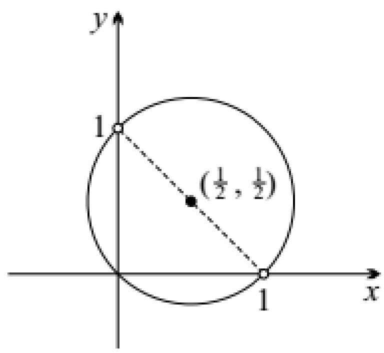
(B)



(C)



(D)



8 How many vertical tangents can be drawn on the graph of $x^2 + y^2 + 4xy - 4 = 0$?

(A) 0

(B) 1

(C) 2

(D) more than 2

9 Let $n \geq 2$ be an integer and $p_n(x)$ be the polynomial

$$p_n(x) = (x - 1) + (x - 2) + \ldots + (x - n)$$

What is the remainder when $p_n(x)$ is divided by $p_{n-1}(x)$?

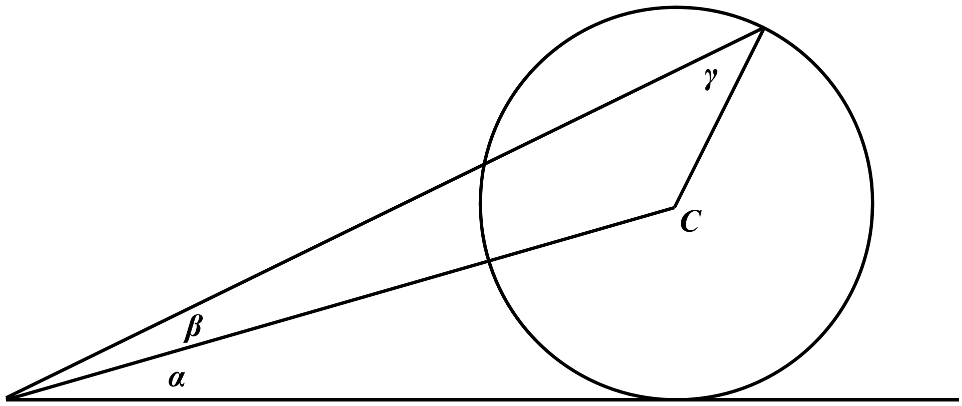
(A) $\frac{n}{2}$

(B) $-\frac{n}{2}$

(C) $\frac{n+1}{2}$

(D) $\frac{n^2 + n}{2}$

10 The circle in the diagram has centre C. Three angles α, β and γ are also indicated.



The angles α, β and γ are related by the equation;

(A) $\cos\alpha = \sin(\beta + \gamma)$

(B) $\sin(\alpha + \beta) = \cos\gamma\sin\alpha$

(C) $\sin\beta(1 - \cos\alpha) = \sin\gamma$

(D) $\sin\beta = \sin\alpha\sin\gamma$

END OF SECTION I

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours 45 minutes for this section

Answer each question on the appropriate answer sheet. Each answer sheet must show your NESA#. Extra paper is available.

In Questions 11 to 16, your responses should include relevant mathematical reasoning and/or calculations.

Marks

Question 11 ***(15 marks)*** Use a *separate* answer sheet

(a) For the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$, find

(i) the eccentricity *1*

(ii) the coordinates of the foci *1*

(b) Let $w=-1+\sqrt{3}\,i$ and $z=1-i$

(i) Find wz in the form $a+ib$ *1*

(ii) Find w and z in mod-arg form *2*

(iii) Hence find the exact value of $\sin\frac{5\pi}{12}$ *2*

(c)(i) Find *a*, *b* and *c* such that *2*

$$\frac{16}{(x^2+4)(2-x)}=\frac{ax+b}{x^2+4}+\frac{c}{2-x}$$

(ii) Find $\int \frac{16}{(x^2+4)(2-x)}\,dx$ *2*

(d)(i) On an Argand diagram shade in the region containing all of the points representing complex numbers *z* such that *2*

$$|z| \le 3 \quad \text{and} \quad \frac{\pi}{4} \le \arg(z+3) \le \frac{\pi}{2}$$

(ii) Find the possible values of $|z|$ and $\arg z$ for all such complex numbers *2*

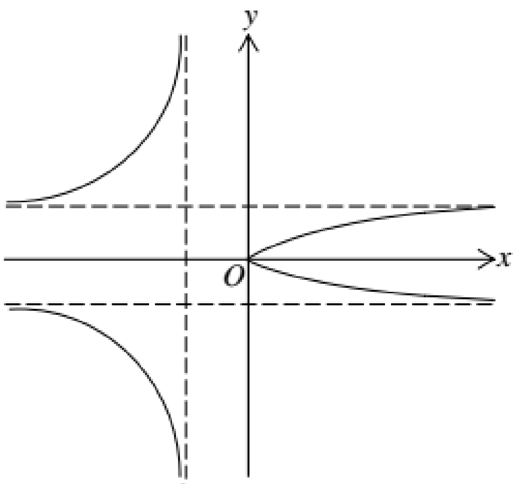
Marks

<u>Question 12</u> ***(15 marks)*** Use a *separate* answer sheet

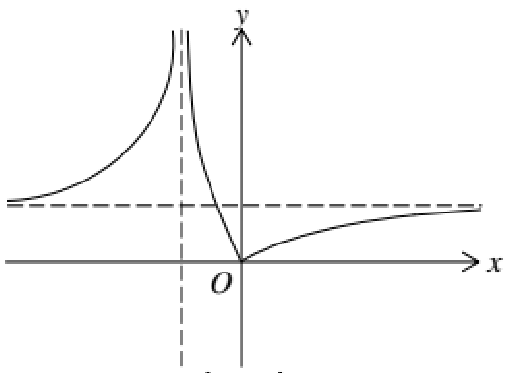
(a) Using the substitution $t = \tan\dfrac{x}{2}$, or otherwise, find *3*

$$\int \frac{dx}{1 + 3\sin x}$$

(b) The graphs of $y^2 = f(x)$ and $y = |f(x)|$ are given below.



$y^2 = f(x)$



$y = |f(x)|$

Draw separate one-third page sketches of the following curves, clearly indicating any important features such as turning points or asymptotes.

(i) $y = f(x)$ *1*

(ii) $y = f(|x|)$ *1*

(c) The equation $x^3 - 3x^2 + 9 = 0$ has roots α, β and γ.

(i) Find the polynomial equation with roots α^2, β^2 and γ^2 *2*

(ii) Find the value of $\alpha^2 + \beta^2 + \gamma^2$ *1*

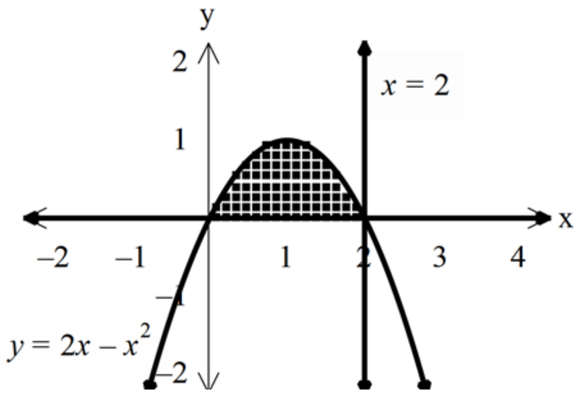
(iii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$ *1*

Question 12 continues on page 9

Marks

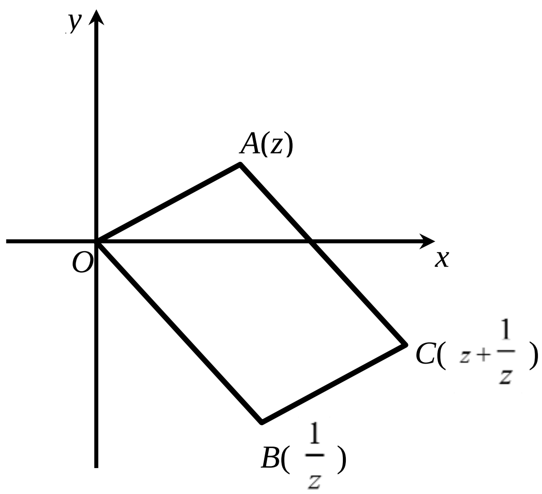
<u>Question 12</u> (continued)

(d) The region bounded by $y = 2x - x^2$ and $y = 0$ is rotated about the line $x = 2$



Using the method of cylindrical shells, find the volume of the solid generated. *3*

(e) The origin O and the points A, B and C representing the complex numbers z, $\frac{1}{z}$ and $z + \frac{1}{z}$ respectively, are joined to form a quadrilateral.



Write down a possible set of conditions for z so that the quadrilateral $OABC$ would be a

(i) rhombus *1*

(ii) square *2*

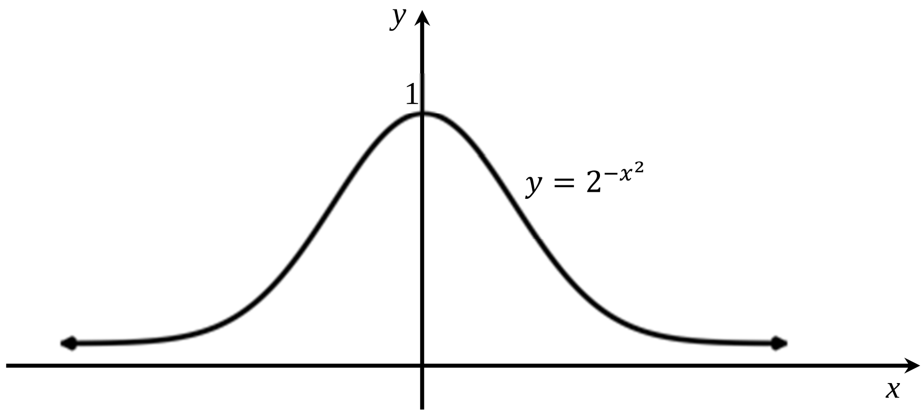
End of Question 12

Marks

<u>Question 13</u> ***(15 marks)*** Use a *separate* answer sheet

(a) Find $\lim\limits_{\theta \to 0} \dfrac{1 - \cos\theta}{\theta}$ *2*

(b) The diagram shows the graph of $y = 2^{-x^2}$ *2*



Sketch the graph of $y = 2^{2x - x^2}$

(c) (i) Show that $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ *1*

(ii) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \ln(1 + \tan x)\,dx$ *3*

(d) An object of mass 20 kg is dropped in a medium where the resistance at speed *v* m/s has a magnitude of 2*v* Newtons. The acceleration due to gravity is 10 $\mathrm{m/s^2}$

(i) By using a force diagram, show that the equation of motion is $\ddot{x} = \dfrac{100 - v}{10}$ *1*

(ii) Find an expression for the velocity at time *t* seconds after the object is dropped. *2*

(iii) Find the terminal velocity of the object. *1*

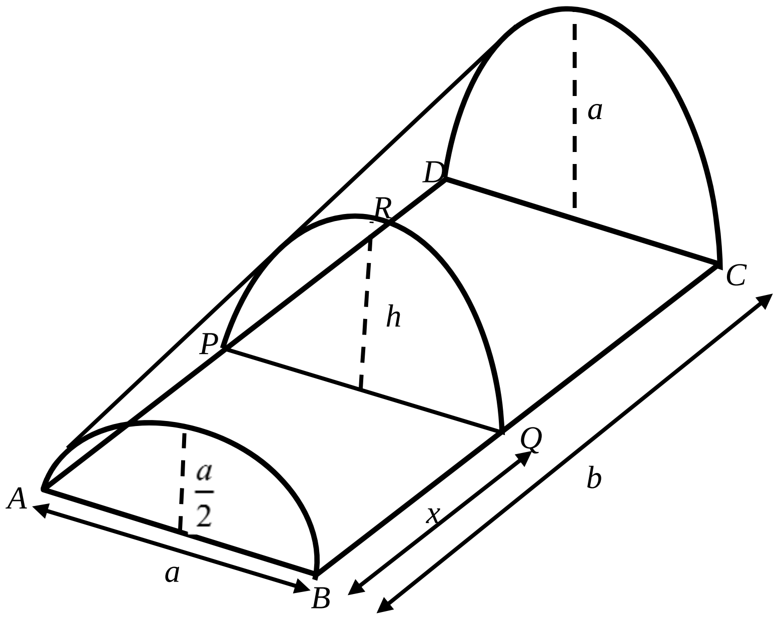
(iv) Find the distance the object has fallen, correct to the nearest metre, before reaching half its terminal velocity. *3*

Marks

<u>Question 14</u> ***(15 marks)*** Use a *separate* answer sheet

(a) Factorise $x^5 - 1$ as the product of real linear and quadratic factors. You may leave your answer in terms of trigonometric ratios. *3*

(b) The diagram shows a solid with a rectangular base *ABCD* of length *b* metres and width *a* metres. The end with *AB* as a base is a semicircle and the other end is a semiellipse whose major axis is twice the length of its minor axis.



(i) Consider the slice of the solid with semielliptical face *PQR* and thickness Δx metres. The slice is parallel to the ends and $BQ = AP = x$ metres. *1*

Let the perpendicular height of the slice *PQR* be *h* metres.

Show that $h = \dfrac{a}{2}\left(\dfrac{x}{b} + 1\right)$

(ii) Hence show the cross-sectional area of the slice *PQR* is given by *1*

$$A(x) = \frac{\pi a^2}{8}\left(\frac{x}{b} + 1\right)$$

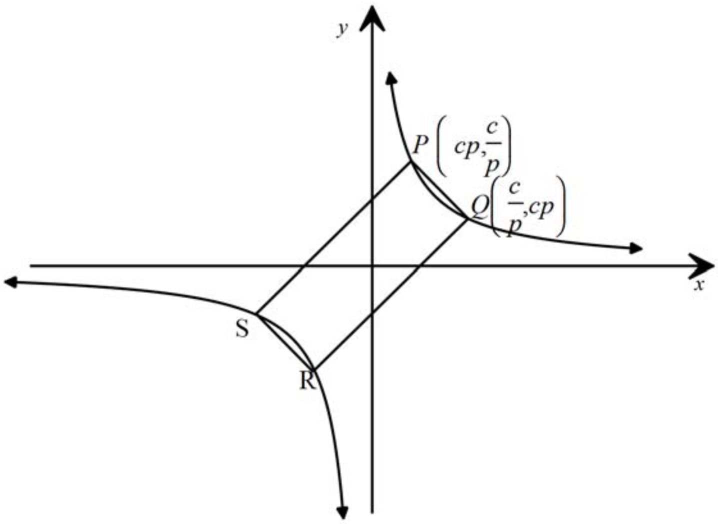
(iii) Find the volume of the solid *2*

Question 14 continues on page 12

Marks

<u>**Question 14**</u> (continued)

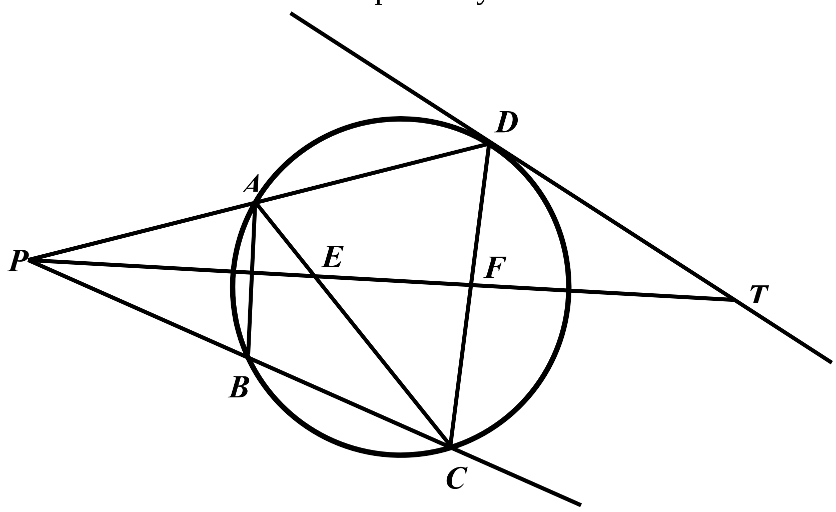
(c) $P\left(cp, \dfrac{c}{p}\right)$ and $Q\left(\dfrac{c}{p}, cp\right)$ are two distinct points on a rectangular hyperbola $xy = c^2$. R and S are two other points such that P, Q, R and S are the vertices of a rectangle.



(i) Write down the coordinates of $R$and S in terms of p *1*

(ii) Prove that it is impossible for these four points to be the vertices of a square *2*

(d) $ABCD$ is a cyclic quadrilateral. DA produced and CB produced, meet at P. T is a point on the tangent at D.
PT cuts CA and CD at E and F respectively. $TF = TD$.



(i) Copy the diagram and show that $AEFD$ is a cyclic quadrilateral *3*

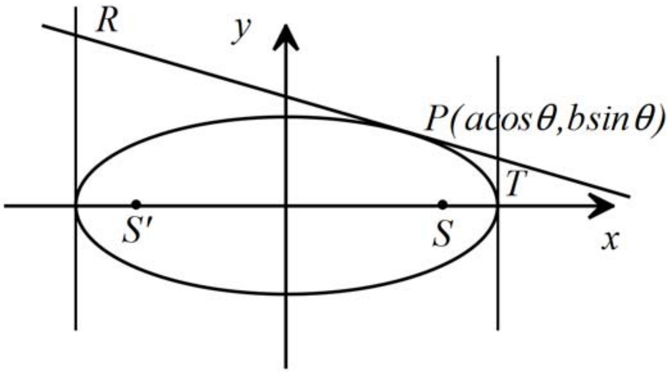
(ii) Show that $AEBP$ is a cyclic quadrilateral *2*

End of Question 14

Marks

<u>Question 15</u> ***(15 marks)*** Use a *separate* answer sheet

(a) The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a > b > 0$, as shown in the diagram below.



The tangent at P meets the tangents at the end of the major axis at R and T. The points S and S' are the foci.

(i) Show that the equation of the tangent at P is given by $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ *1*

(ii) Show that RT subtends a right angle at S *3*

(iii) Show that R, T, S and S' are concyclic. *1*

(b) Let a and b be real numbers. Consider the cubic equation

$$x^3 + 2bx^2 - a^2x - b^2 = 0$$

(i) Show that if $x = 1$ is a solution, then $1-\sqrt{2} \le b \le 1+\sqrt{2}$ *2*

(ii) Show that there is no value of b for which $x = 1$ is a repeated root *2*

(c) If $I_n = \int_0^{\frac{\pi}{2}} \sin^n\theta \, d\theta$ where $n \ge 2$

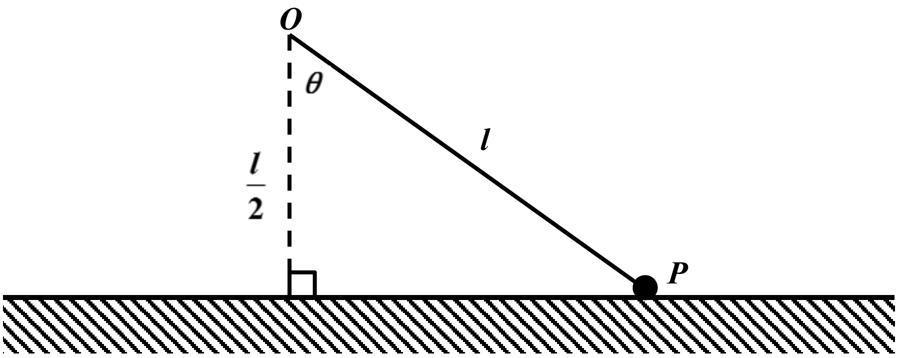
(i) Show that $I_n = \frac{n-1}{n} I_{n-2}$ *3*

(ii) Hence, or otherwise, evaluate $\int_0^2 (4-x^2)^{\frac{5}{2}} \, dx$ *3*

Marks

Question 16 ***(15 marks)*** Use a *separate* answer sheet

(a)



One end of a light inextensible string of length l metres is attached to a fixed point O which is at a height $\frac{l}{2}$ metres above a smooth horizontal table.

A particle P of mass m kg is attached to the other end of the string and rests on the table with the string taut, The particle moves in a circle on the table with constant speed v m/s.

(i) Draw a diagram showing all of the forces acting on the particle P *1*

(ii) Show that the tension, T, in the string has magnitude *2*

$$T = \frac{4mv^2}{3l}$$

(iii) Show that the normal force, N, exerted by the table on P has magnitude *2*

$$N = m\left(g - \frac{2v^2}{3l}\right)$$

(iv) Hence show that *2*

$$v < \sqrt{\frac{3gl}{2}}$$

(v) Explain what would happen if the particle exceeds this speed *1*

(b) Prove by the process of mathematical induction that $(1+x)^n - nx - 1$ is divisible by x^2 for all integers $n \geq 2$ *3*

Question 16 continues on page 15

Marks

Question 16 (continued)

(c) Angus and Benny have a large bag of coins which they use to play a game called HT(2). In this game, Angus and Benny take turns placing one coin at a time on the table, each to the right of a previous one; thus they build a row of coins that builds to the right. Angus always places the first coin. Each coin is placed head-up (H) or tail-up (T), and cannot be flipped or moved once it has been placed.

A player loses the game if they place a coin that results in two adjacent coins having the same pattern of heads and tails as another adjacent pair somewhere in the row. For example,

Benny lost this game by producing a second instance of **HT**.

A	B	A	B	A	B
H	**H**	**T**	T	**H**	**T**

and Angus lost this game by producing a second instance of **TT**.

A	B	A	B	A
T	H	**T**	**T**	**T**

(overlapping pairs can count as a repeat)

(i) What is the smallest number of coins that might be placed in a game of HT(2)? *1*

(ii) What is the largest number of coins that might be placed in a game of HT(2)? *1*

HT(n) has the same rules as HT(2), except the game is lost by the player who creates an unbroken sequence of n heads and tails that appears elsewhere in the row. For example

Benny lost this game of HT(3) by producing a second instance of **THT**.

A	B	A	B	A	B	A	B
H	H	T	**T**	**H**	**T**	**H**	**T**

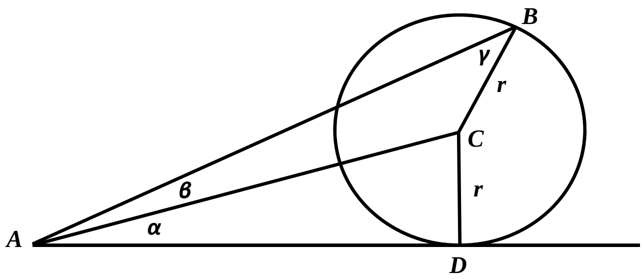
(iii) In these games, a maximum time of one minute is allowed for each turn. *2*

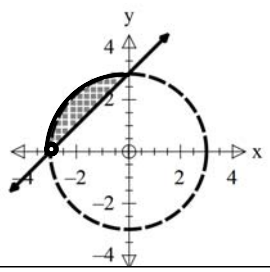
Can we be certain that a game of HT(6) will be finished within two hours? Justify your answer

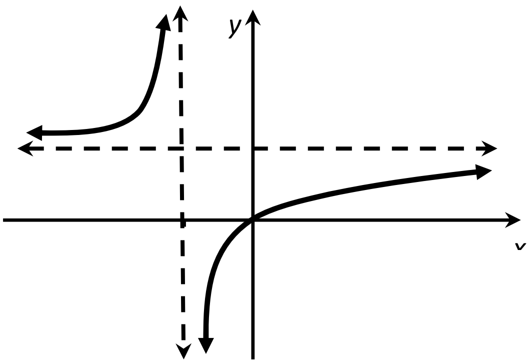
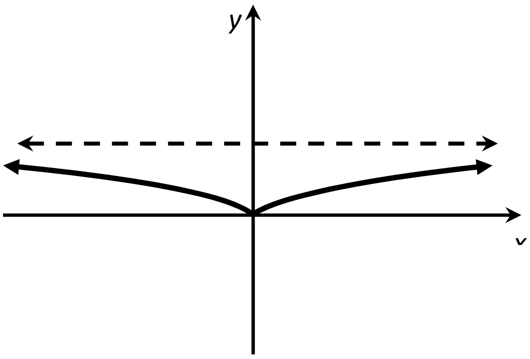
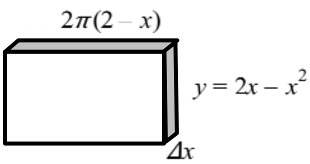
End of paper

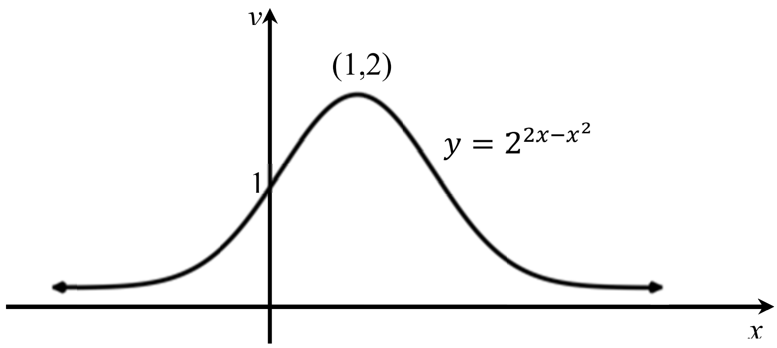
BAULKHAM HILLS HIGH SCHOOL
YEAR 12 EXTENSION 2 TRIAL HSC 2017 SOLUTIONS

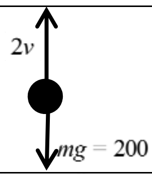
Solution	Marks	Comments
SECTION I		
1. C - $(1+w-w^2)^{2017} = (1+\omega+\omega^2-2\omega^2)^{2017}$ $= (-2w^2)^{2017}$ $\quad (1+\omega+\omega^2=0)$ $= (-2)^{2017}w^{4034}$ $= -2^{2017}\omega^2$ $\quad \left(\omega^{4032}=(w^3)^{1344}=1\right)$	**1**	
2. B – $\int_0^1 (x^2-a)^2\,dx$ $\quad$ minimum $= -\frac{\Delta}{4a}$ $=\int_0^1 (x^4-2ax^2+a^2)\,dx$ $\quad = -\frac{\frac{4}{9}-\frac{4}{5}}{4}$ $=\left[\frac{1}{5}x^5-\frac{2}{3}ax^3+a^2x\right]_0^1$ $\quad =\frac{4}{45}$ $=\frac{1}{5}-\frac{2}{3}a+a^2$	**1**	
3. D – $t^3 = x-1$ $\dot{x}=3t^2$ $\ddot{x}=6t$ which is not constant	**1**	
4. A - $[f(x)]^2 \geq 0$ for all x $\Rightarrow$ eliminates (C) and (D) $[f(x)]^2$ has stationary points at the x-intercepts NOT sharp points $\Rightarrow$ eliminates (B)	**1**	
5. D - $\arg(\alpha z^2)=\arg(\alpha)+2\arg(z)$ from diagram $\therefore \arg z < \arg\alpha$ $\Rightarrow$ eliminates (A) $\lvert\alpha z^2\rvert < \lvert\alpha\rvert < 1$ $\therefore$ $\lvert z\rvert > \lvert z^2\rvert$ $\Rightarrow$ eliminates (B) and (C)	**1**	
6. B - $\sin x \geq \frac{1}{2}$ $\qquad \sin 2x \geq \frac{1}{2}$ $\frac{\pi}{6}\leq x\leq\frac{5\pi}{6}$ $\qquad \frac{\pi}{6}\leq 2x\leq\frac{5\pi}{6}$ and $\frac{13\pi}{6}\leq 2x < +\frac{17\pi}{6}$ $\frac{\pi}{12}\leq x\leq\frac{5\pi}{12}$ $\qquad \frac{13\pi}{12}\leq x\leq\frac{17\pi}{12}$ $\therefore$ one (or both) hold true for $\frac{\pi}{12}\leq x\leq\frac{5\pi}{6}$ and $\frac{13\pi}{12}\leq x\leq\frac{17\pi}{12}$ fraction $=\frac{\frac{13\pi}{12}}{2\pi}=\frac{13}{24}$	**1**	
7. C – ω is imaginary $\Rightarrow$ $\arg\left(\frac{z+1}{z+i}\right)=\pm\frac{\pi}{2}$ which represents a circle, diameter $(-1,0)$ and $(0,-1)$ but not including those points.	**1**	
8. A – $x^2+y^2+4xy-4=0$ $2x+2y\frac{dy}{dx}+4x\frac{dy}{dx}+4y=0$ $\frac{dy}{dx}=-\frac{x+2y}{2x+y}$ Vertical tangents occur when $\frac{dy}{dx}$ is undefined $2x+y=0$ $\qquad x^2+(-2x)^2+4x(-2x)-4=0$ $y=-2x$ $\qquad x^2+4x^2-8x^2-4=0$ $3x^2=-4$ no solutions	**1**	

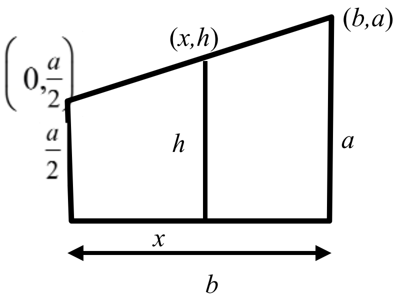
Solution	Marks	Comments
9. B – $p_n(x) = (x-1) + (x-2) + ...(x-n)$ $\quad p_{n-1}(x) = (n-1)x - \frac{(n-1)n}{2}$ $= nx - (1 + 2 + ... + n)$ $= nx - \frac{n(n+1)}{2}$ Using the remainder theorem $R(x) = p_n\left[\frac{\frac{(n-1)n}{2}}{n-1}\right]$ $= p_n\left(\frac{n}{2}\right)$ $= n\left(\frac{n}{2}\right) - \frac{n(n+1)}{2}$ $= \frac{n^2 - n^2 - n}{2}$ $= -\frac{n}{2}$	1	
10. D –  $\frac{AC}{\sin\gamma} = \frac{r}{\sin\beta}$ $\quad \frac{r}{AC} = \sin\alpha$ $AC = \frac{r}{\sin\alpha}$ $\frac{r}{\sin\alpha\sin\gamma} = \frac{r}{\sin\beta}$ $\sin\beta = \sin\alpha\sin\gamma$	1	
SECTION II		
QUESTION 11		
11 (a) (i) $e^2 = \frac{a^2 - b^2}{a^2}$ $e^2 = \frac{4-3}{4}$ $e^2 = \frac{1}{4} \Rightarrow$ eccentricity $= \frac{1}{2}$	1	**1 mark** • Correct answer
11 (a) (ii) foci : $= (\pm ae, 0)$ $= \left(\pm 2 \times \frac{1}{2}, 0\right)$ $= (\pm 1, 0)$	1	**1 mark** • Correct answer
11 (b) (i) $wz = (-1 + \sqrt{3}\,i)(1 - i)$ $= -1 + i + \sqrt{3}\,i + \sqrt{3}$ $= (\sqrt{3} - 1) + (\sqrt{3} + 1)i$	1	**1 mark** • Correct answer
11 (b) (ii) $\lvert w\rvert = 2$ and $\arg w = \tan^{-1}\frac{\sqrt{3}}{-1}$ $\quad \lvert z\rvert = \sqrt{2}$ and $\arg w = \tan^{-1}\frac{\sqrt{3}}{-1}$ $= \frac{2\pi}{3}$ $\quad = \frac{2\pi}{3}$ $w = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ and $z = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$	2	**2 marks** • Correct solution **1 mark** • Finds either *w* or *z* • Calculates both moduli correctly • Calculates both arguments correctly

Solution	Marks	Comments
11 (b) (iii) $wz = 2\sqrt{2}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$ Equating imaginary part with (i) $2\sqrt{2}\sin\frac{5\pi}{12} = \sqrt{3}+1$ $\sin\frac{5\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}$	**2**	**2 marks** • Correct solution **1 mark** • Finds wz in mod-arg form
11 (c) (i) $(ax+b)(2-x)+c(x^2+4)=16$ let $x=2$ $8c=16$ $c=2$ let $x=2i$ $(2ai+b)(2-2i)=16$ $4ai+4a+2b-2bi=16$ $4a-2b=0$ $4a+2b=16$ $8a=16$ $a=2$ and $b=4$ $a=2 \quad b=4 \quad c=2$	**2**	**2 marks** • Correct answers **1 mark** • Finds two of the required pronumerals
11 (c) (ii) $\int \frac{16}{(x^2+4)(2-x)}dx = \int\left(\frac{2x}{x^2+4}+\frac{4}{x^2+4}+\frac{2}{2-x}\right)dx$ $= \ln(x^2+4) + 2\tan^{-1}\left(\frac{x}{2}\right) - 2\ln(2-x) + c$	**2**	**2 marks** • Correct solution **1 mark** • Finds two correct primitives
11 (d) (i)  	**2**	**2 marks** • Correct solution **1 mark** • Shades a region inside a circle radius 3 centred at the origin
11 (d) (ii) minimum $\lvert z\rvert$ is perpendicular distance to origin $=\frac{3}{\sqrt{2}}$ $\frac{3}{\sqrt{2}} \le \lvert z\rvert \le 3$ and $\frac{\pi}{2} \le \arg z < \pi$	**2**	**2 marks** • Correct solution **1 mark** • Finds minimum modulus • Finds argument extremities
QUESTION 12		
12(a) $\int \frac{dx}{1+3\sin x} = \int \frac{\frac{2dt}{1+t^2}}{1+3\left(\frac{2t}{1+t^2}\right)}$ $\qquad t=\tan\frac{x}{2}$, $dt = \frac{2dt}{1+t^2}$ $= \int \frac{2dt}{1+t^2+6t}$ $= \int \frac{2dt}{(t+3)^2-8}$ $= \frac{2}{4\sqrt{2}}\ln\left\lvert\frac{t+3-2\sqrt{2}}{t+3+2\sqrt{2}}\right\rvert + c$ $= \frac{1}{2\sqrt{2}}\ln\left\lvert\frac{\tan\frac{x}{2}+3-2\sqrt{2}}{\tan\frac{x}{2}+3+2\sqrt{2}}\right\rvert + c$	**3**	**3 marks** • Correct solution **2 marks** • Obtains the correct primitive in terms of the substituted variable **1 mark** • Obtains the correct integrand in terms of the substituted variable.

Solution	Marks	Comments
12 (b) (i)  	1	**1 mark** • Correct answer
12 (b) (ii)  	1	**1 mark** • Correct answer
12 (c) (i) let $y = x^2$ $x = y^{\frac{1}{2}}$ $y^{\frac{3}{2}} - 3y + 9 = 0$ $y^{\frac{3}{2}} = 3y - 9$ $y^3 = 9y^2 - 54y + 81$ $y^3 - 9y^2 + 54y - 81 = 0$	2	**2 marks** • Correct solution **1 mark** • Finds a correct nonpolynomial equation
12(c) (ii) Using equation from (i) $\Sigma\alpha^2 = 9$ **OR** Using original equation $\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$ $= (-3)^2 - 2(0)$ $= 9$	1	**1 mark** • Correct answer
12(c) (iii) $\alpha^3 - 3\alpha^2 + 9 = 0$ $\beta^3 - 3\beta^2 + 9 = 0$ $\gamma^3 - 3\gamma^2 + 9 = 0$ $\Sigma\alpha^3 - 3\Sigma\alpha^2 + 27 = 0$ $\Sigma\alpha^3 = 3\Sigma\alpha^2 - 27$ $= 3(9) - 27$ $= 0$	1	**1 mark** • Correct answer
12 (d)  $A(x) = 2\pi(2-x)(2x-x^2)$ $\Delta V = 2\pi(4x - 4x^2 + x^3)\Delta x$ $V = \lim_{\Delta x \to 0} \sum_{x=0}^{2} 2\pi(4x - 4x^2 + x^3)\Delta x$ $= 2\pi\int_0^2 (4x - 4x^2 + x^3)\,dx$ $= 2\pi\left[2x^2 - \frac{4}{3}x^3 + \frac{1}{4}x^4\right]_0^2$ $= 2\pi\left[2(2)^2 - \frac{4}{3}(2)^3 + \frac{1}{4}(2)^4 - 0\right]$ $= \frac{8\pi}{3}\text{ units}^3$	3	**3 marks** • Correct solution **2 marks** • Establishes the primitive function **1 mark** • Correctly expresses the volume as a limiting sum

Solution	Marks	Comments
12 (e) (i) in a rhombus, adjacent sides are equal $\lvert z\rvert = \left\lvert \frac{1}{z} \right\rvert$, $z \neq \pm 1 , \pm i$ $\lvert z\rvert^2 = 1$ $\lvert z\rvert = 1$	1	**1 mark** • Correct answer
12 (e) (ii) $\frac{1}{z} = \frac{\bar{z}}{\lvert z^2\rvert} = \bar{z}$ **,** so AB is vertical, as the diagonals are perpendicular then OC is horizontal. Thus C lies on the real axis Conditions are $Im\left(z + \frac{1}{z}\right) = 0$ and $\lvert z\rvert = 1$ ***OR*** a square is a rhombus with $\angle AOB = 90°$ $z = \frac{i}{z}$ OR $z = -\frac{i}{z}$ $z^2 = i$ $\quad$ $z^2 = -i$ $z = \pm\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$ $\quad$ $z = \pm\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$	2	**2 marks** • Correct possible solution **1 mark** • Recognises A and B are conjugates • Uses the idea of rotation by $90°$ is multiplication by i • $z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ or similar as a single result.
QUESTION 13		
13 (a) $\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} = \lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta} \times \frac{1 + \cos\theta}{1 + \cos\theta}$ $= \lim_{\theta \to 0} \frac{1 - \cos^2\theta}{\theta(1 + \cos\theta)}$ $= \lim_{\theta \to 0} \frac{\sin^2\theta}{\theta(1 + \cos\theta)}$ $= \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \times \frac{\sin\theta}{1 + \cos\theta}$ $= 1 \times 0$ $= 0$	2	**2 marks** • Correct solution **1 mark** • Uses $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$, or similar result
13 (b) $y = 2^{-x^2} = 2^{-(x-1)^2 + 1} = 2 \times 2^{-(x-1)^2}$ Original curve has been stretched vertically by a factor of 2 and translated horizontally 1 unit to the right. 	2	**2 marks** • Correct graph **1 mark** • Recognises graph involves a horizontal shift
13 (c) (i) $\int_0^a f(x)dx = -\int_a^0 f(a-u)du$ $\quad$ $u = a - x \Rightarrow x = a - u$, $du = -dx$ $= \int_0^a f(a-u)du$ $\quad$ when $x = 0$, $u = a$; $x = a$, $u = 0$ $= \int_0^a f(a-x)dx$	1	**1 mark** • Correctly shows result

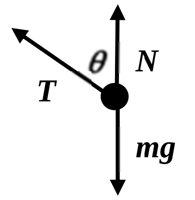
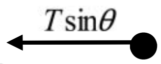
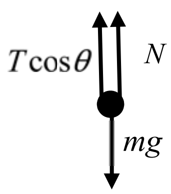
Solution	Marks	Comments
13 (c) (ii) $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)dx = \int_0^{\frac{\pi}{4}} \ln\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx$ $= \int_0^{\frac{\pi}{4}} \ln\left(1+\frac{1-\tan x}{1+\tan x}\right)dx$ $= \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right)dx$ $= \int_0^{\frac{\pi}{4}} \ln 2\, dx - \int_0^{\frac{\pi}{4}} \ln(1+\tan x)dx$ $2\int_0^{\frac{\pi}{4}} \ln(1+\tan x)dx = [x\ln\ 2]_0^{\frac{\pi}{4}}$ $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)dx = \frac{1}{2}\times\frac{\pi}{4}\ln 2$ $= \frac{\pi}{8}\ln 2$	3	**3 marks** • Correct solution **2 marks** • Manipulates the integrand to be an equation in terms of $\int \ln(1+\tan x)dx$ **1 mark** • Use the property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$
13(d) (i) $m\ddot{x} = mg - 2v$ $20\ddot{x} = 200 - 2v$ $\ddot{x} = \frac{100-v}{10}$  	1	**1 mark** • Correct solution
13 (d) (ii) $\frac{dv}{dt} = \frac{100-v}{10}$ $10\int_0^v \frac{dv}{100-v} = \int_0^t dt$ $t = -10\Big[\ln(100-v)\Big]_0^v$ $t = 10\ln\left(\frac{100}{100-v}\right)$ $\frac{t}{10} = \ln\left(\frac{100}{100-v}\right)$ $e^{\frac{t}{10}} = \frac{100}{100-v}$ $100 - v = 100e^{-\frac{t}{10}}$ $v = 100 - 100e^{-\frac{t}{10}}$	2	**2 marks** • Correct solution **1 mark** • Finds t as a function of v.
13 (d) (iii) Terminal velocity occurs when $\ddot{x} = 0$ $v = 100$ OR $\lim_{t\to\infty} v$ $= \lim_{t\to\infty} 100 - 100e^{-\frac{t}{10}}$ $= 100$ m/s	1	**1 mark** • Correct answer
13 (d) (iv) $v\frac{dv}{dx} = \frac{100-v}{10}$ $10\int_0^{50} \frac{v}{100-v}\,dv = \int_0^x dx$ $x = 10\int_0^{50}\left(-1+\frac{100}{100-v}\right)$ dv $x = 10\Big[-v - 100\ln(100-v)\Big]_0^{50}$ $= -500 - 1000\ln\left(\frac{50}{100}\right)$ $= 1000\ln 2 - 500 = 193.157 = 193$ metres (to nearest metre)	3	**3 marks** • Correct solution **2 marks** • Finds x as a function of v (or t with correct value for length of time) **1 mark** • uses $\ddot{x} = v\frac{dv}{dx}$ in an attempt to find the solution *Note: no rounding penalty* • finds correct length of time ($t = 10\ln 2$)

Solution	Marks	Comments
QUESTION 14		
14 (a) $x^5-1=0$ $x^5=1$ $x=\text{cis}\left(\frac{2\pi k}{5}\right)$ where $k=0\,,\pm1\,,\pm2$ $x=1\,,\text{cis}\left(\frac{2\pi}{5}\right),\text{cis}\left(-\frac{2\pi}{5}\right),\text{cis}\left(\frac{4\pi}{5}\right),\text{cis}\left(-\frac{4\pi}{5}\right)$ $x^5-1=(x-1)\left(x-\text{cis}\frac{2\pi}{5}\right)\left(x-\text{cis}\left(-\frac{2\pi}{5}\right)\right)\left(x-\text{cis}\frac{4\pi}{5}\right)\left(z-\text{cis}\left(-\frac{4\pi}{5}\right)\right)$ $=(x-1)\left(x^2-2x\cos\frac{2\pi}{5}+1\right)\left(x^2-2x\cos\frac{4\pi}{5}+1\right)$	3	**3 marks** • Correct solution **2 marks** • Creates five linear factors using the five roots of unity **1 mark** • Finds the five roots of unity (credit paid for generalised form) • Groups two conjugate roots together to create a quadratic factor
14 (b) (i)  $m=\dfrac{a-\frac{a}{2}}{b-0}$ $=\dfrac{a}{2b}$ $h-a=\frac{a}{2b}(x-b)$ $h-a=\frac{a}{2b}x-\frac{a}{2}$ $h=\frac{a}{2b}x+\frac{a}{2}$ $h=\frac{a}{2}\left(\frac{x}{b}+1\right)$	1	**1 mark** • Correct solution
14 (b) (ii) $A(x)=\frac{\pi ab}{2}$ $=\frac{\pi}{2}\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)\left(\frac{x}{b}+1\right)$ $=\frac{\pi a^2}{8}\left(\frac{x}{b}+1\right)$	1	**1 mark** • Correct solution
14 (b) (iii) $\Delta V=\frac{\pi a^2}{8}\left(\frac{x}{b}+1\right)\Delta x$ $V=\lim\limits_{\Delta x\to 0}\sum\limits_{x=0}^{b}\frac{\pi a^2}{8}\left(\frac{x}{b}+1\right)\Delta x$ $=\frac{\pi a^2}{8}\int_0^b\left(\frac{x}{b}+1\right)dx$ $=\frac{\pi a^2}{8}\left[\frac{x^2}{2b}+x\right]_0^b$ $=\frac{\pi a^2}{8}\left(\frac{b}{2}+b\right)$ $=\frac{3\pi a^2 b}{16}$ units^3	2	**2 marks** • Correct solution **1 mark** • Creates an integral to find volume by expressing as a sum of similar slices
14(c) (i) By symmetry **;** $R\left(-cp,-\frac{c}{p}\right)$ and $S\left(-\frac{c}{p},-cp\right)$	1	**1 mark** • Correct answer

Solution	Marks	Comments
14 (c) (ii) $m_{PR} = \dfrac{\dfrac{c}{p}+\dfrac{c}{p}}{cp+cp} = \dfrac{\dfrac{2c}{p}}{2cp} = \dfrac{1}{p^2}$ $\quad m_{QS} = \dfrac{cp+cp}{\dfrac{c}{p}+\dfrac{c}{p}} = \dfrac{2cp}{\dfrac{2c}{p}} = p^2$ $m_{PR} \times m_{QS} = \dfrac{1}{p^2} \times p^2 = 1 \neq -1$ As the diaginals **cannot** meet at right angles, the rectangle **cannot** be a square	**2**	**2 marks** • Correct solution **1 mark** • Finds the slope of one of the diagonals • Finds the length of two adjacent sides
14 (d) (i) $\angle TDF = \angle DAE$ (alternate segment theorem) $\angle TDF = \angle TFD$ ($\angle$'s opposite = sides in Δ are =) $\therefore \angle DAE = \angle TFD$ Thus *AEFD* is cyclic (exterior $\angle$ = opposite interior $\angle$)	**3**	**3 marks** • Correct proof with diagram **2 marks** • Correct proof without diagram • Correct proof with poor reasoning • Uses a relevant circle geometry theorem and includes diagram **1 mark** • Uses a relevant circle geometry theorem • redraws diagram
14 (d) (ii) $\angle PBA = \angle ADC$ (exterior $\angle$ cyclic quadrilateral *BADC*) $\angle FEC = \angle ADC$ (exterior $\angle$ cyclic quadrilateral *EADF*) $\angle FEC = \angle AEP$ (vertically opposite $\angle$'s) $\therefore \angle PBA = \angle PEA$ Thus AEBP is cyclic ($\angle$'s standing on same arcAP are =)	**2**	**2 marks** • Correct proof **1 mark** • Uses a relevant circle geometry theorem
QUESTION 15		
15 (a) (i) $x = \text{a}\cos\theta$ $\quad y = \text{b}\sin\theta$ $\quad \dfrac{dy}{dx} = \dfrac{dy}{d\theta} \times \dfrac{d\theta}{dx}$ $\dfrac{dx}{d\theta} = -\text{a}\sin\theta$ $\quad \dfrac{dy}{d\theta} = \text{b}\cos\theta$ $\quad = -\dfrac{\text{b}\cos\theta}{\text{a}\sin\theta}$ $y - \text{b}\sin\theta = -\dfrac{\text{b}\cos\theta}{\text{a}\sin\theta}(x - \text{a}\cos\theta)$ $ay\sin\theta - \text{ab}sin^2\theta = -bx\cos\theta + ab\cos^2\theta$ $bx\cos\theta + ay\sin\theta = ab(\sin^2\theta + \cos^2\theta)$ $\dfrac{x}{a}\cos\theta + \dfrac{y}{b}\sin\theta = 1$	**1**	**1 mark** • Correct solution

<table>
<tr><th>Solution</th><th>Marks</th><th>Comments</th></tr>
<tr><td>15 (a) (ii) $x = a;\ \cos\theta + \frac{y}{b}\sin\theta = 1$
$y = \frac{b(1-\cos\theta)}{\sin\theta}$
$\therefore T\left(a, \frac{b(1-\cos\theta)}{\sin\theta}\right)$
Similarly $R\left(-a, \frac{b(1+\cos\theta)}{\sin\theta}\right)$
$m_{RS} = \frac{\frac{b(1+\cos\theta)}{\sin\theta}}{-a-ae} = \frac{b}{-a}\left(\frac{1+\cos\theta}{(1+e)\sin\theta}\right)$
$m_{TS} = \frac{\frac{b(1-\cos\theta)}{\sin\theta}}{a-ae} = \frac{b}{a}\left(\frac{1-\cos\theta}{(1-e)\sin\theta}\right)$
$m_{RS} \times m_{TS} = \frac{b^2}{-a^2}\left(\frac{1-\cos^2\theta}{(1-e^2)\sin^2\theta}\right) = \frac{b^2\sin^2\theta}{-b^2\sin^2\theta} = -1$
$\therefore ST \perp RS$ i.e. $\angle RST = 90°$</td><td>3</td><td>3 marks
• Correct solution
2 marks
• Finds m_{RS} or m_{TS}
1 mark
• Finds the coordinates for R or T</td></tr>
<tr><td>15 (a) (iii) Using the result found in (ii) with the other focus
$\angle RS'T = \angle RST = 90°$
Thus RTSS' are concyclic as chord RT subtends = $\angle$'s at s and S'</td><td>1</td><td>1 mark
• Correct solution</td></tr>
<tr><td>15 (b) (i) when $x = 1$; $1 + 2b - a^2 - b^2 = 0$
$a^2 = 1 + 2b - b^2$
however $a^2 \geq 0$
$b^2 - 2b + 1 \geq 0$
$-b^2 + 2b - 1 \leq 0$
$\frac{2-\sqrt{8}}{2} \leq b \leq \frac{2+\sqrt{8}}{2}$
$1-\sqrt{2} \leq b \leq 1+\sqrt{2}$</td><td>2</td><td>2 marks
• Correct solution
1 mark
• makes use of the factor theorem in a valid attempt to show the desired result</td></tr>
<tr><td>15 (b) (ii) $P'(x) = 3x^2 + 4bx - a^2$
$P'(1) = 3 + 4b - a^2$
To be a repeated root both $P'(x) = 0$ and $P(x) = 0$
$P'(1) = 0$ From part (i); $a^2 = 1 + 2b - b^2$
$3 + 4b - a^2 = 0$
$a^2 = 3 + 4b$
$3 + 4b = 1 + 2b - b^2$
$b^2 + 2b + 2 = 0$
$\Delta = 4 - 8 < 0$
There are no values of b for which both $P'(x) = 0$ and $P(x) = 0$
i.e. if $x = 1$, there is no value of b that will create a repeated root.</td><td>2</td><td>2 marks
• Correct solution
1 mark
• Uses the result $P'(x) = 0$ in a valid attempt to show the desired result</td></tr>
</table>

Solution	Marks	Comments
15 (c) (i) $I_n = \int_0^{\frac{\pi}{2}} \sin^n\theta\, d\theta$ $= \int_0^{\frac{\pi}{2}} \sin^{n-1}\theta \times \sin\theta\, d\theta$ $u = \sin^{n-1}\theta$ $v = -\cos\theta$ $du = (n-1)\sin^{n-2}\theta \times \cos\theta\, d\theta$ $dv = \sin\theta\, d\theta$ $I_n = \left[-\cos\theta\sin^{n-1}\theta\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta \times \cos^2\theta\, d\theta$ $= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta(1-\sin^2\theta)\, d\theta$ $= (n-1)\int_0^{\frac{\pi}{2}} (\sin^{n-2}\theta - \sin^n\theta)\, d\theta$ $= (n-1)I_{n-2} - (n-1)I_n$ $nI_n = (n-1)I_{n-2}$ $I_n = \frac{n-1}{n}I_{n-2}$	**3**	**3 marks** • Correct solution **2 marks** • Substantial progress towards a solution using logical techniques **1 mark** • Attempts to create the reduction formula by using integration by parts, or similar merit.
15 (c) (ii) $\int_0^2 (4-x^2)^{\frac{5}{2}}\, dx$ $x = 2\cos\theta$ $dx = -2\sin\theta$ when $x = 0$, $\theta = \frac{\pi}{2}$ when $x = 2$, $\theta = 0$ $= -\int_{\frac{\pi}{2}}^{0} (4-4\cos^2\theta)^{\frac{5}{2}} \times 2\sin\theta\, d\theta$ $= 2^5 \times 2\int_0^{\frac{\pi}{2}} \sin^6\theta\, d\theta$ $= 64I_6$ $= 64 \times \frac{5}{6}I_4$ $= \frac{160}{3} \times \frac{3}{4}I_2$ $= 40 \times \frac{1}{2}I_0$ $= 20\int_0^{\frac{\pi}{2}} \theta\, d\theta$ $= 20\left[\theta\right]_0^{\frac{\pi}{2}}$ $= 20 \times \frac{\pi}{2}$ $= 10\pi$	**3**	**3 marks** • Correct solution **2 marks** • Reduces I_6 to an easily managed integral **1 mark** • Using an appropriate substitution, obtains a multiple of I_6 • Uses the reduction formula to reduce their integral to an easily managed integral

Solution	Marks	Comments
QUESTION 16		
16 (a) (i)  	1	**1 mark** • Correct diagram *Note: penalise if resultant force is draw on diagram*
16 (a) (ii) horizontal $F = \frac{mv^2}{r}$  $T\sin\theta = \frac{mv^2}{r}$ $\frac{Tr}{l} = \frac{mv^2}{r}$ $T = \frac{mv^2}{r} \times \frac{l}{r}$ $= \frac{mv^2 l}{1} \times \frac{4}{3l^2}$ $= \frac{4mv^2}{3l}$ $r^2 = l^2 - \frac{l^2}{4}$ $= \frac{3l^2}{4}$ $r = \frac{\sqrt{3}\,l}{2}$	2	**2 marks** • Correct solution **1 mark** • Deriving the horizontal motion equation • Calculating the radius in terms of l
16 (a) (iii) vertical $F = 0$  $N + T\cos\theta - mg = 0$ $N = mg - T\cos\theta$ $= mg - \frac{4mv^2}{3l} \times \frac{1}{2}$ $= m\left(g - \frac{2v^2}{3l}\right)$	2	**2 marks** • Correct solution **1 mark** • Deriving the vertical motion equation
16 (a) (iv) $N > 0$ $m\left(g - \frac{2v^2}{3l}\right) > 0$ $\frac{2v^2}{3l} < g$ $v^2 < \frac{3gl}{2}$ $v < \sqrt{\frac{3gl}{2}}$	2	**2 marks** • Correct solution **1 mark** • Realises $N > 0$
16 (a) (v) The particle would lift off the table and perform like a regular conical pendulum	1	**1 mark** • Correct answer

Solution	Marks	Comments
16 (b) When $n = 2$; $(1+x)^2 - 2x - 1 = 1 + 2x + x^2 - 2x - 1$ $= x^2$ Which is divisible by x^2 Hence the result is true for $n = 2$ Assume the result is true for $n = k$ where k is an integer i.e. $(1+x)^k - kx - 1 = x^2 P(x)$, where $P(x)$ is a polynomial Prove the result is true for $n = k + 1$ i.e. $(1+x)^{k+1} - (k+1)x - 1 = x^2 Q(x)$, where $Q(x)$ is a polynomial ***PROOF:*** $(1+x)^{k+1} - (k+1)x - 1 = (1+x)[x^2 P(x) + kx + 1] - (k+1)x - 1$ $= x^2(x+1)\, P(x) + kx + 1 + kx^2 + x - kx - x - 1$ $= x^2(x+1)\, P(x) + kx^2$ $= x^2[(x+1)\, P(x) + k]$ $= x^2\, Q(x)$, where $Q(x) = (x+1)\, P(x) + k$ which is a polynomial Hence the result is true for $n = k + 1$, if it is true for $n = k$ Since the result is true for $n = 2$, then it is true for all positive integers $n \geq 2$ by induction.	3	There are 4 key parts of the induction; 1. Proving the result true for $n = 2$ 2. Clearly stating the assumption and what is to be proven 3. Using the assumption in the proof 4. Correctly proving the required statement **3 marks** • Successfully does all of the 4 key parts **2 marks** • Successfully does 3 of the 4 key parts **1 mark** • Successfully does 2 of the 4 key parts
16 (c) (i) 3 coins (**HHH** or **TTT)**	1	**1 mark** • Correct answer
16 (c) (ii) 6 coins There are four distinct pairs (HH, TT, HT, TH) 5 coins would contain four adjacent pairs, so when the sixth coin is placed, one of the pairs MUST be repeated. e.g. **HTTHH** (contains all four distinct pairs) (HT – TT – TH – HH) next coin must repeat one of these (H $\Rightarrow$ **HH** , T $\Rightarrow$ **HT**)	1	**1 mark** • Correct answer
16 (c) (iii) There are $2^6 = 64$ distinct sequences. First sequence will begin at coin 1 Second distinct sequence will begin at coin 2 Third distinct sequence will begin at coin 3 . . . Final distinct sequence will begin at coin 64 If it begins at coin 64, it must end at coin (64+5=69) Thus a maximum of 70 coins will be played, which would take a maximum of 70 minutes to play. So **YES** we can be certain that a game would be finished within two hours.	2	**2 marks** • Correct solution **1 mark** • Calculates the number of distinct sequences