BAULKHAM HILLS HIGH SCHOOL

2018

YEAR 12 TRIAL
HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen.
- NESA-approved calculators may be used
- A reference sheet is provided at the back of this paper.
- In Questions 11-16, show relevant mathematical reasoning and/or calculations.
- Marks may be deducted for careless or badly arranged work

Total marks – 100
Exam consists of 11 pages.

This paper consists of TWO sections.

Section 1 – Page 2-4 (10 marks)
Questions 1 - 10

- Attempt Questions 1 - 10
  Allow about 15 minutes for this section.

Section II – Pages 5 – 11 (90 marks)

- Attempt questions 11 - 16
  Allow about 2 hours 45 minutes for this section.

Section I - Multiple Choice (10 marks)
Allow about 15 minutes for this section.

Use the multiple choice page for Question 1 – 10

1 If $(a + bi)^2 = i$, where a and b are real numbers, then:

(A) $a = -\frac{1}{2}$, $b = -\frac{1}{2}$

(B) $a = -\frac{1}{\sqrt{2}}$, $b = \frac{1}{\sqrt{2}}$

(C) $a = \frac{1}{\sqrt{2}}$, $b = \frac{1}{\sqrt{2}}$

(D) $a = \frac{1}{\sqrt{2}}$, $b = -\frac{1}{\sqrt{2}}$

2 The polynomial $P(x) = x^3 + 3x^2 - 24x + 28$ has a double root. What is the value of this double root?

(A) -8

(B) -7

(C) 4

(D) 2

3 How many points do the graphs of $\frac{(x+1)^2}{9} + \frac{y^2}{4} = 1$ and $x^2 - y^2 = -4$ have in common?

(A) 0

(B) 1

(C) 2

(D) 4

4 Which expression is equal to

$$\int \frac{dx}{\sqrt{7-6x-x^2}} \ ?$$

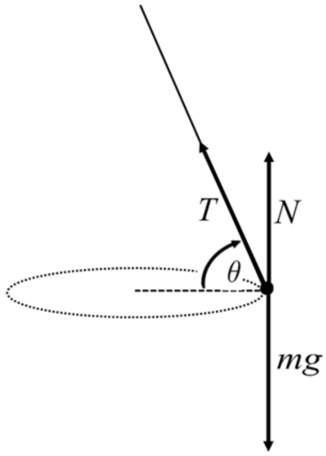
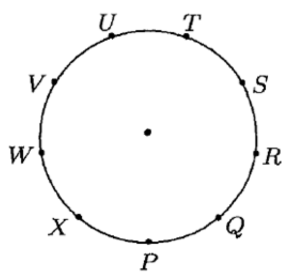
(A) $\sin^{-1}\left(\frac{x+3}{2}\right) + c$

(B) $\sin^{-1}\left(\frac{x+3}{4}\right) + c$

(C) $\sin^{-1}\left(\frac{x-3}{2}\right) + c$

(D) $\sin^{-1}\left(\frac{x-3}{4}\right) + c$

5	Given $P(z) = z^3 + (1+i)z^2 + (1+i)z + c$ (where c is constant) has a real zero $z = -1$ and another zero $z = \alpha$. The third root could be: (A) $z = 1 - \alpha$ (B) $z = \bar{\alpha}$ (C) $z = -\alpha$ (D) $z = \frac{1}{\alpha}$
6	Which of the following functions would be neither odd nor even? (A) $y = x^2 \sin x$ (B) $y = \sin(x^2)$ (C) $y = (\sin x)^2$ (D) $y = x^2 + \sin x$
7	The point (2,3) lies on the curve $2x^2 + 4xy - y^2 = 23$. Find the gradient of the tangent to the curve at this point. (A) $-\frac{5}{2}$ (B) $-\frac{5}{4}$ (C) -4 (D) -10
8	At time t, the position of a particle moving on the Cartesian plane is given by $x = 3t$, $y = e^t$. Its acceleration is: (A) Constant in both magnitude and direction. (B) Constant in magnitude only. (C) Constant in direction only. (D) Constant in neither magnitude nor direction.

9	A particle P of mass *m* kg moves at a constant speed *v* in a circle of radius *r*, on a smooth horizontal table, while attached to a string. The string is fixed at the top and inclined at angle of θ to the table. Hence particle P is acted upon by three forces: tension *T*, normal reaction *N* and weight *mg*.  Which of the following shows the correct resolution of the forces on P ? (A) $T\cos\theta + N = mg\ ,\ T\sin\theta = \frac{mv^2}{r}$ (B) $T\cos\theta - N = mg\ ,\ T\sin\theta = \frac{mv^2}{r}$ (C) $T\sin\theta + N = mg\ ,\ T\cos\theta = \frac{mv^2}{r}$ (D) $T\sin\theta - N = mg\ ,\ T\cos\theta = \frac{mv^2}{r}$
10	The nine points P, Q, R, S, T, U, V, W and X are equally spaced around the circumference of a circle. How many distinct triangles can be formed using three of these points as its vertices, with the condition that the centre of the circle must lie in the interior of each such triangle?  (A) 21 (B) 30 (C) 42 (D) 48
	End of Section 1

Section II (90 marks)
Allow about 2 hours 45 minutes for this section.

Answer each question on the appropriate page in the writing booklet.

Question 11 (15 marks) **Marks**

a) For the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$, find:

(i) the eccentricity. **1**
(ii) the coordinates of the foci. **1**

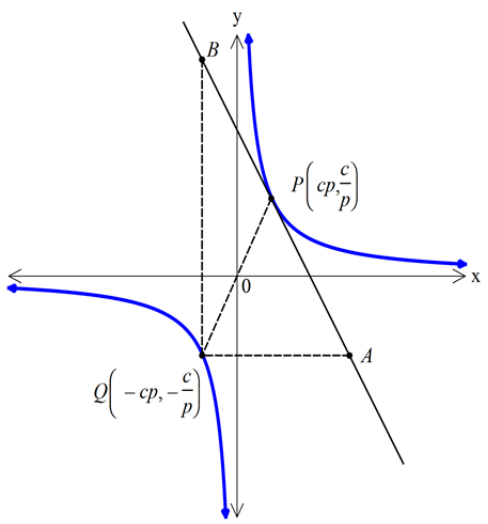
b) Let $z = 1 + i = \sqrt{2}\,\text{cis}\frac{\pi}{4}$ and $w = \sqrt{3} + i$.

(i) Find $\frac{z}{w}$ in the form form $a + ib$, where a and b are real numbers. **1**
(ii) Express w in modulus-argument form. **1**
(iii) Hence or otherwise, find the exact value of $\cos\frac{\pi}{12}$. **2**

c) If $\left|z + \sqrt{3} - i\right| = 1$:

(i) Sketch the locus of z on an Argand diagram. **2**
(ii) Find all possible values for $\arg z$. **2**
(iii) Find the maximum value for $|z|$. **1**

d) $P\left(cp, \frac{c}{p}\right)$ and $Q(-cp, -\frac{c}{p})$ are two points lying on the rectangular hyperbola $xy = c^2$. The tangent at P meets the line which passes through Q and is parallel to the x-axis at point A, and the tangent also meets the line which passes through Q and is parallel to the y-axis at B.

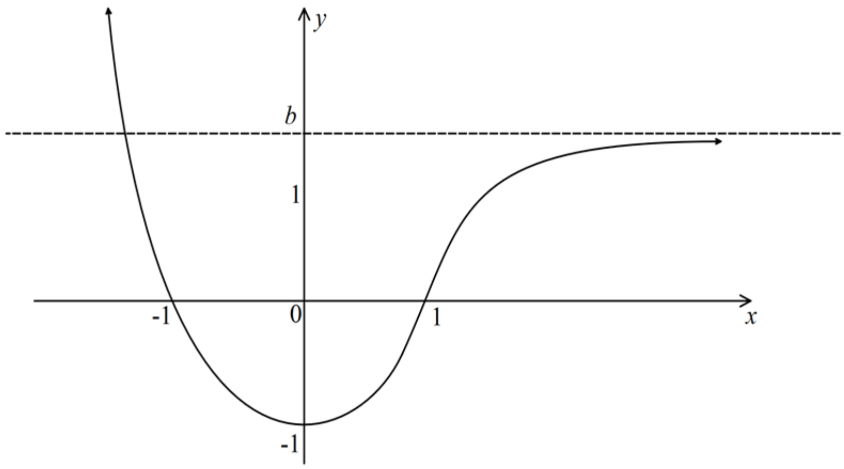


(i) What is the eccentricity of the hyperbola? **1**
(ii) Given that the equation of the tangent at P is $x + p^2y = 2cp$, show that P is the midpoint of AB. **2**
(iii) Find the equation of the locus of A. **1**

End of Question 11

Question 12 (15 marks)

a) The diagram below shows the function $y = f(x)$.



(i) Sketch $y^2 = f(x)$, indicating important features such as turning points, intercepts and asymptotes. **3**

(ii) On a separate diagram, sketch $y = f(|x|)$. **1**

b) The polynomial equation $x^3 - 3x - 2 = 0$ has roots α, β and γ.

(i) Find a polynomial equation with roots α^2, β^2 and γ^2. **2**

(ii) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. **1**

(iii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$. **2**

c) (i) Find: **3**

$$\int \frac{e^x + e^{2x}}{1 + e^{2x}} dx$$

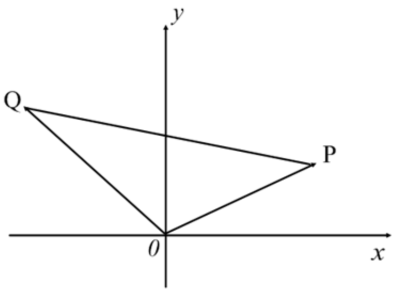
(ii) Evaluate: **3**

$$\int_0^{\frac{\pi}{4}} x \sec^2 x \, dx$$

End of Question 12

Question 13 (15 marks)

a) The Argand diagram below shows a triangle *OPQ*, where points *P* and *Q* represent the complex numbers *z* and *w* respectively.

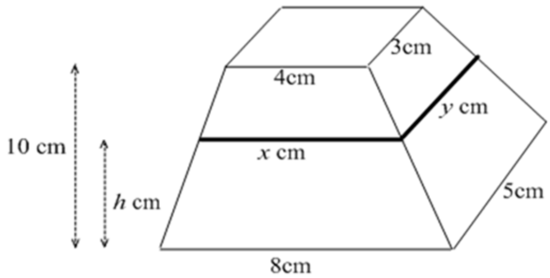


(i) Explain why $|z - w| \leq |z| + |w|$. **1**

(ii) The point *R* represents $z + w$. If $|z-w|=|z+w|$, what type of quadrilateral is *OPRQ*? Give a reason for your answer. **2**

b) The region bounded by the curve $y = x^4 + 1$, the *y*-axis, the *x*-axis and the line $x = 1$ is rotated about the line $x = 3$.
Use the method of cylindrical shells to find the exact volume of the solid generated. **3**

c) A block of wood is a frustum, 10 cm high. The cross-sections parallel to its ends are rectangles which vary from 5cm by 8cm at the bottom, to 3cm by 4cm at the top. All of its faces are plane shapes.



The cross-section shown is a rectangle of length *x* cm and width *y* cm, at a height *h* cm above the base of the block.

(i) Find an expression for *x* in terms of *h* **2**

(ii) Given $y = \frac{25-h}{5}$ (you do NOT need to prove this result), find the volume of the block **3**

d) Evaluate, without the use of a calculator: **2**

$$\lim_{x \to 2} \frac{\sqrt{6-x} - 2}{2 - x}$$

e) Five girls and three boys are to be seated around a circular table. How many arrangements are possible if at least two boys are sitting next to each other? **2**

End of Question 13

Question 14 (15 marks)

a) Given $\frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$, prove by mathematical induction that **3**

$$S_n = \sum_{r=1}^{n} \frac{1}{r^2} \leq 2 - \frac{1}{n}$$

for all positive integers *n*.

(You may also use the result that $\frac{1}{(n+1)^2} < \frac{1}{n(n+1)}$ when $n > 0$, without further proof.)

b) Given that

$$I_n = \int_{\pi/6}^{\pi/4} \cot^n x \, dx \quad \text{for all positive integers } n:$$

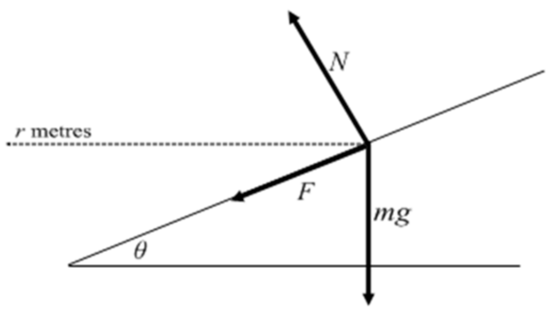
(i) Show that $I_1 = \frac{1}{2}\log_e 2$. **1**

(ii) Show that **3**

$$I_{n-2} + I_n = \frac{1}{n-1}\left[\left(\sqrt{3}\right)^{n-1} - 1\right] \quad \text{for } n = 2,3,4\ldots..$$

(iii) Evaluate I_5. **2**

c) A car of mass m kg is travelling around a circular banked track of radius *r* metres. It experiences a normal reaction force *N* perpendicular to the track and a frictional force *F* parallel to the track. The track is banked at an angle θ to the horizontal and the acceleration due to gravity is *g* m/s^2.



By resolving forces vertically and horizontally it can be shown that

$N\cos\theta - F\sin\theta = mg$ and $F\cos\theta + N\sin\theta = \frac{mv^2}{r}$ (DO NOT PROVE THESE)

(i) Find an expression for $F^2 + N^2$ in terms of *m, g, v* and *r*. **2**

(ii) If the car is traveling at 72 km/h and the ratio $F : N = 6 : 1$, show that **2**

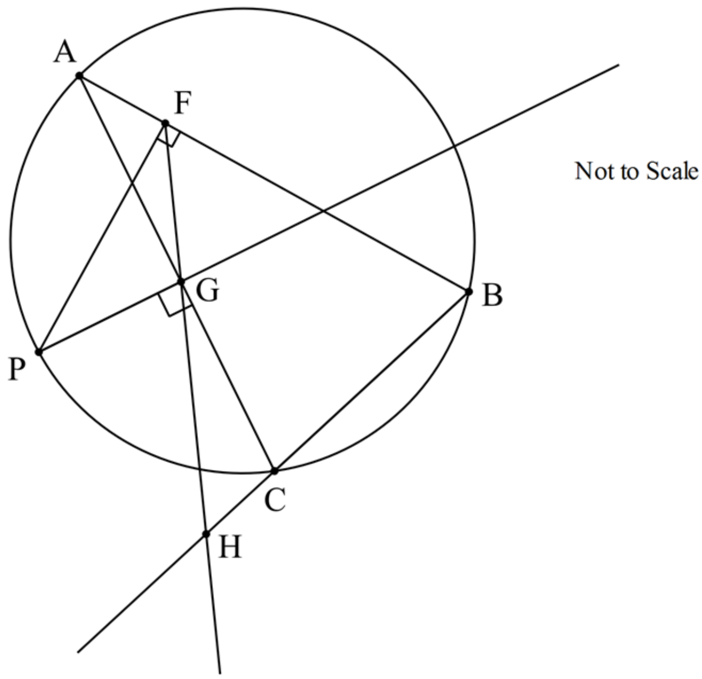
$$N = \frac{m}{\sqrt{37}} \cdot \sqrt{g^2 + \frac{160000}{r^2}}$$

(iii) The driver wishes to travel around the track so that there is no frictional force acting on the car. If the radius f the banked track is 300m, find the speed at which the car should travel, given that $\theta = 5°$ and $g = 10$ m/s^2. **2**

End of Question 14

Question 15 (15 marks)

a) In the diagram below, P is a point on the circle which is drawn through the vertices of ΔABC. From P, perpendiculars are drawn, meeting AB at F and AC at G. FG produced meets BC produced at H.



Copy or trace the diagram into your answer booklet.

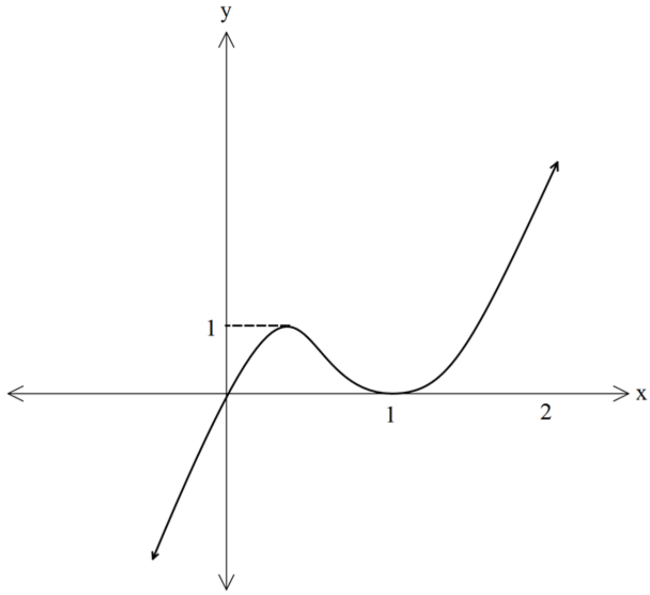
(i) Prove that $\angle PGH = \angle PAF$. **2**

(ii) Prove that $PH \perp BC$ **3**

b) Find $\int \frac{dx}{3-\cos x}$ **3**

c) The ellipse with equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (where $a>b$) has eccentricity 0.4 and it intersects the x-axis at the points A and A'. The distance from one of the foci to point A is 1. What are the possible values of a ? **2**

Question 15 continues on the next page

d)	The diagram shows the graph of a function $y = f(x)$. 	
	Carefully sketch the following curves on separate diagrams:	
	(i) $y = \log_e f(x)$	**2**
	(ii) $y = f(2-x)$	**1**
	(iii) $\lvert y \rvert = f(x)$	**2**
	End of Question 15	

Question 16 (15 marks)

a) A particle of mass *m* kg falls from rest in a resistive medium. Its velocity at time *t* seconds is *v* m/s and it approaches a terminal velocity of *R* m/s. The resistive force upon it is *mkv* newtons.

(i) Show that the acceleration of the particle is given by $\ddot{x}=k(R-v)$ **2**

(ii) Show that the time taken for the particle to reach 50% of its terminal velocity is $\frac{1}{k}\log_e 2$ seconds. **2**

(iii) Find the distance fallen during this this time. **3**

b) Use the identity:

$$(1+x)^{2n+1}(1-x)^{2n}=(1+x)(1-x^2)^{2n}$$

to find a simpler expression for:

$$\binom{2n+1}{0}\binom{2n}{0}-\binom{2n+1}{1}\binom{2n}{1}+\cdots+\binom{2n+1}{2n}\binom{2n}{2n}$$ **3**

c) (i) If $\tan x\tan(\theta-x)=k$, prove that **2**

$$\frac{1+k}{1-k}=\frac{\cos(2x-\theta)}{\cos\theta}$$

(ii) Hence or otherwise, find all solutions for **3**

$$\tan x\,\tan\left(\frac{\pi}{3}-x\right)=2+\sqrt{3}.$$

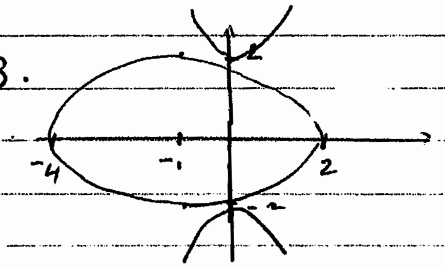
End of Examination

X2. Trial 2018
SOLUTIONS.

Section I.

1. $(a+bi)^2 = a^2 - b^2 + 2iab$
$= 0 + i$
$\therefore a^2 - b^2 = 0$ — all
$2ab = 1 \quad \therefore ab = \frac{1}{2}$ (C)

2. $P(x)$ has a double root
$P'(x)$ has a single root $\Big\}$ in common
$P'(x) = 3x^2 + 6x - 24$
$= 3(x^2 + 2x - 8)$
$= 3(x+8)(x-2)$
$x = -8$ or $x = 2$
Test: $-512 + 192 + 192 + 28 \neq 0$ Test: $8 + 12 - 48 + 28 = 0$
$\therefore x = 2$ only (D)

3.


0 pts. in common

(A)

4. $$\int \frac{dx}{\sqrt{7-6x-x^2}} = \int \frac{dx}{\sqrt{16-(x+3)^2}} \qquad \left[\begin{array}{l} 7-6x-x^2 \\ = 16-(x^2+6x+9) \end{array}\right]$$
$$= \sin^{-1}\left(\frac{x+3}{4}\right) + c \qquad \text{(B)}$$

5. Sum of roots $= -(1+i)$
Product of roots $= -1$ $\Big\}$ (D)
A: $-1 + \alpha + 1 - \alpha = 0 \neq -(1+i)$
B. $-1 + \alpha + \bar{\alpha} = -1 + 2\,\mathrm{Re}(\alpha) \neq -(1+i)$
C. $-1 + \alpha - \alpha = -1 \neq -(1+i)$
D. $-1 + \alpha + \frac{1}{\alpha}$ ~~may~~ $= -1 + i$ is possible, and product $= -1$.

6. (D) (even + odd = neither)

7. $4x + 4x.\frac{dy}{dx} + y.4 - 2y.\frac{dy}{dx} = 0$

$$\frac{dy}{dx}(4x - 2y) = -4(x+y)$$

$$\frac{dy}{dx} = \frac{-4(x+y)}{4x-2y} = \frac{-4(5)}{8-6} = -10 \quad \text{(D)}$$

8. ~~(E)~~ $\quad x = 3t \quad \dot{x} = 3 \quad \ddot{x} = 0 \leftarrow$ constant

$\quad y = e^t \quad \dot{y} = e^t \quad \ddot{y} = e^t \leftarrow$ varies, but >0.

8. Constant direction (direction of y) but varying magnitude (C)

9. (C)

10. List all Δs containing vertex P.

PXT

PWT PWS

PVT PVS PVR

PUT PUS PUR PUQ

$\therefore$ 10 such Δ's through P.

Now if we list all Δs with P, Q, R, S . . . X as vertices, we have $9 \times 10 = 90$ Δs.

But each Δ will be counted 3 times

(eg. PXT will also be counted as XTP and TXP)

$$\therefore \frac{9 \times 10}{3} = 30 \quad \text{(B)}$$

Part II

Question 11

a) (i) $\frac{x^2}{16} - \frac{y^2}{9} = 1$

$b^2 = a^2(e^2 - 1)$

$\frac{9}{16} = e^2 - 1$

$e^2 = \frac{25}{16} \quad \therefore e = \frac{5}{4}$ ← (1)

(ii) Foci $(\pm ae, 0) = (\pm 5, 0)$ ← (1)

b) (i) $\frac{z}{w} = \frac{1+i}{\sqrt{3}+i} \times \frac{\sqrt{3}-i}{\sqrt{3}-i}$

$= \frac{(\sqrt{3}+1) + i(\sqrt{3}-1)}{4}$ ← (1)

$= \frac{\sqrt{3}+1}{4} + i . \frac{\sqrt{3}-1}{4}$

(ii) $w = 2 \text{ cis } \pi/6$ ← (1)

(iii) $\frac{z}{w} = \frac{\sqrt{2} \text{ cis } \pi/4}{2 \text{ cis } \pi/6} = \frac{\sqrt{2}}{2} . \text{cis } \frac{\pi}{12}$

$= \frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{12} + i \sin\frac{\pi}{12}\right)$

Equating real parts : $\frac{\sqrt{2}}{2} . \cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{4}$

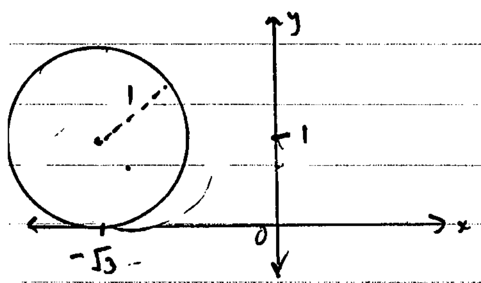
$\therefore \cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{4} \times \frac{2}{\sqrt{2}}$

$= \frac{\sqrt{3}+1}{2\sqrt{2}}$.

(2) correct soln
(1) • attempts to equate real parts.
or other valid working

c) (i) $|z + (\sqrt{3} - i)| = 1$

$|z - (-\sqrt{3} + i)| = 1$



A circle $\begin{cases} \text{centre } (-\sqrt{3}, 1) \\ \text{radius } 1 \end{cases}$

← (2) Draw correct circle

(1) Draw an circle with correct radius or centre

(ii) $\frac{2\pi}{3} \le \arg z \le \pi$.

(2) correct answer
(1) partially correct answer.

(iii) 3 units.

(1) correct answer

11 d) (i) $e = \sqrt{2}$ since rectangular hyperbola. (1) correct answer.

(ii) At A, $y = -\frac{c}{p}$ $\quad x + p^2\left(-\frac{c}{p}\right) = 2cp$

$$x - cp = 2cp$$

$$x = 3cp \qquad A\left(3cp, -\frac{c}{p}\right)$$

At B, $x = -cp$ $\quad -cp + p^2 y = 2cp$

$$p^2 y = 3cp$$

$$y = \frac{3c}{p} \qquad B\left(-cp, \frac{3c}{p}\right)$$

Now midpt of AB

$$= \left(\frac{3cp - cp}{2}, \frac{\frac{3c}{p} - \frac{c}{p}}{2}\right)$$

$$= \left(cp, \frac{c}{p}\right)$$

= coord's of P.

(2) correct soln
(1) find A or B.

(iii) At A, $x = 3cp$, $p = \frac{x}{3c}$

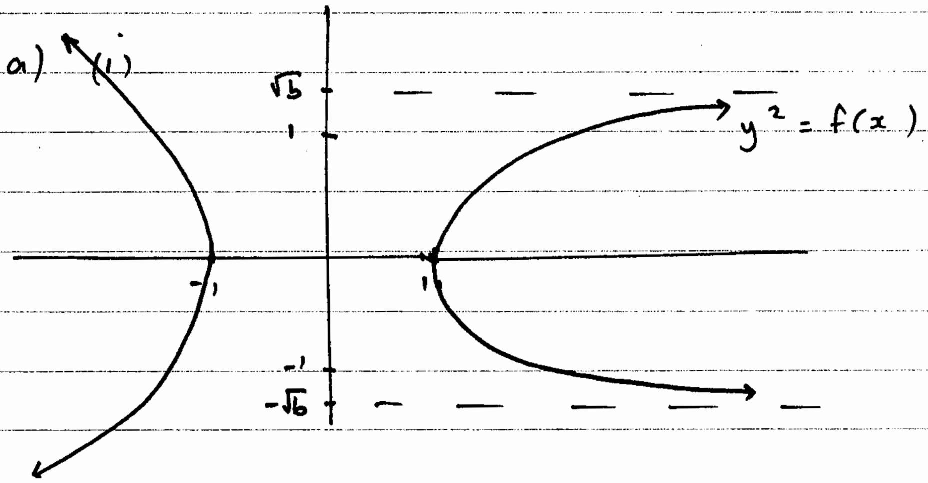
$$y = -\frac{c}{p}, \quad y = -c \times \frac{3c}{x} = \frac{-3c^2}{x}$$

$$\therefore \underline{xy = -3c^2}$$

(1) correct soln.

Question 12

a) (i)



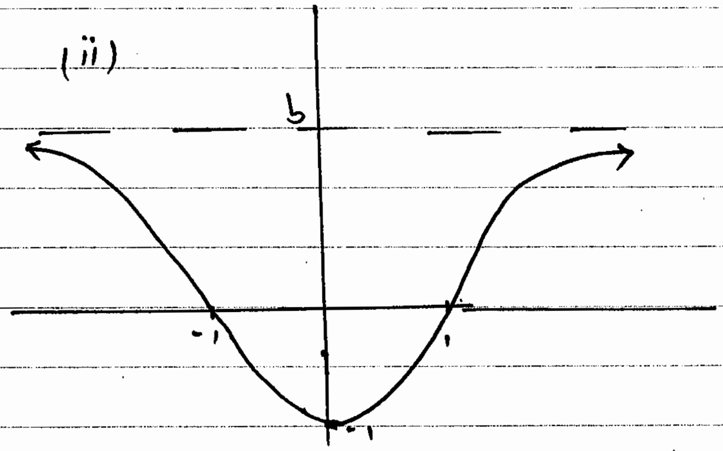
(3) correct graph

(2) draw $y = \sqrt{f(x)}$ and its reflection but missing one of the following:

- vertical tangents
- correct use of horiz. asymptotes

(1) draw $y = \sqrt{f(x)}$

(ii)



(1) correct graph.

b) $P(x) = x^3 - 3x - 2$; roots α, β, γ

(i) Replace x with $\sqrt{x}$:

$$x\sqrt{x} - 3\sqrt{x} - 2 = 0$$

$$\sqrt{x}(x-3) = 2$$

$$x(x-3)^2 = 4$$

$$x(x^2 - 6x + 9) = 4$$

$$x^3 - 6x^2 + 9x - 4 = 0$$

(1) Replace x with $\sqrt{x}$

(2) a correct polynomial equation (any equivalent)

(ii) $\alpha^2 + \beta^2 + \gamma^2 = -\frac{b}{a}$ (for new eqn) $= 6$

(1) correct answer

(iii)
$$\alpha^3 = 3\alpha + 2$$
$$\beta^3 = 3\beta + 2$$
$$\gamma^3 = 3\gamma + 2$$

$$\therefore \alpha^3 + \beta^3 + \gamma^3 = 3(\alpha + \beta + \gamma) + 6$$
$$= 3(0) + 6$$
$$= 6$$

(2) correct solution + working

(1) obtain $3(\alpha+\beta+\gamma)+6$ OR bald answer

c) (i) $\displaystyle\int \frac{e^{x}+e^{2x}}{1+e^{2x}} \cdot dx$

$= \displaystyle\int \frac{e^{x}(1+e^{x})}{1+(e^{x})^{2}} \cdot dx$ Let $u = e^{x}$

$du = e^{x} . dx$

$= \displaystyle\int \frac{1+u}{1+u^{2}} \cdot du$

$= \displaystyle\int \frac{1}{1+u^{2}} + \frac{u}{1+u^{2}} \cdot du$

(3) correct soln.

(2) partially correct result ~~integrand~~

(1) use substn method

$= \tan^{-1} u + \frac{1}{2} \ln |1+u^{2}| + c$

$= \tan^{-1}(e^{x}) + \frac{1}{2} \ln |1+e^{2x}| + c$

|...| not nec.

(ii) $\displaystyle\int_{0}^{\pi/4} x \sec^{2} x \cdot dx$

$= \Big[x \tan x\Big]_{0}^{\pi/4} - \displaystyle\int_{0}^{\pi/4} \tan x \cdot dx$

$= \frac{\pi}{4}(1) - 0 + \Big[\ln|\cos x|\Big]_{0}^{\pi/4}$

(3) correct evaluation

(2) obtain ~~correct~~ $x \tan x$ and $\ln|\cos x|$

$= \frac{\pi}{4} + \left(\ln \frac{1}{\sqrt{2}} - \ln 1\right)$

$= \frac{\pi}{4} + \ln \frac{1}{\sqrt{2}}$

(1) attempt IBP, ~~obtain $x \tan x$~~

Question 13.

a) (i) $|z|$, $|w|$ and $|z-w|$ are the three side lengths of ΔOPQ.

$\therefore$ $|z-w| \leq |z| + |w|$ because any side of a Δ is less than or equal to the sum of the other two sides (equality when O, P, Q collinear). (1) correct explanation

(ii) OPRQ is a parallelogram with equal diagonals since $|\vec{OP}| = |z+w|$

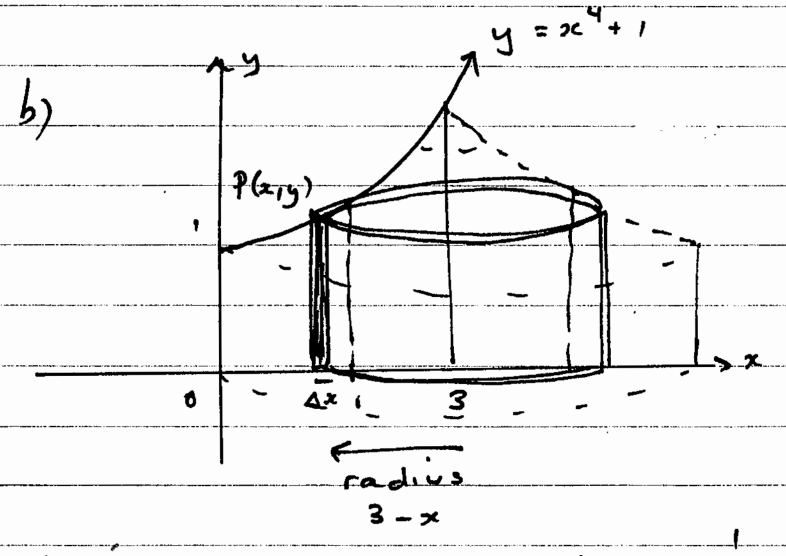
$\therefore$ OPRQ is a rectangle.

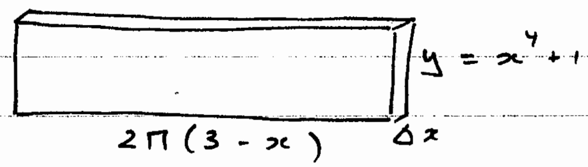
(2) Rectangle with reason

(1) Diagonals equal

(Parallelogram and diagonals equal)

b)





$$\Delta V = 2\pi(3-x)(x^4+1) \cdot \Delta x$$

$$\therefore \quad V = \lim_{\Delta x \to 0} \sum_{x=0}^{1} 2\pi(3-x)(x^4+1) \cdot \Delta x$$

(1) · Development of Σ formula
· Correct primitives · but no dev. of Σ formula.

$$= 2\pi \int_0^1 (3-x)(x^4+1) \cdot dx$$

$$= 2\pi \int_0^1 3x^4 + 3 - x^5 - x \cdot dx$$

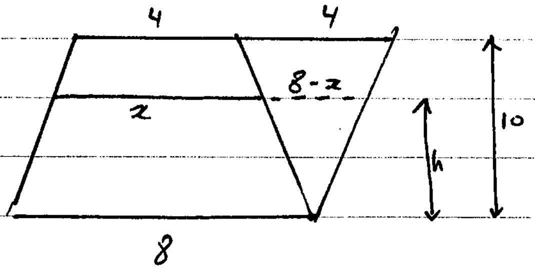
(3) exact volume

(2) · exact vol. but missing development of Σ formula.
· correct primitives

$$= 2\pi \left[\frac{3x^5}{5} + 3x - \frac{x^6}{6} - \frac{x^2}{2} \right]_0^1$$

$$= 2\pi \left(\frac{3}{5} + 3 - \frac{1}{6} - \frac{1}{2} \right) = \frac{88\pi}{15} \text{ units}^3$$

c) (i)



$$\frac{8-x}{4} = \frac{h}{10}$$

$$80 - 10x = 4h$$

$$\cancel{x = 800}$$

$$x = \frac{80-4h}{10}$$

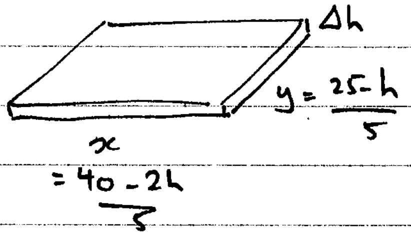
$$= \frac{40-2h}{5}$$

(2) correct expression

(1) correct proportion statement

$$\left(\text{or} \quad 8 - \tfrac{2}{5}h\right).$$

(ii)



$$\Delta V = \frac{40-2h}{5} \cdot \frac{25-h}{5} \cdot \Delta h$$

$$= \frac{1}{25}(40-2h)(25-h) \cdot \Delta h$$

$$V \doteq \lim_{\Delta h \to 0} \sum_{h=0}^{10} \frac{2}{25}(20-h)(25-h) \cdot \Delta h$$

$$V = \frac{2}{25}\int_0^{10} 500 - 45h + h^2 \cdot dh$$

$$= \frac{2}{25}\left[500h - \frac{45h^2}{2} + \frac{h^3}{3}\right]_0^{10}$$

$$= \frac{2}{25}\left(5000 - \frac{4500}{2} + \frac{1000}{3}\right)$$

$$= \frac{740}{3} \text{ cm}^3$$

$$\left(\text{or} \quad 246\tfrac{2}{3} \text{ cm}^3\right).$$

(3) correct solution

(2) correct der. + ~~integrand~~ primitive

(1) correct integrand development

d) $\lim_{x \to 2} \dfrac{\sqrt{6-x} - 2}{2-x}$

$= \lim_{x \to 2} \dfrac{\sqrt{6-x} - 2}{2-x} \cdot \dfrac{\sqrt{6-x} + 2}{\sqrt{6-x} + 2}$

$= \lim_{x \to 2} \dfrac{\cancel{6-x-4}\ \cancel{2-x}}{\cancel{(2-x)}(\sqrt{6-x} + 2)}$

$= \dfrac{1}{\sqrt{4} + 2}$

$= \dfrac{1}{4}$

(2) correct solution

(1) · partially correct
· rationalise numerator
· bald answer.

e) There are $7!$ ways without restriction

No. of ways where no boys are together is

$4! \times (\cancel{5} \times \cancel{2}4 \times 3)$

ways of seating the girls

ways of seating the boys between them.

$\therefore\ 7! - 4! \times 60 = \underline{3600}$

(2) correct solution

(1) ~~sign~~ some correct working / explanation.

Question 14.

a) If $n=1$, $S_1 = \frac{1}{1^2} = 1$

$2 - \frac{1}{1} = 1$ $\therefore$ True for $n=1$

Assume true for $n=k$ (where k is an integer).

ie. Assume
$$S_k = \sum_{r=1}^{k} \frac{1}{r^2} \leq 2 - \frac{1}{k}$$

Now prove also true for $n=k+1$

ie. Prove
$$S_{k+1} = \sum_{r=1}^{k+1} \frac{1}{r^2} \leq 2 - \frac{1}{k+1}$$

Now $S_{k+1} = S_k + \frac{1}{(k+1)^2}$

$\leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$ by assumption

$< 2 - \frac{1}{k} + \frac{1}{k(k+1)}$ (using the "given" ~~assumption~~ result)

$= 2 - \left(\frac{1}{k} - \frac{1}{k(k+1)}\right)$

$= 2 - \frac{k+1-1}{k(k+1)}$

$= 2 - \frac{1}{k+1}$ $\therefore$ If true for $n=k$, then statement is also true for $n=k+1$

Now statement is true for $n=1$

$\therefore$ Also true for $n=2,3,4\ldots$ and by induction, true for all positive integers n.

(3) correct proof.

(2) use assumption + ~~given~~ set up.

(1) Test ~~one~~ $n=1$ and set up.

b) $I_n = \int_{\pi/6}^{\pi/4} \cot^n x \, . dx$

(i) $I_1 = \int_{\pi/6}^{\pi/4} \frac{\cos x}{\sin x} \, . dx$

$= \left[\ln |\sin x| \right]_{\pi/6}^{\pi/4}$

$= \ln (\sin \frac{\pi}{4}) - \ln (\sin \pi/6)$

$= \ln (\frac{1}{\sqrt{2}}) - \ln (\frac{1}{2})$ $\leftarrow$ (1) or any equivalent

$= \ln (\frac{1}{\sqrt{2}} \times \frac{2}{1})$

$= \ln \sqrt{2}$

$= \frac{1}{2} \ln 2.$

(ii) $I_{n-2} + I_n$

$= \int_{\pi/6}^{\pi/4} \cot^{n-2} x + \cot^n x \, . dx$

$= \int_{\pi/6}^{\pi/4} \cot^{n-2} x \, (1 + \cot^2 x) . dx$

$= \int_{\pi/6}^{\pi/4} \cot^{n-2} x \, . \, \operatorname{cosec}^2 x \, . dx$

$= -\frac{1}{n-1} \left[\cot^{n-1} x \right]_{\pi/6}^{\pi/4}$

$= -\frac{1}{n-1} \left(\cot^{n-1} \frac{\pi}{4} - \cot^{n-1} \frac{\pi}{6} \right)$

$$= -\frac{1}{n-1}\left(1 - (\sqrt{3})^{n-1}\right)$$

$$= \frac{(\sqrt{3})^{n-1} - 1}{n-1}$$

$$= \frac{1}{n-1}\left[(\sqrt{3})^{n-1} - 1\right]$$

(3) correct solution.

(2) correct primitive: $\frac{-1}{n-1}[\cot^{n-1} x]$

(1) obtain integrand; $\cot^{n-2} x \ \operatorname{cosec}^2 x$

(iii) $$I_5 + I_3 = \frac{\sqrt{3}^4 - 1}{4} = 2$$

$$I_3 + I_1 = \frac{\sqrt{3}^2 - 1}{2} = 1$$

$$I_1 = \frac{1}{2}\ln 2.$$

$$\therefore \quad I_3 = 1 - \frac{1}{2}\ln 2$$

$$I_5 = 2 - \left(1 - \frac{1}{2}\ln 2\right)$$

$$= \underline{1 + \frac{1}{2}\ln 2}.$$

(2) correct answer

(1) ◦ obtain I_3.

◦ $I_5 = 2 - (\text{incorrect } I_3)$

c) (i) By squaring the given equations:

$$N^2\cos^2\theta + F^2\sin^2\theta - 2NF\sin\theta\cos\theta = m^2g^2$$

$$N^2\sin^2\theta + F^2\cos^2\theta + 2NF\sin\theta\cos\theta = \frac{m^2v^4}{r^2}$$

By addition:

$$N^2(1) + F^2(1) = m^2g^2 + \frac{m^2v^4}{r^2}$$

$$\therefore \quad \underline{N^2 + F^2 = m^2g^2 + \frac{m^2v^4}{r^2}}$$

(2) correct solution

(1) attempts to square and add eqns.

(ii) $F : N = 6 : 1$ $\quad v = 72\text{ km/h}$

$\therefore F = 6N$ $\quad = 20\text{ m/s}.$

$$(6N)^2 + N^2 = \frac{20^4 m^2}{r^2} + m^2 g^2$$

$$\therefore \underline{37 N^2} = m^2\left(\frac{20^4}{r^2} + g^2\right)$$

$$N^2 = \frac{m^2}{37}\left(\frac{160\,000}{r^2} + g^2\right)$$ ← (2) correct soln. (1) partially correct

$$N = \frac{m}{\sqrt{37}}\sqrt{\frac{160\,000}{r^2} + g^2}$$

(iii) Let $F=0$ in the 2 eqns obtained by resolving forces: (Also $g=10$, $\theta = 5^\circ$)

$$N \sin 5^\circ = \frac{mv^2}{300} \quad \ldots (1)$$

$$N \cos 5^\circ = 10m \quad \ldots (2)$$

$$\therefore \tan 5^\circ = \frac{v^2}{3000}$$

$$v^2 = 3000 \tan 5^\circ$$

$$\underline{v \doteq 16.2\text{ m/s}.} \quad (\text{or } 58.3\text{ km/h}).$$

(2) correct answer including correct units.

(1) · correct numerical answer but no units

· finding expressions with $\sin 5^\circ$, $\cos 5^\circ$

Question 15.

a) (i) $\angle PFA = \angle PGA = 90°$ (given)

$\therefore$ PGFA is a cyclic quadrilateral (equal $\angle$s subtended at points G, F on same side of interval PA)

$\angle PGH = \angle PAF$ (exterior $\angle$ of cyclic quadrilateral = interior opposite $\angle$)

(2) correct proof

(1) establish cyclic quad.

(ii) $\angle PCH = \angle PAB$ (exterior $\angle$ of cyclic quadrilateral = interior opposite $\angle$)

$\therefore \angle PCH = \angle PGH$ (both $= \angle PAF$)

$\therefore$ PHCG is a cyclic quadrilateral (equal $\angle$s subtended at points C, G on same side of interval PH)

$\therefore \angle PGC + \angle PHC = 180°$

(opposite $\angle$s of cyclic quadrilateral supplementary)

$\therefore \angle PHC = 180 - 90 = 90°$

ie. $\underline{PH \perp BC}$

(3) correct proof

(2) establish cyclic ~~cast~~ quadrilateral

(1) using a correct cyclic quad. to conclude.

b)
$$\int \frac{dx}{3 - \cos x} = \int \frac{\frac{2}{1+t^2}}{3 - \frac{1-t^2}{1+t^2}} \cdot dt$$

$$= \int \frac{2}{3 + 3t^2 - 1 + t^2} \cdot dt$$

$$= \int \frac{2}{2 + 4t^2} \cdot dt$$

(3) Correct solution

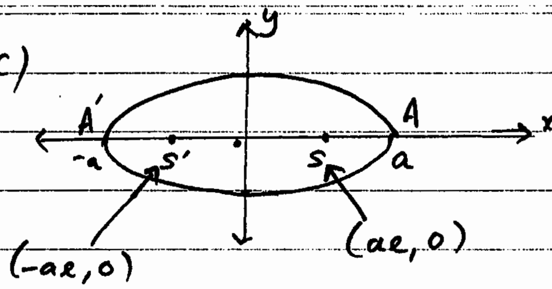
(2) correct integrand

(1) use t-results.

$$= \int \frac{1}{2t^2 + 1} \cdot dt$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\sqrt{2} \tan \frac{x}{2}\right) + c .$$

c)



$a - ae = 1$ or $a + ae = 1$

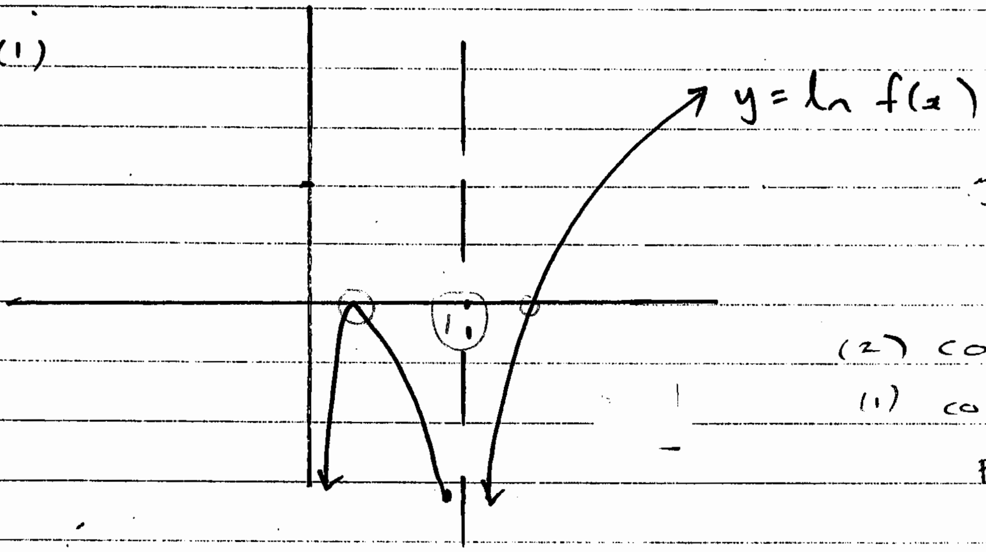
$a(1-e) = 1$ $\quad$ $a(1+e) = 1$

$a = \frac{1}{0.6} = \frac{5}{3}$ $\quad$ $a = \frac{1}{1.4} = \frac{5}{7}$

(2) both solutions.

(1) obtain one or both eqns ($a \pm ae = 1$

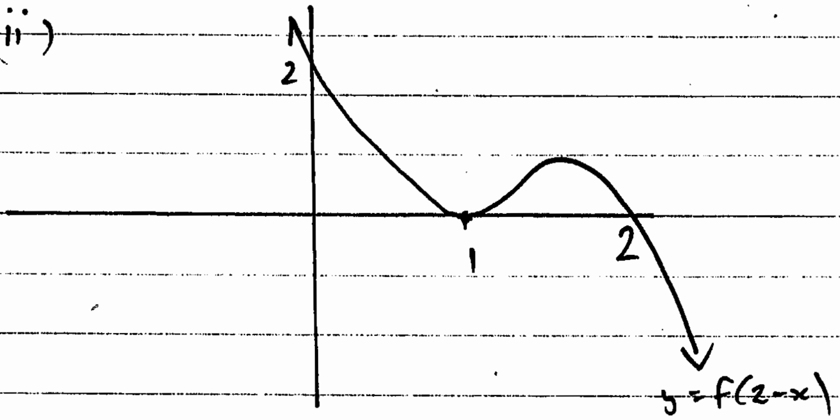
d) (i)



(2) correct graph

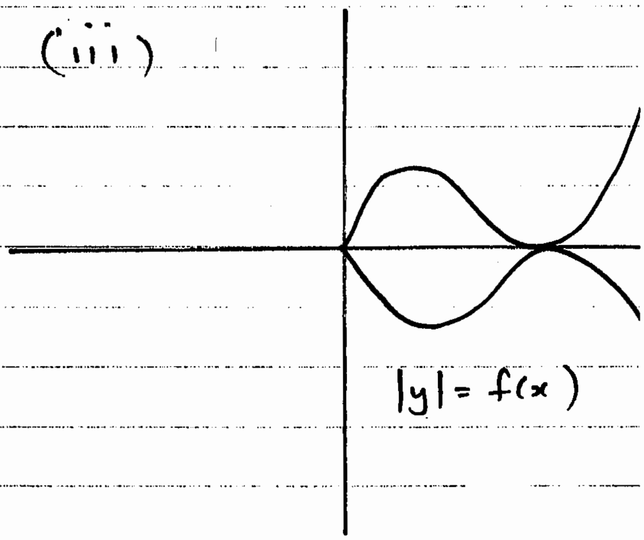
(1) correct asymptote and position of intercepts.

(ii)



(1) correct graph

(iii)

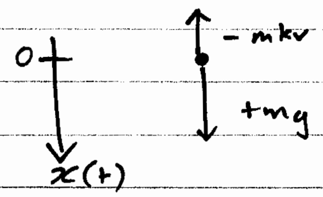


(2) correct graph

(1) symmetry in x axis

Question 16.

a)(i)



$$m\ddot{x} = mg - mkv$$
$$\ddot{x} = g - kv$$

When $\ddot{x} = 0$, $v = R$:

$$0 = g - kR$$
$$g = kR$$
$$\therefore \quad \ddot{x} = kR - kv$$
$$= k(R - v)$$

(2) correct soln

(1) establishes eqn. of motion.

(ii) $\ddot{x} = \dfrac{dv}{dt} = k(R-v)$

$$\int_0^{0.5R} \frac{dv}{R-v} = \int_0^T k \, . \, dt$$

$$\left[kt\right]_0^t = \left[-\ln(R-v)\right]_0^{0.5R}$$

$$kt = -\ln 0.5R + \ln R$$
$$= \ln \frac{R}{0.5R}$$
$$= \ln 2$$
$$\therefore \quad t = \frac{1}{k} \ln 2 .$$

(2) Correct soln

(1) correct primitive

(iii) $\ddot{x} = v . \dfrac{dv}{dx} = k(R-v)$

$$\int_0^{0.5R} \frac{v}{R-v} . dv = \int_0^x k . dx$$

$$kx = \int_0^{0.5R} -1 + \frac{R}{R-v} . dv$$

$$= \left[-v - R\ln(R-v)\right]_0^{0.5R}$$

$$x = \frac{1}{k}\left(-0.5R - R\ln 0.5R - (0 - R\ln R)\right)$$

$$= \frac{1}{k}\left(R\ln R - R\ln 0.5R - 0.5R\right)$$

$$= \frac{1}{k}\left(R\ln\frac{R}{0.5R} - 0.5R\right)$$

$$= \frac{1}{k}\left(R\ln 2 - 0.5R\right)$$

$$= \frac{R}{k}\left(\ln 2 - 0.5\right) \text{ metres.}$$

(3) correct soln
(2) correct primitive with c or limits.
(1) attempts to integrate using $v.\frac{dv}{dx}$ or $\frac{dv}{dt}$

b) From the given identity,

$$LHS = \left[\binom{2n+1}{0} + \binom{2n+1}{1}x^1 + \binom{2n+1}{2}x^2 + \dots + \binom{2n+1}{2n}x^{2n} + \binom{2n+1}{2n+1}x^{2n+1}\right]$$

$$\times$$

$$\left[\binom{2n}{0} - \binom{2n}{1}x^1 + \binom{2n}{2}x^2 - \dots - \binom{2n}{2n-1}x^{2n-1} + \binom{2n}{2n}x^{2n}\right]$$

(this last term will be positive since $2n$ is even)

The coefficient of x^{2n} will be

$$\binom{2n+1}{0}\binom{2n}{2n} + \binom{2n+1}{1}\cdot -\binom{2n}{2n-1} + \binom{2n+1}{2}\cdot\binom{2n}{2n-2} + \dots$$

$$\dots + \binom{2n+1}{2n-1}\cdot -\binom{2n}{1} + \binom{2n+1}{2n}\cdot\binom{2n}{0}$$

$$= \binom{2n+1}{0}\binom{2n}{2n} - \binom{2n+1}{1}\binom{2n}{2n-1} + \binom{2n+1}{2}\binom{2n}{2n-2} + \dots$$

$$\dots - \binom{2n+1}{2n-1}\binom{2n}{1} + \binom{2n+1}{2n}\binom{2n}{0}$$

$$= \binom{2n+1}{0}\binom{2n}{0} - \binom{2n+1}{1}\binom{2n}{1} + \binom{2n+1}{2}\binom{2n}{2} + \cdots$$

$$\cdots - \binom{2n+1}{2n-1}\binom{2n}{2n-1} + \binom{2n+1}{2n}\binom{2n}{2n}$$

$$\left[\begin{array}{ll} \text{since} & \binom{n}{r} = \binom{n}{n-r}, \\ \text{and} & \binom{2n}{r} = \binom{2n}{2n-r} \end{array}\right]$$

and this is the required expression.

Now

$$RHS = (1+x)(1-x^2)^{2n}$$

$$= (1+x)\left(\binom{2n}{0} - \binom{2n}{1}x^2 + \binom{2n}{2}x^4 + \cdots \binom{2n}{n}(-x^2)^n\right)$$

and the only term containing x^{2n} will be

$$1 \times \binom{2n}{n} \times (-x^2)^n$$

$$= \binom{2n}{n} \cdot (-1)^n \cdot x^{2n}$$

$\therefore$ Required expression $= \binom{2n}{n} \cdot (-1)^n$.

(3) correct soln.

(2) Find coefficient of x^{2n} (or x^{2n+1}) and either use symmetry or simplifies correctly.

(1) Either find coefficient of x^{2n} (or x^{2n+1}) or use symmetry $\left({}^nC_r = {}^nC_{n-r}\right)$

c) (i) $LHS = \dfrac{1+k}{1-k}$

$$= \frac{1+\tan x \tan(\theta - x)}{1-\tan x \tan(\theta - x)} \times \frac{\cos x \cos(\theta - x)}{\cos x \cos(\theta - x)}$$

$$= \frac{\cos x \cos(\theta - x) + \sin x \sin(\theta - x)}{\cos x \cos(\theta - x) - \sin x \sin(\theta - x)}$$

$$= \frac{\cos(x - (\theta - x))}{\cos(x + (\theta - x))}$$

$$= \frac{\cos(2x - \theta)}{\cos \theta}$$

$= RHS.$

(2) correct soln.
(1) significant progress
eg. expressing LHS in terms of sin + cos.

(ii) If $\tan x \tan(\theta - x) = k = 2+\sqrt{3}$ and $\theta = \frac{\pi}{3}$,

then $\dfrac{1+k}{1-k} = \dfrac{\cos(2x-\theta)}{\cos\theta}$

is equivalent to

$$\frac{3+\sqrt{3}}{-1-\sqrt{3}} = \frac{\cos(2x - \frac{\pi}{3})}{\cos \pi/3}$$

$$\frac{3+\sqrt{3}}{-(1+\sqrt{3})} = 2\cos(2x - \frac{\pi}{3})$$

(3) correct soln.
(2) significant progress
(1) Sub. k and $\frac{\pi}{3}$ into eqn from part (i).

Now $\dfrac{3+\sqrt{3}}{1+\sqrt{3}} = \sqrt{3}$ (rationalising denominator)

$$\therefore \cancel{2}\cos(2x - \frac{\pi}{3}) = \frac{-\sqrt{3}}{2}$$

$$2x - \frac{\pi}{3} = 2n\pi \pm \frac{5\pi}{6}$$

(where n is an integer)

$2x = 2n\pi + \frac{5\pi}{6} + \pi/3$ or $2x = 2n\pi - \frac{5\pi}{6} + \frac{\pi}{3}$

$= 2n\pi + \frac{7\pi}{6}$ $\quad$ $= 2n\pi - \pi/2$

$\therefore$ $x = n\pi + \frac{7\pi}{12}$ or $x = n\pi - \frac{\pi}{4}$.