BAULKHAM HILLS HIGH SCHOOL

2020

YEAR 12 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations
- Marks may be deducted for careless or badly arranged work

Total marks: 100

Section I – 10 marks (pages 2 – 6)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 7 – 15)

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10

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1 Let $z = a + ib$, where a and b are real non-zero numbers.

If $z + \frac{1}{z} \in \mathbb{R}$, which of the following must be **true**?

(A) $\arg z = \frac{\pi}{4}$

(B) $a = -b$

(C) $a = b$

(D) $|z| = 1$

2 Which of the following statements is **true** when an object performs simple harmonic motion about a central point O?

(A) the acceleration is always directed away from O

(B) the acceleration and velocity are always in opposite directions

(C) the acceleration and displacement from O are always in the same direction

(D) the graph of acceleration against displacement is a straight line

3 Given the vectors $\underset{\sim}{u} = 3\underset{\sim}{i} - 4\underset{\sim}{j} + 12\underset{\sim}{k}$ and $\underset{\sim}{v} = 2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$, the vector projection of $\underset{\sim}{u}$ on $\underset{\sim}{v}$ is;

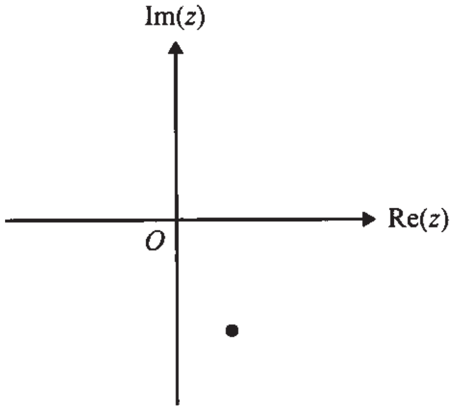
(A) $-\frac{14}{3}\left(2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right)$

(B) $-\frac{14}{9}\left(2\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}\right)$

(C) $-\frac{14}{13}\left(3\underset{\sim}{i} - 4\underset{\sim}{j} + 12\underset{\sim}{k}\right)$

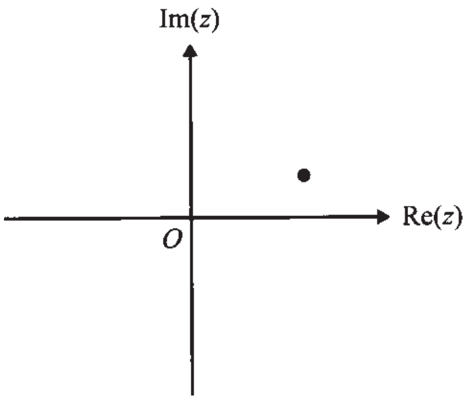
(D) $-\frac{14}{169}\left(3\underset{\sim}{i} - 4\underset{\sim}{j} + 12\underset{\sim}{k}\right)$

4 The complex number z, where $|z| = 1$ is represented in the following diagram

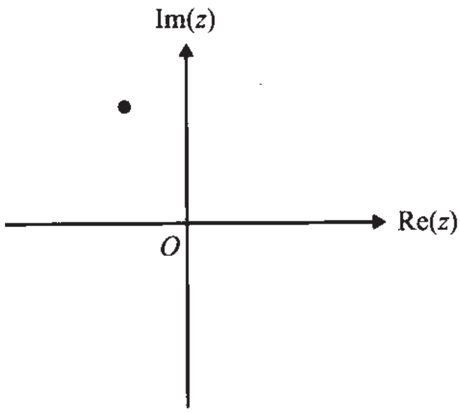


Which of the following diagrams would best represent the complex number $-\bar{z}$?

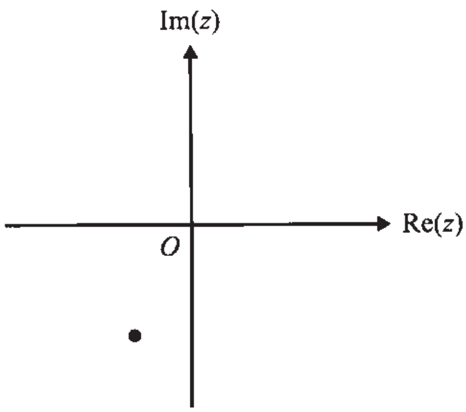
(A)



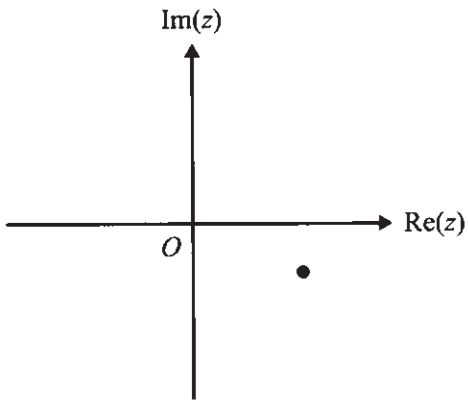
(B)



(C)



(D)



5 A constant force of magnitude F Newtons accelerates a particle of mass 2 kg in a straight line from rest to 12 m/s over a distance of 16 metres.

It follows that;

(A) $F = 4.5$

(B) $F = 9$

(C) $F = 12$

(D) $F = 18$

6 Consider the statement;

"Every day next week, Max will do at least one maths problem"

If this statement is **not** true, which of the following statements is certainly true?

(A) Some day next week, Max will do no maths problems

(B) Some day next week, Max will do more than one maths problem

(C) On no day next week will Max do more than one maths problem

(D) Every day next week, Max will do no maths problems

7 The algebraic fraction $\dfrac{x}{3(x+c)^2}$, where c is a non-zero real number, can be written in partial fraction form, where A and B are real numbers, as;

(A) $\dfrac{A}{3x+c} + \dfrac{B}{(x+c)^2}$

(B) $\dfrac{A}{3x+c} + \dfrac{B}{x+c}$

(C) $\dfrac{A}{x+c} + \dfrac{B}{(x+c)^2}$

(D) $\dfrac{A}{3(x+c)} + \dfrac{B}{x+c}$

8 What is the value of $i^{1!} + i^{2!} + i^{3!} + \ldots + i^{2020!}$?

(A) 2016

(B) $2015 + i$

(C) $2014 + 2i$

(D) $2018 + 2i$

9 P, Q and R are three collinear points with position vectors $\underset{\sim}{p}$, $\underset{\sim}{q}$ and $\underset{\sim}{r}$ respectively, where Q lies between P and R.

If $|\overrightarrow{QR}| = \frac{1}{2}|\overrightarrow{PQ}|$, then $\underset{\sim}{r} =$

(A) $\frac{3}{2}\underset{\sim}{q} - \frac{1}{2}\underset{\sim}{p}$

(B) $\frac{3}{2}\underset{\sim}{p} - \frac{1}{2}\underset{\sim}{q}$

(C) $\frac{1}{2}\underset{\sim}{q} - \frac{3}{2}\underset{\sim}{p}$

(D) $\frac{1}{2}\underset{\sim}{p} - \frac{3}{2}\underset{\sim}{q}$

10 When the substitution $x = 5\sin\theta$ is used, the integral equivalent to

$$\int \frac{x^2}{(25 - x^2)\sqrt{25 - x^2}}\,dx \text{ is;}$$

(A) $\displaystyle\int \frac{\sin^2\theta}{\cos^3\theta}\,d\theta$

(B) $\displaystyle\int 5\tan^2\theta\,d\theta$

(C) $\displaystyle\int \sqrt{5}\tan^2\theta\,d\theta$

(D) $\displaystyle\int \tan^2\theta\,d\theta$

END OF SECTION I

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section

Answer each question on the appropriate answer sheet. Each answer sheet must show your NESA#. Extra paper is available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the pages labelled Question 11 in the answer booklet

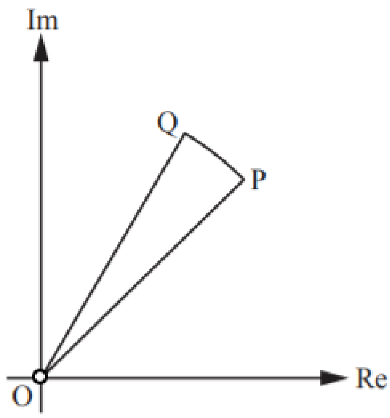
(a) Let $z = 5 + 2i$ and $w = \overline{z}$. Find ;

(i) w *1*

(ii) $w - z$ *1*

(iii) $\dfrac{w}{z}$ *2*

(b) The sketch below shows the region $\{z : |z| \leq 4\sqrt{2}\} \wedge \left\{ z : \dfrac{\pi}{4} \leq \arg z \leq \dfrac{\pi}{3} \right\}$



(i) Find in mod-arg form, the complex number represented by the point P. *1*

(ii) Find, in the form $x + iy$, the complex number represented by Q. *1*

(iii) Determine whether $3 + 5i$ lies within this region. *2*

Question 11 continues on page 8

Question 11 (continued)

(c)(i) Find A, B and C such that; **2**

$$\frac{A}{(x-3)^2}+\frac{B}{x-3}+\frac{C}{x+2}\equiv\frac{16x-43}{(x-3)^2(x+2)}$$

(ii) Find $\displaystyle\int\frac{16x-43}{(x-3)^2(x+2)}\,dx$ **2**

(d) The equations of two lines are;

$$\underset{\sim}{r}=\begin{pmatrix}2\\-3\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\4\\1\end{pmatrix}\text{ and }\underset{\sim}{r}=\begin{pmatrix}-4\\6\\-2\end{pmatrix}+\mu\begin{pmatrix}4\\-11\\3\end{pmatrix}$$

(i) Explain why these lines are not parallel. **1**

(ii) Determine whether these lines are skew or whether they intersect. **2**

End of Question 11

Question 12 (14 marks) Use the pages labelled Question 12 in the answer booklet

(a) Consider this statement about two positive integers a and b;

" *If $a^2 + b^2$ is even then either both a and b are even or both a and b are odd* "

(i) Write down the contrapositive of the statement. *1*

(ii) Prove the original statement by contraposition. *2*

(b) Two complex numbers z and w, satisfy the inequalities;

$$|z - 3 - 2i| \leq 2 \text{ and } |w - 7 - 5i| \leq 1$$

(i) Sketch both regions on the same Argand diagram. *2*

(ii) Find the least possible value of $|z - w|$ *1*

(c) Find $\int x^2 \ln x \, dx$ *3*

(d) A particle is travelling in simple harmonic motion such that

$$\ddot{x} = -4x$$

where x represents the displacement from the origin in metres.

(i) Show that $x = \alpha\cos(2t + \beta)$ is a possible equation for the displacement of the particle, where α and β are constants and t represents time in seconds. *2*

(ii) The particle is initially observed to have a velocity of 2 m/s at a displacement of 4 metres from the origin. Find the amplitude of the motion. *2*

(iii) Determine the maximum speed of the particle. *1*

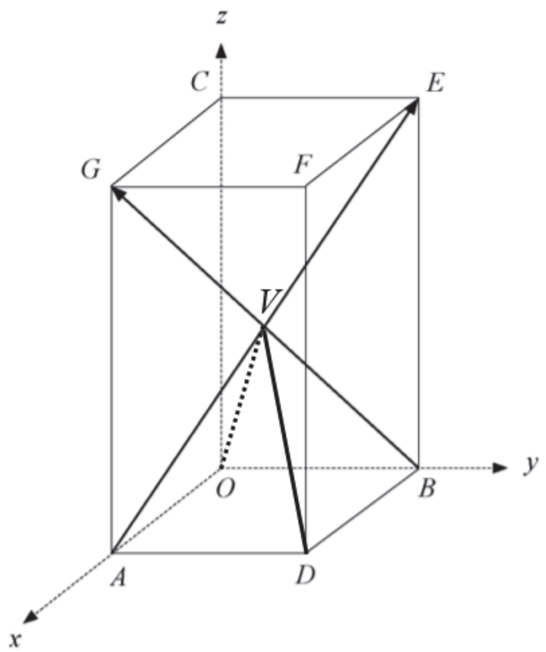
Question 13 (15 marks) Use the pages labelled Question 13 in the answer booklet

(a) Solve $e^{2i\theta} - e^{-2i\theta} = i\sqrt{3}$ where $-\pi < \theta \le \pi$ 3

(b) Evaluate $\displaystyle\int_0^{\frac{2\pi}{3}} \frac{dx}{5 + 4\cos x}$ 4

(c) Use the substitution $u = x\sin x + \cos x$ to find $\displaystyle\int \frac{x\,dx}{x\tan x + 1}$ 3

(d) A rectangular prism with square base *OADB* is shown below.



O is the origin and points *A, B* and *C* have position vectors of

$\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 8 \end{pmatrix}$ respectively.

(i) Determine the vector equation of the line containing *A* and *E*. 2

(ii) The vectors $\overrightarrow{AE}$ and $\overrightarrow{BG}$ intersect at *V*. *V* is joined to *O, A, D* and *B* to create a square pyramid. 3

Using vectors, calculate the angle that one of the triangular faces of the pyramid makes with the base, to the nearest degree.

Question 14 (16 marks) Use the pages labelled Question 14 in the answer booklet

(a) (i) Explain why $z^4 = (z-1)^4$ does not have four roots. *1*

(ii) Solve $z^4 = (z-1)^4$ *4*

(b) A particle of mass m kg is projected vertically with an initial velocity of u m/s.

It experiences a force due to air resistance that is equal to $\dfrac{mv^2 g}{u^2}$ Newtons,

where v is the velocity of the particle and g is the acceleration due to gravity.

(i) Find the time taken for the particle to reach its maximum height in terms of u and g. *3*

(ii) Show that the maximum height reached by the particle is given by *3*

$$H = \frac{u^2 \ln 2}{2g}$$

(c) Let $I_n = \displaystyle\int_0^1 \frac{x^n\,dx}{\sqrt{1+x^2}}$, where $n \in \mathbb{Z} \wedge n \geq 0$

(i) Find the value of I_1 *1*

(ii) Show that $I_n = \dfrac{\sqrt{2} - (n-1)I_{n-2}}{n}$ $\forall\ n > 1$ *3*

(iii) By setting up a suitable inequality, prove that $\displaystyle\lim_{n \to \infty} I_n = 0$ *1*

Question 15 (15 marks) Use the pages labelled Question 15 in the answer booklet

(a) Under what condition will the arithmetic mean of any set of numbers equal the geometric mean? *1*

(b) Heron's formula states that the area of a triangle is given by *3*

$$A = \sqrt{s(s-x)(s-y)(s-z)}$$

where x, y and z are the lengths of the sides of the triangle and s is the semi-perimeter.

Show that the triangle of maximum area with constant perimeter is an equilateral triangle.

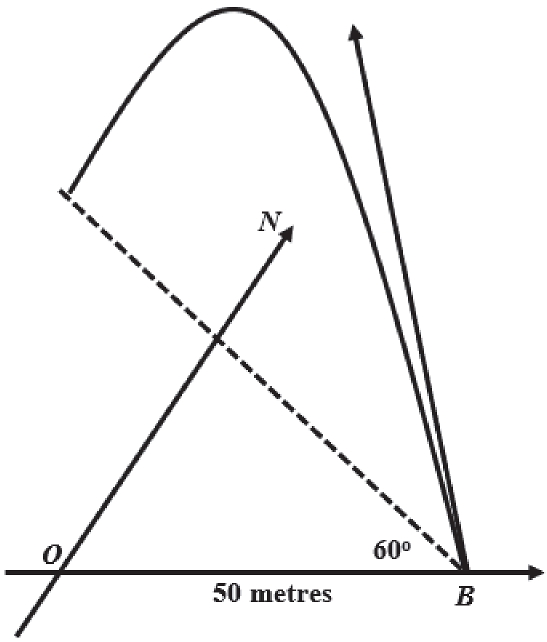
(c)(i) Show that the line $\underset{\sim}{r} = \underset{\sim}{b} + \lambda \underset{\sim}{m}$, where $\underset{\sim}{m}$ is a unit vector, intersects the sphere with centre O and radius a units at two points if *2*

$$a^2 > \underset{\sim}{b} \cdot \underset{\sim}{b} - (\underset{\sim}{b} \cdot \underset{\sim}{m})^2$$

(ii) Prove that a tangent to the sphere will always be perpendicular to the sphere's radius at the point of contact. *2*

Question 15 continues on page 13

(d) The points O and B lie in a horizontal field. The point B lies 50 metres east of O. A particle is projected from B at speed 25 m/s at an angle $\alpha = \tan^{-1}\frac{1}{2}$ above the horizontal and on a bearing of $330°$ from B; it passes to the north of O.



In this question you may assume the trajectory of a projectile fired with speed V m/s at an angle of θ to the horizontal is represented by the parametric equations

$$x = Vt\cos\theta \text{ and } y = Vt\sin\theta - 5t^2$$

(Do NOT prove these results)

(i) Taking unit vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ in the directions east, north and vertically upwards, show that the position vector, $\overrightarrow{OP}$, of the particle is given by **3**

$$(50 - 5\sqrt{5}t)\underset{\sim}{i} + 5\sqrt{15}t\,\underset{\sim}{j} + (5\sqrt{5}t - 5t^2)\underset{\sim}{k}$$

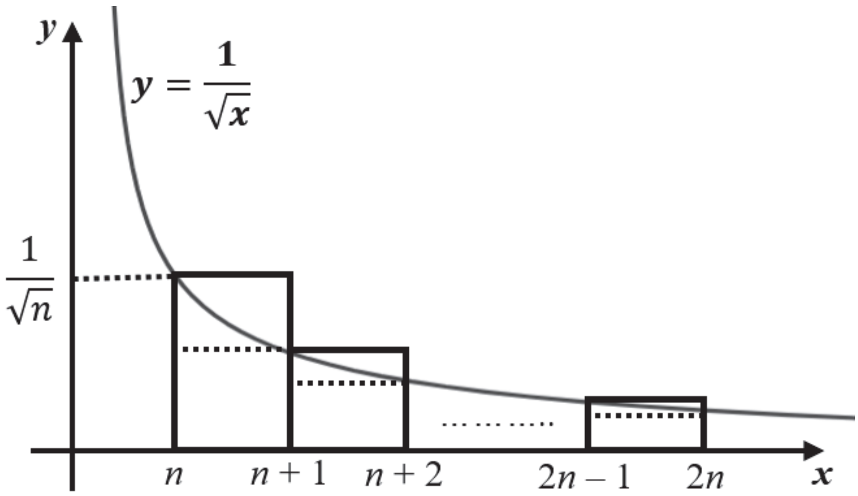
(ii) Show that $|\overrightarrow{OP}| = 5(t^2 - \sqrt{5}t + 10)$ **2**

(iii) Show that the particle reaches its maximum height when it is closest to the point O. **2**

End of Question 15

Question 16 (15 marks) Use the pages labelled Question 16 in the answer booklet

(a) In the diagram below, the graph of $y = \dfrac{1}{\sqrt{x}}$ has been drawn, and n rectangles have been constructed between $x = n$ and $x = 2n$, each of width 1 unit



(i) Show that $\displaystyle\int_n^{2n} \frac{dx}{\sqrt{x}} = 2\sqrt{n}(\sqrt{2} - 1)$ *1*

(ii) Let $S_n = \dfrac{1}{\sqrt{n+1}} + \dfrac{1}{\sqrt{n+2}} + \ldots + \dfrac{1}{\sqrt{2n}}$, show that; *4*

$$2\sqrt{n}(\sqrt{2} - 1) + \frac{1-\sqrt{2}}{\sqrt{2n}} < S_n < 2\sqrt{n}(\sqrt{2} - 1)$$

(iii) Hence find, correct to three decimal places *2*

$$\frac{1}{\sqrt{1000000}} + \frac{1}{\sqrt{1000001}} + \frac{1}{\sqrt{1000002}} + \ldots + \frac{1}{\sqrt{2000000}}$$

Question 16 continues on page 15

Question 16 (continued)

(b) The real numbers $a_1\ , a_2\ , a_3\ , \ldots$ are all positive. For each positive n, A_n and G_n are defined by

$$A_n = \frac{a_1 + a_2 + \ldots + a_n}{n} \quad \text{and} \quad G_n = (a_1\ a_2 \ldots a_n)^{\frac{1}{n}}$$

(i) Show that, for any given positive integer k, *3*

if $(\lambda_k)^{k+1} - (k+1)\lambda_k + k \geq 0$, where $\lambda_k = \left(\dfrac{a_{k+1}}{G_k}\right)^{\frac{1}{k+1}}$

then $(k+1)(A_{k+1} - G_{k+1}) \geq k(A_k - G_k)$

(ii) Let $f(x) = x^{k+1} - (k+1)x + k$, where $x > 0$ and $k \in \mathbb{Z}^+$ *2*

Show that $f(x) \geq 0$

(iii) Hence prove by mathematical induction *3*

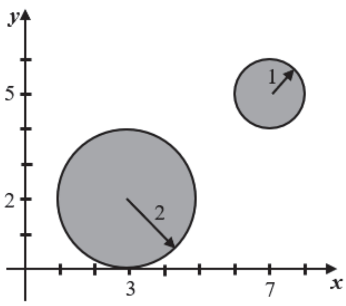
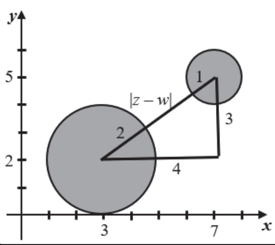
$$A_n \geq G_n \ \ \forall\ n : n \in \mathbb{Z}^+$$

End of paper

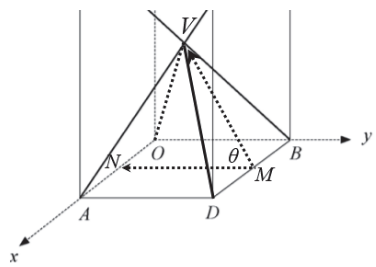
BAULKHAM HILLS HIGH SCHOOL
YEAR 12 EXTENSION 2 TRIAL HSC SOLUTIONS

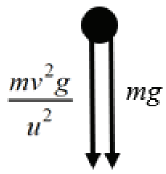
Solution	Marks	Comments
SECTION I		
1. D - $\frac{1}{z} = \frac{\overline{z}}{\lvert z\rvert^2}$ so in order for $z + \frac{1}{z} \in \mathbb{R}$, $\lvert z\rvert$ must equal 1	**1**	
2. D – $\ddot{x} = -n^2 x$, which represents a linear function in terms of x, thus the graph of acceleration vs displacement must be a straight line	**1**	
3. B – $\text{proj}_{\underset{\sim}{v}}\underset{\sim}{u} = \frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{\underset{\sim}{v}\cdot\underset{\sim}{v}}\underset{\sim}{v}$ $= \frac{6-8-12}{2^2+2^2+1^2}\left(2\underset{\sim}{i}+2\underset{\sim}{j}-\underset{\sim}{k}\right)$ $= -\frac{14}{9}\left(2\underset{\sim}{i}+2\underset{\sim}{j}-\underset{\sim}{k}\right)$	**1**	
4. C – $\overline{z}$ reflects in the real axis, so z would move to quadrant 1 $-\overline{z}$ reflects through the origin, (or rotates $180°$), so $\overline{z}$ would move to quadrant 3	**1**	
5. B - $m\ddot{x} = F$ $v\frac{dv}{dx} = \frac{F}{m}$ $\int_0^{12} v\,dv = \frac{F}{2}\int_0^{16} dx$ $\left[v^2\right]_0^{12} = F\left[x\right]_0^{16}$ $144 = 16F$ $F = 9$	**1**	
6. A – If the statement is false, then there must exist a day where it is not the case that Max does at least one problem. The complementary event to doing at least one problem, is to do no problems at all. So it must certainly be true that there exists a day next week the Max does the complementary event	**1**	
7. C – $\frac{x}{3(x+c)^2} \equiv \frac{\frac{x}{3}}{(x+c)^2}$ $= \frac{A}{x+c} + \frac{B}{(x+c)^2}$	**1**	
8. B – $i^{1!} + i^{2!} + i^{3!} + \ldots + i^{2020!}$ $= i + i^2 + i^6 + i^{24} + i^{120} + i^{720} + \ldots + i^{2020!}$ Every term after the third term has an index that is a multiple of four and $\therefore = 1$ $= i - 1 - 1 + 2017 \times 1$ $= 2015 + i$	**1**	
9. A – $\lvert\overrightarrow{QR}\rvert = \frac{1}{2}\lvert\overrightarrow{PQ}\rvert$ $\underset{\sim}{r} - \underset{\sim}{q} = \frac{1}{2}\left(\underset{\sim}{q} - \underset{\sim}{p}\right)$ $\underset{\sim}{r} = \frac{3}{2}\underset{\sim}{q} - \frac{1}{2}\underset{\sim}{p}$	**1**	

Solution	Marks	Comments
10. D – $\int \frac{x^2}{(25-x^2)\sqrt{25-x^2}}dx = \int \frac{25\sin^2\theta\, 5\cos\theta d\theta}{(25-25\sin^2\theta)\sqrt{25-25\sin^2\theta}}$ $x = 5\sin\theta$, $dx = 5\cos\theta d\theta$ $= \int \frac{125\sin^2\theta\cos\theta d\theta}{25\cos^2\theta\, 5\cos\theta}$ $= \int \frac{\sin^2\theta d\theta}{\cos^2\theta}$ $= \int \tan^2\theta d\theta$	1	
SECTION II		
QUESTION 11		
11(a) (i) $w = \overline{5+2i}$ $= 5-2i$	1	**1 mark** • Correct answer
11 (a) (ii) $w - z = 5-2i-(5+2i)$ $= -4i$ ***OR*** $w - z = \overline{z} - z$ $= 2Re(\overline{z})$ $= -4i$	1	**1 mark** • Correct answer
11 (a) (iii) $\frac{w}{z} = \frac{5-2i}{5+2i} \times \frac{5-2i}{5-2i}$ $= \frac{25-20i-4}{25+4}$ $= \frac{21}{29} - \frac{20}{29}i$ ***OR*** $\frac{w}{z} = \frac{\overline{z}}{z}$ $= \frac{\overline{z}^2}{\lvert z\rvert^2}$ $= \frac{(5-2i)^2}{29}$ $= \frac{21}{29} - \frac{20}{29}i$	2	**2 marks** • Correct solution **1 mark** • Multiplies numerator and denominator by $\overline{z}$
11 (b) (i) $4\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$	1	**1 mark** • Correct answer
11 (b) (ii) $4\sqrt{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ $= 4\sqrt{2}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$ $= 2\sqrt{2} + 2\sqrt{6}i$	1	**1 mark** • Correct answer *Note: accept factorised form*
11 (b) (iii) $\lvert 3+5i\rvert = \sqrt{3^2+5^2}$ $= \sqrt{34} > 4\sqrt{2}\ (= \sqrt{32})$ $\therefore$ $3+5i$ is outside the region	2	**2 marks** • Correct solution **1 mark** • Finds $\lvert 3+5i\rvert$
11 (c) (i) $16x - 43 = A(x+2) + B(x+2)(x-3) + C(x-3)^2$ $x = 3;$ $5 = 5A$, $A = 1$ $x = -2;$ $-75 = 25C$, $C = -3$ coefficient of x^2; $0 = B + C$, $B = -C$, $B = 3$	2	**2 marks** • Correct solution **1 mark** • Evaluates two of *A*, *B* or *C*
11 (c) (ii) $\int \frac{16x-43}{(x-3)^2(x+2)}dx = \int \left[\frac{1}{(x-3)^2} + \frac{3}{x-3} - \frac{3}{x+2}\right]dx$ $= -\frac{1}{x-3} + 3\ln\lvert x-3\rvert - 3\ln\lvert x+2\rvert + c$ $= -\frac{1}{x-3} + 3\ln\left\lvert\frac{x-3}{x+2}\right\rvert + c$	2	**2 marks** • Correct solution **1 mark** • Finds one correct primitive

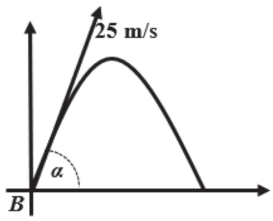
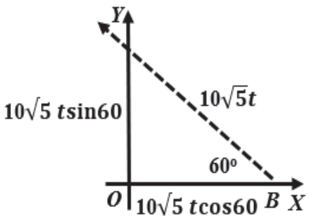
Solution	Marks	Comments
11 (d) (i) the direction vectors are not multiples of each other i.e. $\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} \neq k \begin{pmatrix} 4 \\ -11 \\ 3 \end{pmatrix}$ where $k \in \mathbb{Z}$	1	**1 mark** • Correct explanation
11 (d) (ii) $2 + \lambda = -4 + 4\mu \Rightarrow \lambda - 4\mu = -6$ $1 + \lambda = -2 + 3\mu \Rightarrow \underline{\lambda - 3\mu = -3}$ $\mu = 3 \quad \therefore \lambda = 6$ $-3 + 4\lambda = 6 - 11\mu$ if $\mu = 3$; $-3 + 4\lambda = 6 - 33$ $4\lambda = -24$ $\lambda = -6 \neq 6 \quad \therefore$ lines are skew	2	**2 marks** • Correct solution **1 mark** • Obtains either $\mu = 3 \quad \lambda = 6$ or $\mu = \frac{21}{23} \quad \lambda = -\frac{6}{23}$ or $\mu = \frac{11}{9} \quad \lambda = -\frac{10}{9}$ • Uses two equations to find a set of values for μ and λ into the third equation
QUESTION 12		
12 (a) (i) If statement $X \Rightarrow Y \vee Z$ then contrapositive statement is $\neg Y \wedge \neg Z \Rightarrow \neg X$ If a and b are two positive integers and one of a or b is even and the other is odd then $a^2 + b^2$ is odd.	1	**1 mark** • Correct statement
12 (a) (ii) Let $a = 2m$ and $b = 2n + 1$ where $m, n \in \mathbb{N}$ $a^2 + b^2 = (2m)^2 + (2n+1)^2$ $= 4m^2 + 4n^2 + 4n + 1$ $= 2(2m^2 + 2n^2 + 2n) + 1$ $= 2K + 1$ where $K = 2m^2 + 2n^2 + 2n \in \mathbb{N}$ $\therefore$ if a is even and b is odd then $a^2 + b^2$ is odd thus if a and b are two positive integers and $a^2 + b^2$ is even then by contraposition, one of a and b is even and the other is odd	2	**2 marks** • Correct proof **1 mark** • Defines a and b • Demonstrates the technique of proof by contraposition
12(b) (i) 	2	**2 marks** • Correct solution **1 mark** • Sketches one correct region • Both regions correctly located
12(b) (ii)  $(2 + \|z - w\| + 1)^2 = 3^2 + 4^2$ $= 25$ $\|z - w\| + 3 = 5$ $\|z - w\| = 2$ $\therefore$ minimum $\|z - w\| = 2$ units	1	**2 marks** • Correct solution **1 mark** • Sketches one correct region • Both regions correctly located

Solution	Marks	Comments
12(c) $\int x^2 \ln x\, dx$ $= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 dx$ $= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + c$ $u = \ln x \qquad v = \frac{1}{3}x^3$ $du = \frac{dx}{x} \qquad dv = x^2 dx$	3	**3 marks** • Correct solution **2 marks** • Correctly uses parts or equivalent merit **1 mark** • Attempts to uses parts or equivalent merit
12 (d) (i) $x = \alpha\cos(2t + \beta)$ $\dot{x} = -2\alpha\sin(2t + \beta)$ $\ddot{x} = -4\alpha\cos(2t + \beta)$ $= -4x$	2	**2 marks** • Correct solution **1 mark** • Differentiates twice to obtain $\ddot{x}$ in terms of x
12 (d) (ii) when $t = 0$; $x = 4 \Rightarrow 4 = \alpha\cos\beta$ $\dot{x} = 2 \Rightarrow 2 = -2\alpha\sin\beta$ $-1 = \alpha\sin\beta$ $\alpha^2\sin^2\beta + \alpha^2\cos^2\beta = (-1)^2 + 4^2$ $\alpha^2 = 17$ $\alpha = \sqrt{17}$ $\therefore$ amplitude of the motion is $\sqrt{17}$ metres	2	**2 marks** • Correct solution **1 mark** • Uses the initial conditions to create simultaneous equations for α and β
12 (d) (iii) $-1 \le \sin(2t + \beta) \le 1$ $-2\sqrt{17} \le 2\sqrt{17}\sin(2t + \beta) \le 2\sqrt{17}$ $\therefore$ maximum speed is $2\sqrt{17}$ m/s	1	**1 mark** • Correct answer
QUESTION 13		
13 (a) $e^{2i\theta} - e^{-2i\theta} = i\sqrt{3} \quad , \ -\pi < \theta \le \pi$ $\cos 2\theta + i\sin 2\theta - \cos(-2\theta) - i\sin(-2\theta) = i\sqrt{3}$ $\cos 2\theta + i\sin 2\theta - \cos 2\theta + i\sin 2\theta = i\sqrt{3}$ $2\sin 2\theta = \sqrt{3} \quad , -2\pi < 2\theta \le 2\pi$ $\sin 2\theta = \frac{\sqrt{3}}{2}$ $2\theta = -\frac{5\pi}{3}, -\frac{4\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}$ $\theta = -\frac{5\pi}{6}, -\frac{2\pi}{3}, \frac{\pi}{6}, \frac{\pi}{3}$	3	**3 marks** • Correct solution **2 marks** • Obtains $\sin 2\theta = \frac{\sqrt{3}}{2}$ **1 mark** • Uses Euler's formula
13 (b) $\int_0^{\frac{2\pi}{3}} \frac{dx}{5 + 4\cos x}$ $t = \tan\frac{x}{2}$ $dx = \frac{2dt}{1 + t^2}$ when $x = 0$, $t = 0$ when $x = \frac{2\pi}{3}$, $t = \sqrt{3}$ $= \int_0^{\sqrt{3}} \frac{2\ dt}{5(1 + t^2) + 4(1 - t^2)}$ $= \int_0^{\sqrt{3}} \frac{2dt}{9 + t^2}$ $= \frac{2}{3}\left[\tan^{-1}\frac{t}{3}\right]_0^{\sqrt{3}}$ $= \frac{2}{3}\left(\frac{\pi}{6}\right)$ $= \frac{\pi}{9}$	4	**4 marks** • Correct solution **3 marks** • Obtains the primitive function **2 marks** • Rewrites integrand and limits in terms of t **1 mark** • Substitutes t results into integrand • Calculates the limits with respect to t

Solution	Marks	Comments
13 (c) $\int \frac{x\,dx}{x\tan x + 1}$ $= \int \frac{x\,dx}{\frac{x\sin x}{\cos x} + 1}$ $= \int \frac{x\cos x\,dx}{x\sin x + \cos x}$ $= \int \frac{du}{u}$ $= \ln\lvert u\rvert + c$ $= \ln\lvert x\sin x + \cos x\rvert + c$ $u = x\sin x + \cos x$ $du = (x\cos x + \sin x - \sin x)dx$ $= x\cos x\,dx$	3	**3 marks** • Correct solution **2 marks** • Rewrites integrand in terms of u **1 mark** • uses $\tan x = \frac{\sin x}{\cos x}$, or equivalent merit
13 (d) (i) $AE = \begin{pmatrix} 0 \\ 4 \\ 8 \end{pmatrix} - \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$ $= \begin{pmatrix} -4 \\ 4 \\ 8 \end{pmatrix}$ $\therefore \underset{\sim}{r} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$	2	**2 marks** • Correct solution **1 mark** • Finds a possible direction vector
13 (d) (ii) Let M and N be the midpoints of BD and OA respectively $\overrightarrow{MV} = \begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix}$ and $\overrightarrow{MN} = \begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix}$  $\cos\theta = \frac{\overrightarrow{MV}\cdot\overrightarrow{MN}}{\lvert\overrightarrow{MV}\rvert\lvert\overrightarrow{MN}\rvert}$ $= \frac{0 + 8 + 0}{\sqrt{4 + 16}\sqrt{16}}$ $= \frac{8}{4\sqrt{20}}$ $= \frac{1}{\sqrt{5}}$ $\theta = 63°$ (to nearest degree)	3	**3 marks** • Correct solution **2 marks** • Calculates the dot product of the two vectors **1 mark** • Finds two appropriate vectors • Uses the dot product in a valid attempt
QUESTION 14		
14 (a) (i) Polynomial $z^4 - (z-1)^4$ is of order 3 and a polynomial cannot have more roots than the degree of the polynomial	1	**1 mark** • Correct explanation
14 (a) (ii) $z^4 = (z-1)^4$ $\left(\frac{z}{z-1}\right)^4 = 1$ let $x = \frac{z}{z-1}$ $x^4 = 1$ $x = \pm 1, \pm i$ $\frac{z}{z-1} = 1$ $z = z - 1$ no solutions $\frac{z}{z-1} = -1$ $z = 1 - z$ $2z = 1$ $z = \frac{1}{2}$ $\frac{z}{z-1} = i$ $z = i(z-1)$ $z(1-i) = -i$ $z = \frac{i}{i-1} \times \frac{i+1}{i+1}$ $= \frac{-1+i}{-2}$ $= \frac{1}{2} - \frac{1}{2}i$ as cofficients are real conjugate solutions must appear $\therefore z = \frac{1}{2} + \frac{1}{2}i$ $z = \frac{1}{2}, \frac{1}{2} \pm \frac{1}{2}i$	4	**4 marks** • Correct solution **3 marks** • Finds the non-real solutions **2 marks** • Uses the roots of unity to find possibilities for $\frac{z}{z-1}$ • Finds the real solution and realises the other two solutions are conjugates **1 mark** • Finds the four roots of unity • Finds the real solution

Solution	Marks	Comments
QUESTION 14		
14 (b) (i)  $m\ddot{x} = -mg - \frac{mv^2g}{u^2}$ $\frac{dv}{dt} = -g\left(1 + \frac{v^2}{u^2}\right)$ $\int_0^T dt = -\frac{1}{g}\int_u^0 \frac{u^2}{u^2 + v^2}dv$ $T = -\frac{u^2}{g}\left[\frac{1}{u}\tan^{-1}\frac{v}{u}\right]_u^0$ $= -\frac{u}{g}\left(0 - \frac{\pi}{4}\right)$ $= \frac{\pi u}{4g}$ seconds	3	**3 marks** • Correct solution **2 marks** • Obtains an expression for t in terms of v **1 mark** • Finds a correct expression for $\frac{dv}{dt}$
14 (b) (ii) $v\frac{dv}{dx} = -g\left(\frac{u^2 + v^2}{u^2}\right)$ $\int_0^H dx = -\frac{u^2}{g}\int_u^0 \frac{v\,dv}{u^2 + v^2}$ $H = -\frac{u^2}{2g}[\ln\lvert u^2 + v^2\rvert\ ]_u^0$ $= -\frac{u^2}{2g}(\ln u^2 - \ln 2u^2)$ $= \frac{u^2 \ln 2}{2g}$	3	**3 marks** • Correct proof **2 marks** • Obtains an expression for displacement, in terms of velocity **1 mark** • Uses $\ddot{x} = v\frac{dv}{dx}$ in an attempt to find the given result
14 (c) (i) $I_1 = \int_0^1 \frac{x\,dx}{\sqrt{1 + x^2}}$ $= \left[\sqrt{1 + x^2}\right]_0^1$ $= \sqrt{2} - 1$	1	**1 mark** • Correct answer
14 (c) (ii) $I_n = \int_0^1 \frac{x^n\,dx}{\sqrt{1 + x^2}}$ $u = x^{n-1}$ $v = \sqrt{1 + x^2}$ $du = (n-1)x^{n-2}dx$ $dv = \frac{x\,dx}{\sqrt{1 + x^2}}$ $= \left[x^{n-1}\sqrt{1 + x^2}\right]_0^1 - (n-1)\int_0^1 x^{n-2}\sqrt{1 + x^2}\,dx$ $= \sqrt{2} - (n-1)\int_0^1 \frac{x^{n-2}}{\sqrt{1 + x^2}} \times (1 + x^2)dx$ $= \sqrt{2} - (n-1)\int_0^1 \frac{x^{n-2}\,dx}{\sqrt{1 + x^2}} - (n-1)\int_0^1 \frac{x^n\,dx}{1 + x^2}$ $= \sqrt{2} - (n-1)I_{n-2} - (n-1)I_n$ $nI_n = \sqrt{2} - (n-1)I_{n-2}$ $I_n = \frac{\sqrt{2} - (n-1)I_{n-2}}{n}$	3	**3 marks** • Correct solution **2 marks** • Correct use of integration by parts and attempts an algebraic manipulation on the resulting integrand **1 mark** • Correct use of integration by parts

<table>
<tr><th>Solution</th><th>Marks</th><th>Comments</th></tr>
<tr><td>14 (c) (iii) $I_n = \dfrac{\sqrt{2} - (n-1)I_{n-2}}{n}$
$nI_n = \sqrt{2} - (n-1)I_{n-2}$
$nI_n \le \sqrt{2}$
$I_n \le \dfrac{\sqrt{2}}{n}$
as $I_n \ge 0$; $0 \le I_n \le \dfrac{\sqrt{2}}{n}$
as $n \to 0$; $0 \le I_n \le 0$
$\therefore \lim\limits_{n \to 0} I_n = 0$</td><td>1</td><td>1 mark
• Correct solution</td></tr>
<tr><td colspan="3">QUESTION 15</td></tr>
<tr><td>15 (a) The arithmetic mean will equal the geometric mean when all of the numbers are equal
i.e. $\dfrac{a_1 + a_2 + a_3 + \ldots + a_n}{n} = (a_1 \times a_2 \times a_3 \times \ldots \times a_n)^{\frac{1}{n}}$ when $a_1 = a_2 = a_3 = \ldots = a_n$</td><td>1</td><td>1 mark
• Correct answer</td></tr>
<tr><td>15 (b)
$\dfrac{(s-x)+(s-y)+(s-z)}{3} \ge \sqrt[3]{(s-x)(s-y)(s-z)}$ $(AM \ge GM)$
equality occurs when $(s-x) = (s-y) = (s-z) \Rightarrow x = y = z$ as s is constant
$\dfrac{3s-(x+y+z)}{3} \ge \sqrt[3]{(s-x)(s-y)(s-z)}$
$\dfrac{s}{3} \ge \sqrt[3]{(s-x)(s-y)(s-z)}$ (perimeter $= x + y + z = 2s$)
$(s-x)(s-y)(s-z) \le \dfrac{s^3}{27}$
$s(s-x)(s-y)(s-z) \le \dfrac{s^4}{27}$
$\sqrt{s(s-x)(s-y)(s-z)} \le \dfrac{s^2}{3\sqrt{3}}$
Area $\le \dfrac{s^2}{3\sqrt{3}}$
Thus the maximum area is $\dfrac{s^2}{3\sqrt{3}}$ units^2 and it will occur when $x = y = z$
i.e. the triangle of maximum area with constant perimeter is equilateral</td><td>3</td><td>3 marks
• Correct solution
2 marks
• Shows that area $\ge$ constant amount
1 mark
• Establishes $\dfrac{(s-x)+(s-y)+(s-z)}{3} \ge \sqrt[3]{(s-x)(s-y)(s-z)}$</td></tr>
<tr><td>15(c) (i) At the points of intersection
$|\underset{\sim}{r}| = a$
$\underset{\sim}{r} \cdot \underset{\sim}{r} = a^2$
$(\underset{\sim}{b} + \lambda \underset{\sim}{m}) \cdot (\underset{\sim}{b} + \lambda \underset{\sim}{m}) = a^2$
$\underset{\sim}{b} \cdot \underset{\sim}{b} + 2\lambda \underset{\sim}{b} \cdot \underset{\sim}{m} + \lambda^2 \underset{\sim}{m} \cdot \underset{\sim}{m} = a^2$
as $\underset{\sim}{m}$ is a unit vector $\underset{\sim}{m} \cdot \underset{\sim}{m} = 1$
$\lambda^2 + 2\lambda \underset{\sim}{b} \cdot \underset{\sim}{m} + \left(\underset{\sim}{b} \cdot \underset{\sim}{b} - a^2\right) = 0$
two solutions occur when $\Delta > 0$
$4(\underset{\sim}{b} \cdot \underset{\sim}{m})^2 - 4\left(\underset{\sim}{b} \cdot \underset{\sim}{b} - a^2\right) > 0$
$(\underset{\sim}{b} \cdot \underset{\sim}{m})^2 - \underset{\sim}{b} \cdot \underset{\sim}{b} + a^2 > 0$
$a^2 > \underset{\sim}{b} \cdot \underset{\sim}{b} - (\underset{\sim}{b} \cdot \underset{\sim}{m})^2$</td><td>2</td><td>2 marks
• Correct solution
1 mark
• Finds a quadratic in terms of λ</td></tr>
</table>

Solution	Marks	Comments
15(c) (ii) If $\underset{\sim}{r}$ is a tangent then $a^2 = \underset{\sim}{b}\cdot\underset{\sim}{b} - (\underset{\sim}{b}\cdot\underset{\sim}{m})^2$ thus $\lambda^2 + 2\underset{\sim}{b}\cdot\underset{\sim}{m}\,\lambda + \underset{\sim}{b}\cdot\underset{\sim}{b} - \underset{\sim}{b}\cdot\underset{\sim}{b} + (\underset{\sim}{b}\cdot\underset{\sim}{m})^2 = 0$ $\lambda^2 + 2\underset{\sim}{b}\cdot\underset{\sim}{m}\,\lambda + (\underset{\sim}{b}\cdot\underset{\sim}{m})^2 = 0$ $(\lambda + \underset{\sim}{b}\cdot\underset{\sim}{m})^2 = 0$ $\lambda = -\underset{\sim}{b}\cdot\underset{\sim}{m}$ Let $\underset{\sim}{p}$ be the position vector of the point of contact with the sphere $\underset{\sim}{p} = \underset{\sim}{b} - (\underset{\sim}{b}\cdot\underset{\sim}{m})\,\underset{\sim}{m}$ $\underset{\sim}{m}\cdot\underset{\sim}{p} = \underset{\sim}{b}\cdot\underset{\sim}{m} - (\underset{\sim}{b}\cdot\underset{\sim}{m})\,\underset{\sim}{m}\cdot\underset{\sim}{m}$ $= \underset{\sim}{m}\cdot\underset{\sim}{b} - \underset{\sim}{b}\cdot\underset{\sim}{m}$ $= 0$ as $\underset{\sim}{m}$ is the direction vector of the tangent and $\underset{\sim}{p}$ is the direction vector of the radius radius $\perp$ tangent	2	**2 marks** • Correct solution **1 mark** • Establishes $\lambda = -\underset{\sim}{b}\cdot\underset{\sim}{m}$
15(d) (i)  **Trajectory of the particle** **horizontal component** $x = 25t\cos\alpha$ $= 25t\left(\frac{2}{\sqrt{5}}\right)$ $= 10\sqrt{5}t$ **vertical component** $y = 25t\sin\alpha - 5t^2$ $= 25t\left(\frac{1}{\sqrt{5}}\right) - 5t^2$ $= 5\sqrt{5}t - 5t^2$ **Resolving horizontal motion into orthogonal components**  **Relative to B** **horizontal component** $X = 10\sqrt{5}t\cos 60°$ $= 10\sqrt{5}t\left(\frac{1}{2}\right)$ $= 5\sqrt{5}t$ **vertical component** $Y = 10\sqrt{5}t\sin 60°$ $= 10\sqrt{5}t\left(\frac{\sqrt{3}}{2}\right)$ $= 5\sqrt{15}t$ thus $OB = 50 - 5\sqrt{5}t$ $\therefore \overrightarrow{OP} = (50 - 5\sqrt{5}t)\underset{\sim}{i} + 5\sqrt{15}t\,\underset{\sim}{j} + (5\sqrt{5}t - 5t^2)\underset{\sim}{k}$	3	**3 marks** • Correct solution **2 marks** • Resolves the horizontal motion into orthogonal components **1 mark** • Establishes the horizontal and vertical equations of motion for the displacement, relative to B
15(d) (ii) $\lvert\overrightarrow{OP}\rvert = \sqrt{(50 - 5\sqrt{5}t)^2 + (5\sqrt{15}t)^2 + (5\sqrt{5}t - 5t^2)^2}$ $= 5\sqrt{(10 - \sqrt{5}t)^2 + (\sqrt{15}t)^2 + (\sqrt{5} - t^2)^2}$ $= 5\sqrt{100 - 20\sqrt{5}t + 5t^2 + 15t^2 + 5t^2 - 2\sqrt{5}t^3 + t^4}$ $= 5\sqrt{100 - 20\sqrt{5}t + 25t^2 - 2\sqrt{5}t^3 + t^4}$ Now $(t^2 - \sqrt{5}t + 10)^2 = t^4 + 5t^2 + 100 + 2(-\sqrt{5}t^3 + 10t^2 - 10\sqrt{5}t)$ $= t^4 - 2\sqrt{5}t^3 + 25t^2 - 20\sqrt{5}t + 100$ $\therefore \lvert\overrightarrow{OP}\rvert = 5\sqrt{(t^2 - \sqrt{5}t + 10)^2}$ $= 5(t^2 - \sqrt{5}t + 10)$	2	**2 marks** • Correct solution **1 mark** • Finds an expression for $\lvert\overrightarrow{OP}\rvert$

Solution	Marks	Comments
15(d) (iii) minimum $\lvert\overrightarrow{OP}\rvert$ will occur at $t = -\frac{b}{2a}$ $= \frac{\sqrt{5}}{2}$ when $t = \frac{\sqrt{5}}{2}$; $\dot{y} = 5\sqrt{5} - 10\left(\frac{\sqrt{5}}{2}\right)$ $= 0$ $\therefore$ $\lvert\overrightarrow{OP}\rvert$ is a minimum when the particle is at its maximum height	2	**2 marks** • Correct solution **1 mark** • Finds when P is closest to O • Finds when P is at its maximum height
QUESTION 16		
16(a) (i) $\int_n^{2n} \frac{dx}{\sqrt{x}} = [2\sqrt{x}]_n^{2n}$ $= 2\sqrt{2n} - 2\sqrt{n}$ $= 2\sqrt{n}(\sqrt{2} - 1)$	1	**1 mark** • Correct solution
16(a) (ii) Area of the lower rectangles < Area under the curve $\frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{n+2}} + \ldots + \frac{1}{\sqrt{n}} < \int_n^{2n} \frac{dx}{\sqrt{x}}$ $S_n < 2\sqrt{n}(\sqrt{2} - 1)$ Area under the curve < Area of the upper rectangles $\int_n^{2n} \frac{dx}{\sqrt{x}} < \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{n+2}} + \ldots + \frac{1}{\sqrt{2n-1}}$ $2\sqrt{n}(\sqrt{2} - 1) < \frac{1}{\sqrt{n}} + \left(\frac{1}{\sqrt{n+1}} + \frac{1}{\sqrt{n+2}} + \ldots + \frac{1}{\sqrt{2n-1}} + \frac{1}{\sqrt{2n}}\right) - \frac{1}{\sqrt{2n}}$ $2\sqrt{n}(\sqrt{2} - 1) - \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{2n}} < S_n$ $2\sqrt{n}(\sqrt{2} - 1) + \frac{1 - \sqrt{2}}{\sqrt{2n}} < S_n$ $\therefore 2\sqrt{n}(\sqrt{2} - 1) + \frac{1 - \sqrt{2}}{\sqrt{2n}} < S_n < 2\sqrt{n}(\sqrt{2} - 1)$	4	There are two inequalities that need to be shown; (1) S_n < RHS of result (2) S_n > LHS of result **4 marks** • Correct solution **3 marks** • Correctly shows (2) • Correctly shows (1) and makes significant progress in showing (2) **2 marks** • Correctly shows (1) • Makes significant progress in showing (2) **1 mark** • Validly uses the idea of comparing the area of rectangles with the area under the curve
16(a) (iii) Let $n = 1\,000\,000$ $2\sqrt{1000000}(\sqrt{2} - 1) + \frac{1 - \sqrt{2}}{\sqrt{2000000}} < S_{1000000} < 2\sqrt{1000000}(\sqrt{2} - 1)$ $828.4268 < S_{1000000} < 828.4271$ $\therefore S_{1000000} = 828.427$ (to 3 decimal places) $\frac{1}{\sqrt{1000000}} + \frac{1}{\sqrt{1000001}} + \ldots + \frac{1}{\sqrt{2000000}} = \frac{1}{\sqrt{1000000}} + S_{1000000}$ $= 828.428$ (to 3 decimal places)	2	**2 marks** • Correctly evaluation of the result **1 mark** • evaluates $S_{1000000}$ • Adds 0.01 to their result for S_n

Solution	Marks	Comments
16(b) (i) $(k+1)(A_{k+1} - G_{k+1}) - k(A_k - G_k)$ $= \dfrac{(k+1)(a_1 + a_2 + \dots + a_{k+1})}{(k+1)} - \dfrac{k(a_1 + 1_2 + \dots + a_k)}{k} + kG_k - (k+1)G_{k+1}$ $= a_{k+1} + kG_k - (k+1)G_{k+1}$ $= G_k\left[\dfrac{a_{k+1}}{G_k} + k - (k+1)\dfrac{G_{k+1}}{G_k}\right]$ $= G_k\left[\left\{\left(\dfrac{a_{k+1}}{G_k}\right)^{\frac{1}{k+1}}\right\}^{k+1} + k - (k+1)\dfrac{(a_1\, a_2 \dots a_{k+1})^{\frac{1}{k+1}}}{(a_1\, a_2 \dots a_k)^{\frac{1}{k}}}\right]$ $= G_k\left[(\lambda_k)^{k+1} + k - (k+1)a_{k+1}^{\frac{1}{k+1}}(a_1\, a_2 \dots a_k)^{-\frac{1}{k(k+1)}}\right]$ $= G_k\left[(\lambda_k)^{k+1} + k - (k+1)\left\{a_{k+1}(a_1\, a_2 \dots a_k)^{-\frac{1}{k}}\right\}^{\frac{1}{k+1}}\right]$ $= G_k\left[(\lambda_k)^{k+1} + k - (k+1)\left(\dfrac{a_{k+1}}{G_k}\right)^{\frac{1}{k+1}}\right]$ $= G_k\left[(\lambda_k)^{k+1} + k - (k+1)\lambda_k\right]$ ≥ 0 if $(\lambda_k)^{k+1} - (k+1)\lambda_k + k \geq 0$ as $G_k > 0$ $\therefore (k+1)(A_{k+1} - G_{k+1}) \geq k(A_k - G_k)$	3	**3 marks** • Correct solution **2 marks** • Establishes $\dfrac{G_{k+1}}{G_k} = \lambda_k$ **1 mark** • Attempts to show LHS − RHS ≥ 0 • simplifies $(k+1)A_{k+1} - kA_k$ to a_{k+1}
16(b) (ii) $f(x) = x^{k+1} - (k+1)x + k$ $f'(x) = (k+1)x^k - (k+1)$ $f''(x) = k(k+1)x^{k-1}$ Stationary points occur when $f'(x) = 0$ i.e. $(k+1)x^k - (k+1) = 0$ $x^k = 1$ $x = \pm 1$, but $x > 0$ $\therefore x = 1$ when $x = 1$; $f''(1) = k(k+1) > 0$ $\therefore x = 1$ is a minimum turning point as there is one stationary point, at $(1,0)$, which is minimum $f(x) \geq 0$	2	**2 marks** • Correct solution **1 mark** • Identifies $x = 1$ as a stationary point • Uses calculus in a valid attempt

<table>
<tr><th>Solution</th><th>Marks</th><th>Comments</th></tr>
<tr><td>

16(b) (iii) When $n = 1$

$$\text{LHS} = A_1 = \frac{a_1}{1} = a_1 \qquad \text{RHS} = G_1 = a_1^{\,1} = a_1$$

$\therefore$ LHS=RHS

Hence the result is true for $n = 1$

Assume the result is true for $n = k$ where $k \in \mathbb{Z}^+$

i.e. $A_k \geq G_k$

Prove the result is true for $n = k + 1$

i.e. Prove $A_{k+1} \geq G_{k+1}$

PROOF:

by (ii) $(\lambda_k)^{k+1} - (k+1)\lambda_k + k \geq 0 \quad \forall\ \lambda_k \geq 0$

i.e. the condition for (i) is met

thus $(k+1)(A_{k+1} - G_{k+1}) \geq k(A_k - G_k)$

$$(A_{k+1} - G_{k+1}) \geq \frac{k}{k+1}(A_k - G_k)$$

$$A_{k+1} - G_{k+1} \geq 0 \qquad (\text{by assumption } A_k \geq G_k)$$

$$A_{k+1} \geq G_{k+1}$$

Hence the result is true for $n = k + 1$, if it is true for $n = k$

Since the result is true for $n = 1$, then it is true $\forall\ n$ where $n \in \mathbb{Z}^+$ by induction.

</td><td>3</td><td>

There are 4 key parts of the induction;

1. Proving the result true for $n = 1$
2. Clearly stating the assumption and what is to be proven
3. Using the assumption in the proof and acknowledges the condition for (*i*)
4. Correctly proving the required statement

3 marks

- Successfully does all of the 4 key parts

2 marks

- Successfully does 3 of the 4 key parts

1 mark

Successfully does 2 of the 4 key parts

</td></tr>
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