Blacktown Boys' High School

2018

HSC Trial Examination

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided for this paper
- All diagrams are not drawn to scale
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks (pages 3 – 6)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 7 – 12)

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

Assessor: X. Chirgwin

Student Name: ____

Teacher Name: ____

Students are advised that this is a trial examination only and cannot in any way guarantee the content or format of the 2018 Higher School Certificate Examination.

Section I

10 marks
Attempt Questions 1–10

Use the multiple choice answer sheet provided on page 14 for Questions 1–10.

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1 Which conic has eccentricity $\dfrac{\sqrt{21}}{5}$?

A. $\dfrac{x^2}{5^2}+\dfrac{y^2}{2^2}=1$

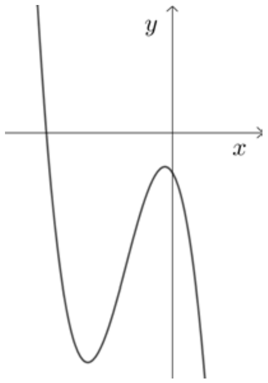
B. $\dfrac{x^2}{5}+\dfrac{y^2}{2}=1$

C. $\dfrac{x^2}{5^2}-\dfrac{y^2}{2^2}=1$

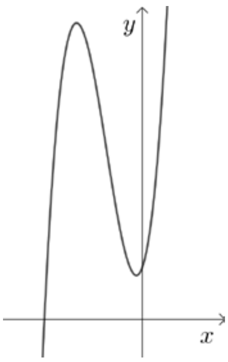
D. $\dfrac{x^2}{5}-\dfrac{y^2}{2}=1$

2 The polynomial $P(x)=x^3-5x^2+7x+13$ has a zero at $x=3+2i$. Which of the graphs below could be the graph of $=P(x)$?

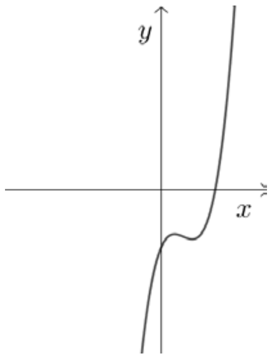
A.



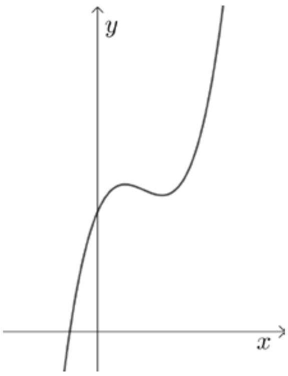
B.



C.



D.



3 What value of z satistifes $z^2 = 32 + 24i$?

A. $2 + 6i$

B. $2 - 6i$

C. $6 + 2i$

D. $6 - 2i$

4 If w is a complex root of $z^5 - 1 = 0$, then $(1 + w - w^2 + w^3 + w^4)^{2018}$ is equal to

A. $2^{2018}w^3$

B. $2^{2018}w^2$

C. $2^{2018}w$

D. 2^{2018}

5 The primitive function of $\cos^3 x$ is

A. $\sin x - \frac{1}{3}\sin^3 x + C$

B. $\frac{1}{3}\sin^3 x - \sin x + C$

C. $\frac{1}{4}\sin^4 x + C$

D. $\frac{1}{4}\cos^4 x + C$

6 Consider the region bounded by the y-axis, the line $y = 3$ and the curve $y = e^x$. If the region is rotated about the line $y = 3$, which expression gives the volume of the solid of revolution?

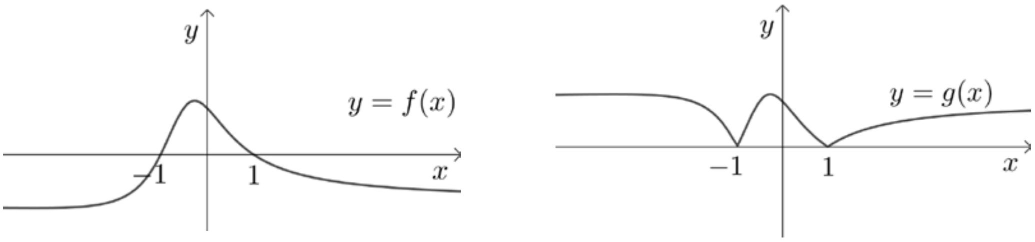
A. $V = \pi \int_1^3 (3 - e^x)^2 \, dx$

B. $V = \pi \int_0^{\ln 3} (3 - e^x)^2 \, dx$

C. $V = \pi \int_1^3 (3 - y) \, dy$

D. $V = \pi \int_0^{\ln 3} e^{2x} \, dx$

7 Which statement is true for these graphs?



A. $g(x) = \sqrt{f(x)}$

B. $|f(x)| = g(x)$

C. $g(x) = [f(x)]^{-1}$

D. $f\left(\sqrt{x}\right) = g(x)$

8 Suppose that $f(x)$ is a non-zero continuous odd function.
Which of the statement below is true?

A. $\int_{-1}^{1}(f(x)\cos x)dx = 2\int_{0}^{1}(f(x)\cos x)dx$

B. $\int_{-1}^{1}(f(x^2)\sin x)dx = 2\int_{0}^{1}(f(x^2)\sin x)dx$

C. $\int_{-1}^{1}(f(x)\tan x)dx = 2\int_{0}^{1}(f(x)\tan x)dx$

D. $\int_{-1}^{1}(f(x^2)e^x)dx = 2\int_{0}^{1}(f(x^2)e^x)dx$

9 A committee of three to be chosen at random from 6 women and n men $(n \geq 1)$.
Find the number of possible committees containing exactly one women.

A. ${}^{n}C_2$

B. 20

C. $6n$

D. $3n(n-1)$

10 The normal at the point $P\left(cp, \frac{c}{p}\right)$ on the rectangular hyperbola $xy = c^2$ has equation $p^3x - py = cp^4 - c$. This normal cuts the hyperbola at a second point $Q\left(cq, \frac{c}{q}\right)$.
What is the relationship between p and ?

A. $p^3q = -1$

B. $pq^4 = -1$

C. $p^2q = -1$

D. $pq = -1$

End of Section I

Section II

90 Marks
Attempt Questions 11–16
Answer each question on a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

a) Let $z = \sqrt{3} - i$ and $w = 4\sqrt{3} + 2i$

i) Find $\overline{w} + z^{-1}$ in the form $a + bi$, where a and b are real. **2**

ii) Express z in modulus-argument form. **2**

iii) Write z^{18} in its simplest form. **2**

b) i) Find the numbers A, B and C such that **2**

$$\frac{2x^2 + 9x + 1}{(x-2)(x^2+5)} = \frac{A}{x-2} + \frac{Bx + C}{x^2 + 5}$$

ii) Hence, or otherwise, find **2**

$$\int \frac{2x^2 + 9x + 1}{(x-2)(x^2+5)}\,dx$$

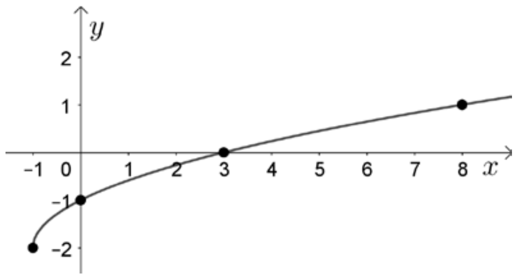
c) Sketch $\dfrac{x^2}{16} + \dfrac{y^2}{11} = 1$ indicating the coordinates of its foci and equations of the directrices. **2**

d) Find the value of $\dfrac{dy}{dx}$ at the point $(-1, 5)$ on the curve $x^3y^2 = 10 + y$. **3**

End of Questions 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

a) The diagram below shows the graph of the function $f(x) = \sqrt{x+1} - 2$.



On separate diagrams sketch the following graphs, showing clearly any intercepts on the coordinate axes, key points, and the equations of any asymptotes:

i) $y = |f(x)|$ **1**

ii) $y = [f(x)]^2$ **2**

iii) $y^2 = f(x)$ **2**

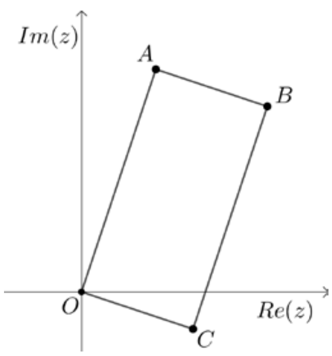
iv) $y = f(x^2)$ **2**

v) $y = \dfrac{1}{f(x)}$ **2**

vi) $y = x \cdot f(x)$ **2**

vii) $y = \log_e[f(x)]$ **2**

b) $OABC$ is a rectangle on the Argand diagram shown below, where O is the origin. If A represents the complex number $2 + 6i$, find the complex numbers represented by B and C, given that the side OC is half the length of OA. **2**



End of Questions 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

a) If α, β and γ are the roots of the polynomial equation $2x^3 - 7x + 3 = 0$, form the polynomial equation with roots:

i) $\alpha + 1, \beta + 1$, and $\gamma + 1$ **2**

ii) α^{-2}, β^{-2}, and γ^{-2} **2**

b) Consider the hyperbolas $H_1 : \dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$ and $H_2 : \dfrac{y^2}{16} - \dfrac{x^2}{9} = 1$.

i) Show that the foci of both hyperbolas lies on the same circle. **2**

ii) Hence, or otherwise, write down the equation of this circle. **1**

c) Evaluate $\displaystyle\int_0^a x \sin(a - x)\, dx$ **3**

d) i) Find, in modulus-argument form, the three cube roots of -64. **2**

ii) Write the two unreal cube roots of -64 in the form $a + bi$, where a and b are real. **1**

iii) If w_1 and w_2 are the unreal cube roots of -64, show that **2**

$$w_1^{6n} + w_2^{6n} = 2^{12n+1}$$

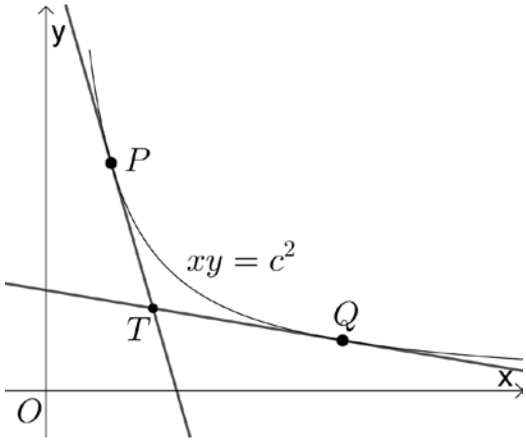
for all integers n.

End of Questions 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

a) Find, by the method of cylindrical shells, the volume of the solid generated when the region bounded by the curve $y = x^2 + 3$, the lines $x = 2$ and the x and y axes is rotated about the line $x = 4$. **3**

b) The distinct points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ are on the same branch of the Hyperbola with the equation $xy = c^2$. The tangents at P and Q meet at the point T.



i) Show that the equation of the tangent at P is $+p^2y = 2cp$. **2**

ii) Show that T has coordinates $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$. **2**

iii) Let P and Q move so that the tangent at P intersects the x-axis at $(cq, 0)$. Show that the locus of T is a hyperbola and state its eccentricity. **4**

c) A sequence of numbers T_n, $n = 1, 2, 3, \ldots$ is defined by $T_1 = 2, T_2 = 0$ and $T_n = 2T_{n-1} - 2T_{n-2}$ for $n = 3, 4, 5, \ldots$. **4**
Use mathematical induction to show that

$$T_n = \left(\sqrt{2}\right)^{n+2} \cos\left(\frac{n\pi}{4}\right), n = 1, 2, 3, \ldots$$

End of Questions 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

a) A chocolate has a circular base of radius 2 cm. If every section perpendicular to this base is an equilateral triangle, find the volume of chocolate needed to make a box of 30 such chocolates. **3**

b) i) Show that $\dfrac{(1-t)^n}{t} = t^{n-1}\left(\dfrac{1}{t}-1\right)^n$ **1**

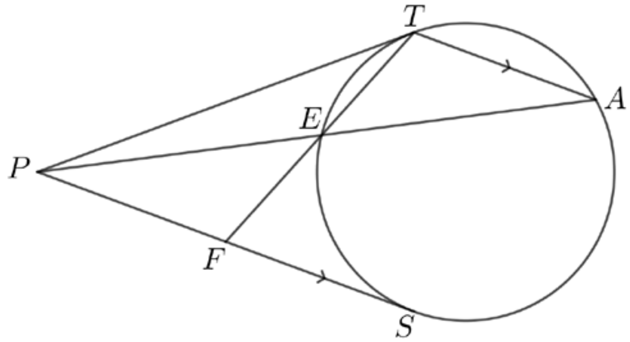
ii) Let $I_n = \displaystyle\int_1^x \frac{(1-t)^n}{t}\,dt$, $n = 1,2,3,\ldots$ **3**

Show that $I_n = \dfrac{(1-x)^n}{n} + I_{n-1}$

c) If a root of the cubic equation $ax^3 + bx^2 + cx + d = 0$ is equal to the reciprocal of another root, show that **3**

$$a^2 - d^2 = ac - bd$$

d) The diagram below shows two tangents PT and PS drawn to a circle from a point P outside the circle. Through T, a chord TA is drawn parallel to the tangent PS. The secant PA meets the circle at E, and TE produced meets PS at F.



i) Prove that ΔEFP is similar to ΔPFT. **2**

ii) Hence show that $PF^2 = TF \times EF$. **1**

iii) Hence, or otherwise, prove that F is the midpoint of PS. **2**

End of Questions 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

a) Four letters are chosen from the letters of the word TELEGRAPH. **2**

These four letters are then placed alongside one another to form a four letter arrangement. Find the number of distinct four letter arrangements which are possible, considering all choices.

b) If $m > 0, n > 0, p > 0, q > 0$, show that

i) $m + n \geq 2\sqrt{mn}$. **1**

ii) $(m+n)(n+p)(m+p) \geq 8mnp$. **2**

iii) $\frac{m}{n} + \frac{n}{p} + \frac{p}{q} + \frac{q}{m} \geq 4$. **2**

c) Given that $f(x) = x^{x^2}$, show that $f'(x) = (2\log_e x + 1)x^{x^2+1}$. **2**

d) It is given that $A > 0, B > 0$ and n is a positive integer.

i) Simplify $\frac{A^{n+1} - A^n B + B^{n+1} - B^n A}{A - B}$. **1**

ii) Deduce that $A^{n+1} + B^{n+1} \geq A^n B + B^n A$. **2**

iii) Show by mathematical induction, that $\left(\frac{A+B}{2}\right)^n \leq \frac{A^n + B^n}{2}$. **3**

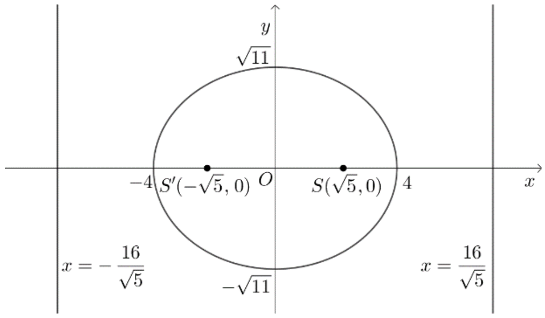
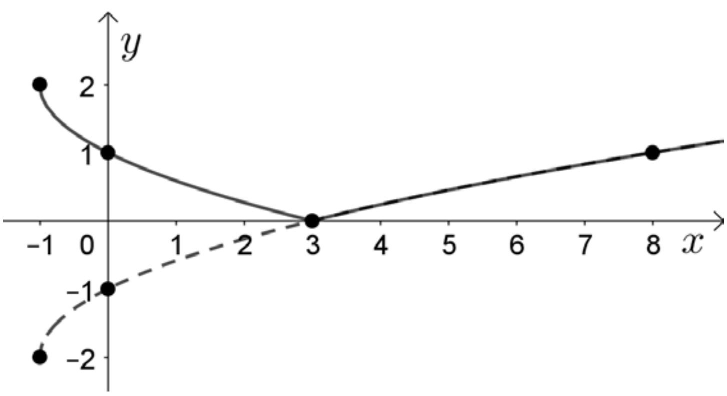
End of Paper

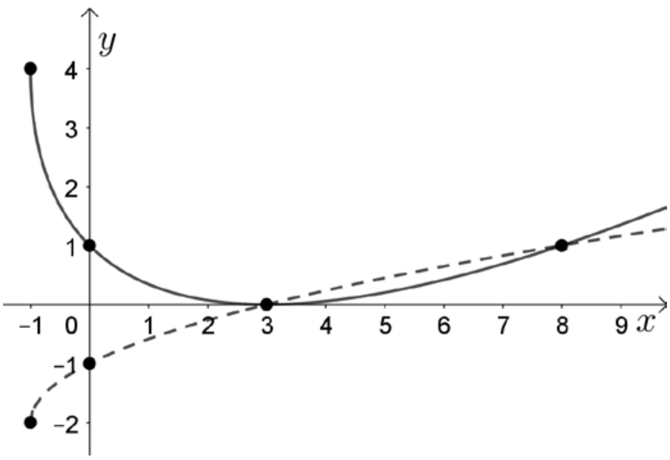
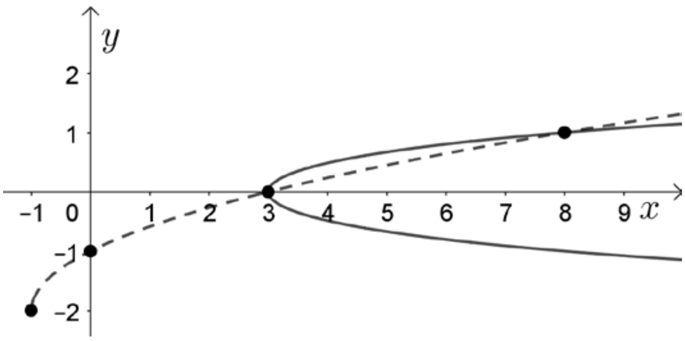
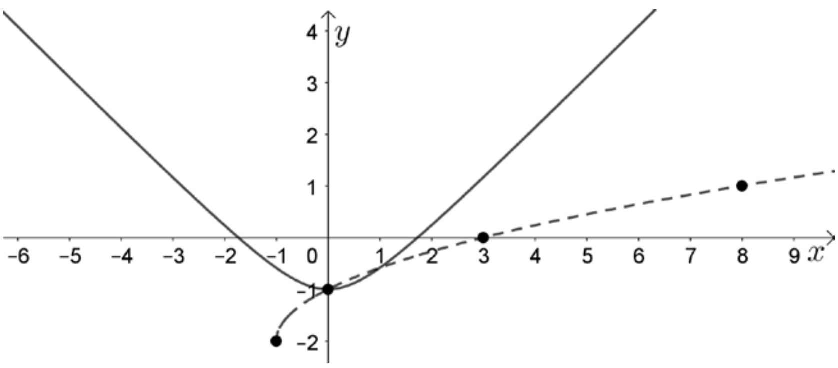
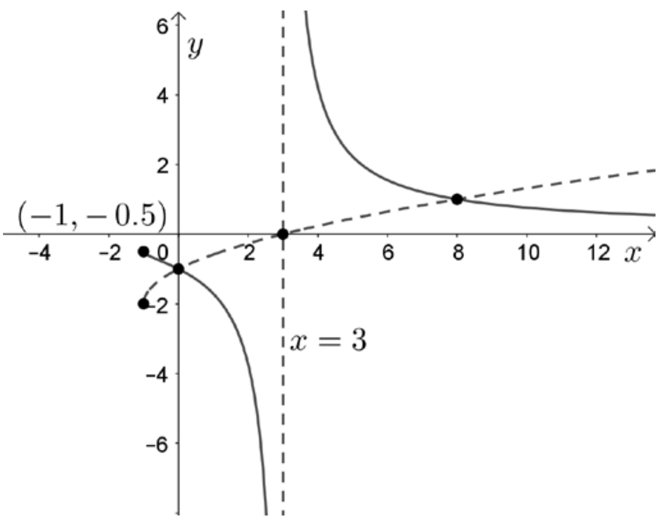
2018 Mathematics Extension 2 Trial Solutions

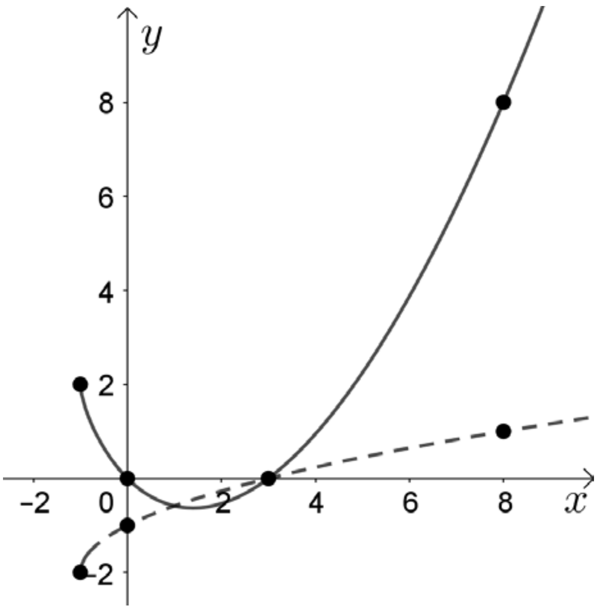
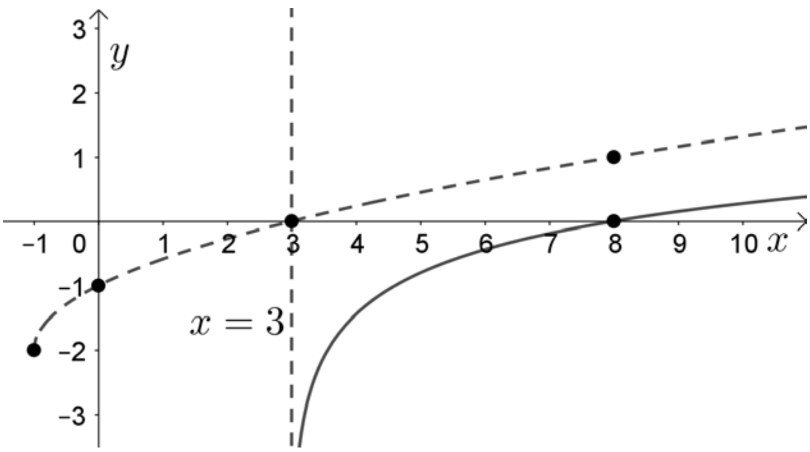
Section 1		
1	**A** Ellipse since $0 < e < 1$ $b^2 = a^2(1-e^2)$ $\frac{b^2}{a^2} = 1 - \left(\frac{\sqrt{21}}{5}\right)^2$ $\frac{b^2}{a^2} = \frac{4}{25}$ $\therefore \frac{x^2}{5^2} + \frac{y^2}{2^2} = 1$	1 Mark
2	**D** The graph has a positive y-intercept so only possibility is B or D. Solving for stationary points, $P'(x) = 0$ $3x^2 - 10x + 7 = 0$ $(x-1)(3x-7) = 0$ $x = 1, x = \frac{7}{3}$ Hence, stationary points occurs in the first quadrant.	1 Mark
3	**C** $(6+2i)^2$ $= 36 + 24i - 4$ $= 32 + 24i$	1 Mark
4	**C** $(1 + w - w^2 + w^3 + w^4)^{2018}$ $= (1 + w + w^2 + w^3 + w^4 - 2w^2)^{2018}$ $= (-2w^2)^{2018}$ $\quad (1 + w + w^2 + w^3 + w^4 = 0)$ $= 2^{2018}w^{4036}$ $= 2^{2018}w^{4035}w$ $= 2^{2018}w$	1 Mark
5	**A** $\int \cos^3 x \, dx$ $= \int \cos x \ \cos^2 x \, dx$ $= \int \cos x \ (1 - \sin^2 x) dx$ $= \int (\cos x - \cos x \sin^2 x) dx$ $= \sin x - \frac{1}{3}\sin^3 x + C$	1 Mark
6	**B** $\Delta V = \pi(3-y)^2 \Delta x$ $V = \pi \int_0^{\ln 3} (3-y)^2 \, dx$ $V = \pi \int_0^{\ln 3} (3-e^x)^2 \, dx$	1 Mark
7	**B** $\lvert f(x) \rvert = g(x)$	1 Mark

8	**C** Odd $\times$ Odd $=$ Even $\therefore \int_{-1}^{1}(f(x)\tan x)dx = 2\int_{0}^{1}(f(x)\tan x)dx$	1 Mark
9	**D** 1 woman from 6 women and 2 men from n men ${}^{6}C_1 \times {}^{n}C_2 = 6 \times \frac{n!}{(n-2)!\,2!}$ ${}^{6}C_1 \times {}^{n}C_2 = 3n(n-1)$	1 Mark
10	**A** Q lies on $p^3x - py = cp^4 - c$ $Q\left(cq, \frac{c}{q}\right)$ $p^3 \times cq - p \times \frac{c}{q} = cp^4 - c$ $cp^3q - \frac{cp}{q} = cp^4 - c$ $p^3q^2 - p = p^4q - q$ $p^3q^2 - p^4q = p - q$ $p^3q(q-p) = -(q-p)$ $p^3q = -1$	1 Mark

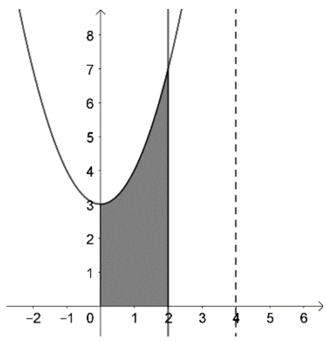
Section 2		
Q11 a) i)	$\overline{w}+z^{-1}$ $=4\sqrt{3}-2i+\dfrac{1}{\sqrt{3}-i}\times\dfrac{\sqrt{3}+i}{\sqrt{3}+i}$ $=4\sqrt{3}-2i+\dfrac{\sqrt{3}+i}{3+1}$ $=\dfrac{17\sqrt{3}}{4}-\dfrac{7}{4}i$	2 Marks Correct solution 1 Mark Obtains the correct expressions for $\overline{w}$ or z^{-1}
Q11 a) ii)	$z=\sqrt{3}-i$ $\lvert z\rvert=\sqrt{3+1}=2$ $\arg z=\tan^{-1}\left(\dfrac{-1}{\sqrt{3}}\right)=-\dfrac{\pi}{6}$ $z=2\left(\cos\left(-\dfrac{\pi}{6}\right)+i\sin\left(-\dfrac{\pi}{6}\right)\right)$ $z=2\left(\cos\dfrac{\pi}{6}-i\sin\dfrac{\pi}{6}\right)$	2 Marks Correct solution 1 Mark Finds modulus or argument
Q11 a) iii)	By de Moivre's theorem and part ii) z^{18} $=2^{18}\left(\cos\left(18\times\dfrac{\pi}{6}\right)-i\sin\left(18\times\dfrac{\pi}{6}\right)\right)$ $=2^{18}(\cos 3\pi-i\sin 3\pi)$ $=-2^{18}$	2 Marks Correct solution 1 Mark Applies de Moivre's theorem
Q11 b) i)	$A(x^2+5)+(Bx+C)(x-2)=2x^2+9x+1$ Let $x=2$ $A\times(2^2+5)+0=2\times2^2+9\times2+1$ $9A=27$ $A=3$ Let $x=0$ $5A-2C=1$ $5\times3-2C=1$ $-2C=-14$ $C=7$ Let $x=1$ $6A+(B+C)(1-2)=2+9+1$ $6\times3+(B+7)\times-1=12$ $-B-7=-6$ $-B=1$ $B=-1$	2 Marks Correct solution 1 Mark Find the correct value of A or B or C with working
Q11 b) ii)	$\displaystyle\int\frac{2x^2+9x+1}{(x-2)(x^2+5)}\,dx$ $\displaystyle=\int\left(\frac{3}{x-2}+\frac{-x+7}{x^2+5}\right)dx$ $\displaystyle=\int\frac{3}{x-2}\,dx-\frac{1}{2}\int\frac{2x}{x^2+5}\,dx+\int\frac{7}{x^2+5}\,dx$ $\displaystyle=3\ln\lvert x-2\rvert-\frac{1}{2}\ln\lvert x^2+5\rvert+\frac{7}{\sqrt{5}}\tan^{-1}\frac{x}{\sqrt{5}}+C$ $\displaystyle=\ln\left\lvert\frac{(x-2)^3}{\sqrt{x^2+5}}\right\rvert+\frac{7}{\sqrt{5}}\tan^{-1}\frac{x}{\sqrt{5}}+C$	2 Marks Correct solution 1 Mark Attempts to integrate using partial fractions and makes significant progress

Q11 c)	$\frac{x^2}{16}+\frac{y^2}{11}=1$ $b^2=a^2(1-e^2)$ $11=16(1-e^2)$ $\frac{11}{16}=1-e^2$ $e^2=1-\frac{11}{16}$ $e=\frac{\sqrt{5}}{4}$ Foci $(\pm ae,0)$ $=\left(\pm 4\times\frac{\sqrt{5}}{4},0\right)$ $=(\pm\sqrt{5},0)$ Equations of directrices: $x=\pm\frac{a}{e}$ $x=\pm\frac{4}{\frac{\sqrt{5}}{4}}$ $x=\pm\frac{16}{\sqrt{5}}$ 	2 Marks Correct solution and graph 1 Mark Finding the correct the foci or equations of directrices
Q11 d)	$x^3y^2=10+y$ $3x^2y^2+x^3\times 2y\frac{dy}{dx}=\frac{dy}{dx}$ $3x^2y^2=\frac{dy}{dx}-2yx^3\frac{dy}{dx}$ $\frac{dy}{dx}(1-2yx^3)=3x^2y^2$ $\frac{dy}{dx}=\frac{3x^2y^2}{1-2yx^3}$ At $(-1,5)$ $\frac{dy}{dx}=\frac{3\times(-1)^2\times 5^2}{1-2\times 5\times(-1)^3}$ $\frac{dy}{dx}=\frac{75}{11}$	3 Marks Correct solution 2 Mark Correct differentiation 1 Mark Attempts to differentiate implicitly using the chain rule
Q12 a) i)	$y=\|f(x)\|$ 	1 Mark Correct graph

Q12 a) ii)	$y = [f(x)]^2$ 	2 Marks Correct graph 1 Mark Correct key points indicated
Q12 a) iii)	$y^2 = f(x)$ 	2 Marks Correct graph 1 Mark Recognises correct domain
Q12 a) iv)	$y = f(x^2)$ 	2 Marks Correct graph 1 Mark Recognises reflection along the y-axis
Q12 a) v)	$y = \dfrac{1}{f(x)}$ 	2 Marks Correct graph 1 Mark Recognises and labels the asymptote

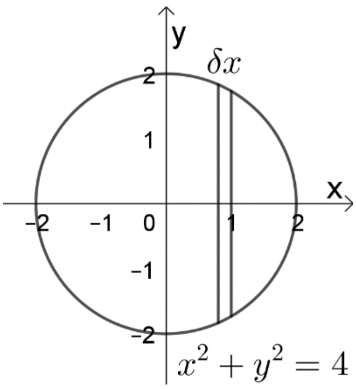
Q12 a) vi)	$y = x \cdot f(x)$  	2 Marks Correct graph 1 Mark Correct key points indicated
Q12 a) vii)	$y = \log_e[f(x)]$  	2 Marks Correct graph 1 Mark Recognises correct domain
Q12 b)	$\overrightarrow{OC} = \frac{1}{2} \times \overrightarrow{OA} \times (-i)$ $\overrightarrow{OC} = \frac{1}{2} \times (2 + 6i) \times (-i)$ $\overrightarrow{OC} = 3 - i$ $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{OC}$ $\overrightarrow{OB} = 2 + 6i + 3 - i$ $\overrightarrow{OB} = 5 + 5i$	2 Marks Correct solution 1 Mark Finds the complex number representing C
Q13 a) i)	$2x^3 - 7x + 3 = 0$ Let $x = \alpha + 1$ $\alpha = x - 1$ $2(x-1)^3 - 7(x-1) + 3 = 0$ $2(x^3 - 3x^2 + 3x - 1) - 7x + 7 + 3 = 0$ $2x^3 - 6x^2 - x + 8 = 0$	2 Marks Correct solution 1 Mark Recognises the relationships of the roots and makes the correct substitution

Q13 a) ii)	$2x^3 - 7x + 3 = 0$ Let $x = \alpha^{-2} = \frac{1}{\alpha^2}$ $\alpha = \pm\frac{1}{\sqrt{x}}$ $2\left(\pm\frac{1}{\sqrt{x}}\right)^3 - 7\left(\pm\frac{1}{\sqrt{x}}\right) + 3 = 0$ $\pm\frac{2}{x\sqrt{x}} - \left(\pm\frac{7}{\sqrt{x}}\right) + 3 = 0$ $\pm\frac{1}{\sqrt{x}}\left(\frac{2}{x} - 7\right) = -3$ $\left[\pm\frac{1}{\sqrt{x}}\left(\frac{2}{x} - 7\right)\right]^2 = (-3)^2$ $\frac{1}{x}\left(\frac{4}{x^2} - \frac{28}{x} + 49\right) = 9$ $4 - 28x + 49x^2 = 9x^3$ $9x^3 - 49x^2 + 28x - 4 = 0$	2 Marks Correct solution 1 Mark Recognises the relationships of the roots and makes the correct substitution
Q13 b) i)	$H_1: \frac{x^2}{9} - \frac{y^2}{16} = 1$ $b^2 = a^2(e^2 - 1)$ $16 = 9(e^2 - 1)$ $e^2 = \frac{16}{9} + 1$ $e = \frac{5}{3}$ Foci of H_1: $(\pm ae, 0) = (\pm 5, 0)$ $H_2: \frac{y^2}{16} - \frac{x^2}{9} = 1$ $a^2 = b^2(e^2 - 1)$ $9 = 16(e^2 - 1)$ $e^2 = \frac{9}{16} + 1$ $e = \frac{5}{4}$ Foci of H_2: $(0, \pm be) = (0, \pm 5)$ The four foci are 5 units from the origin. ∴They all lie on the same circle with centre at origin and radius 5.	2 Marks Correct solution 1 Mark Provides the correct foci for both hyperbolas
Q13 b) ii)	$x^2 + y^2 = 25$	1 Mark Correct solution
Q13 c)	$I = \int_0^a x\sin(a - x)\,dx$ $u = x \quad v' = \sin(a - x)$ $u' = 1 \quad v = \cos(a - x)$ $I = [x\cos(a - x)]_0^a - \int_0^a \cos(a - x)\,dx$ $I = [a\cos(a - a) - 0] - [\sin(a - x)]_0^a$ $I = a - \sin a$	3 Marks Correct solution 2 Marks Correct integration 1 Mark Attempts to integrate by parts

Q13 d) i)	$z^3 = -64$ $z^3 = 64(\cos\pi + i\sin\pi)$ $z^3 = 64(\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi))$ where k is an integer $z = 64^{\frac{1}{3}}\left(\cos\left(\frac{\pi + 2k\pi}{3}\right) + i\sin\left(\frac{\pi + 2k\pi}{3}\right)\right)$ $z = 4\left(\cos\left(\frac{\pi + 2k\pi}{3}\right) + i\sin\left(\frac{\pi + 2k\pi}{3}\right)\right)$ Let $k = -1, 0, 1$ The three cube roots of -64 are: $4\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$, $4\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$, and $4(\cos(\pi) + i\sin(\pi))$	2 Marks Correct solution 1 Mark Applies De Moivre' theorem
Q13 d) ii)	$4\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$ $= 4\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$ $= 2 - 2\sqrt{3}i$ $4\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$ $= 4\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$ $= 2 + 2\sqrt{3}i$	1 Mark Correct solution
Q13 d) iii)	Let $w_1 = 4\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$, $w_2 = 4\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$ $w_1^{6n} + w_2^{6n}$ $= w_1^{6n} + w_2^{6n}$ $= (w_1^3)^{2n} + (w_2^3)^{2n}$ $= (-64)^{2n} + (-64)^{2n}$ $= 64^{2n} + 64^{2n}$ $= (2^6)^{2n} + (2^6)^{2n}$ $= 2 \times 2^{12n}$ $= 2^{12n+1}$	2 Marks Correct solution 1 Mark Recognises the relation of roots
Q14 a)	$V = 2\pi\int_0^2 (4 - x)y\,dx$ $V = 2\pi\int_0^2 (4 - x)(x^2 + 3)\,dx$ $V = 2\pi\int_0^2 (4x^2 + 12 - x^3 - 3x)\,dx$ $V = 2\pi\left[\frac{4x^3}{3} + 12x - \frac{x^4}{4} - \frac{3x^2}{2}\right]_0^2$ $V = 2\pi\left(\frac{4 \times 2^3}{3} + 12 \times 2 - \frac{2^4}{4} - \frac{3 \times 2^2}{2}\right)$ $V = \frac{148\pi}{3}$ $units^3$ 	3 Marks Correct solution 2 Marks Correct primitive function 1 Mark Applies cylindrical shell method

Q14 b) i)	$xy = c^2$ $y = \frac{c^2}{x}$ $\frac{dy}{dx} = -\frac{c^2}{x^2}$ $P\left(cp, \frac{c}{p}\right)$ $m_P = -\frac{c^2}{(cp)^2}$ $m_P = -\frac{1}{p^2}$ Equation of tangent at P $y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$ $p^2y - cp = -x + cp$ $\therefore x + p^2y = 2cp$	2 Marks Correct solution 1 Mark Obtains the correct gradient of tangent at point P
Q14 b) ii)	Similarly, equation of tangent at Q is $x + q^2y = 2cq$ $x + p^2y = 2cp$ (1) $x + q^2y = 2cq$ (2) (1) – (2) $(p^2 - q^2)y = 2c(p - q)$ $y = \frac{2c(p-q)}{(p^2 - q^2)}$ $y = \frac{2c(p-q)}{(p+q)(p-q)}$ $y = \frac{2c}{p+q}$ Substitute into (1) $x + p^2 \times \frac{2c}{p+q} = 2cp$ $x = 2cp - \frac{2cp^2}{p+q}$ $x = \frac{2cp(p+q)}{p+q} - \frac{2cp^2}{p+q}$ $x = \frac{2cp^2 + 2cpq - 2cp^2}{p+q}$ $x = \frac{2cpq}{p+q}$ $\therefore T\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$	2 Marks Correct solution 1 Mark Obtains the correct x or y coordinates of T
Q14 b) iii)	Given $(cq, 0)$ lies on tangent at P: $x + p^2y = 2cp$ $cq + 0 = 2cq$ $q = 2p$ $T\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$ $x = \frac{2cpq}{p+q}$	4 Marks Correct solution 3 Marks Finds the locus of T 2 Marks Obtains x and y in terms p.

	$x = \frac{2cp \times 2p}{p + 2p}$ $x = \frac{4cp^2}{3p}$ $x = \frac{4cp}{3}$ $y = \frac{2c}{p+q}$ $y = \frac{2c}{p+2p}$ $y = \frac{2c}{3p}$ $xy = \frac{4cp}{3} \times \frac{2c}{3p}$ $xy = \frac{8c^2}{9}$ Since $\frac{8c^2}{9}$ is a constant, then $xy = \frac{8c^2}{9}$ is a rectangular hyperbola with eccentricity $\sqrt{2}$.	1 Mark Establishing the relationship of $q = 2p$
Q14 c)	$T_n = (\sqrt{2})^{n+2} \cos\left(\frac{n\pi}{4}\right)$ 1. Prove statement is true for $n = 1, n = 2$ $T_1 = (\sqrt{2})^{1+2} \cos\left(\frac{\pi}{4}\right)$ $T_1 = 2\sqrt{2} \times \frac{1}{\sqrt{2}}$ $T_1 = 2$ $T_2 = (\sqrt{2})^{2+2} \cos\left(\frac{2\pi}{4}\right)$ $T_2 = 4 \times 0$ $T_2 = 0$ $\therefore$ Statement is true for $n = 1$ and $n = 2$. 2. Assume statement is true for $n = k$ (k some positive integer) $T_k = (\sqrt{2})^{k+2} \cos\left(\frac{k\pi}{4}\right)$ 3. Prove statement is true for $n = k + 1$ $i.e. T_{k+1} = (\sqrt{2})^{k+3} \cos\left(\frac{(k+1)\pi}{4}\right)$ $T_{k+1} = 2T_k - 2T_{k-1}$ $T_{k+1} = 2 \times (\sqrt{2})^{k+2} \cos\left(\frac{k\pi}{4}\right) - 2 \times (\sqrt{2})^{(k-1)+2} \cos\left(\frac{(k-1)\pi}{4}\right)$ $T_{k+1} = (\sqrt{2})^{k+4} \cos\left(\frac{k\pi}{4}\right) - (\sqrt{2})^{k+3} \cos\left(\frac{(k-1)\pi}{4}\right)$ $T_{k+1} = (\sqrt{2})^{k+3} \left(\sqrt{2} \cos\left(\frac{k\pi}{4}\right) - \cos\left(\frac{k\pi}{4} - \frac{\pi}{4}\right)\right)$	4 Marks Correct solution 3 Marks Simplifies the T_{k+1} terms and attempts to apply the difference of angles for cosine 2 Marks Establish the statement for T_{k+1} 1 Mark Prove statement is true for $n = 1$ and $n = 2$

	$T_{k+1}=\left(\sqrt{2}\right)^{k+3}\left(\sqrt{2}\cos\left(\frac{k\pi}{4}\right)-\left(\cos\left(\frac{k\pi}{4}\right)\cos\left(\frac{\pi}{4}\right)+\sin\left(\frac{k\pi}{4}\right)\sin\left(\frac{\pi}{4}\right)\right)\right)$ $T_{k+1}=\left(\sqrt{2}\right)^{k+3}\left(\sqrt{2}\cos\left(\frac{k\pi}{4}\right)-\frac{1}{\sqrt{2}}\cos\left(\frac{k\pi}{4}\right)-\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{k\pi}{4}\right)\right)$ $T_{k+1}=\left(\sqrt{2}\right)^{k+3}\left(\frac{1}{\sqrt{2}}\cos\left(\frac{k\pi}{4}\right)-\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{k\pi}{4}\right)\right)$ $T_{k+1}=\left(\sqrt{2}\right)^{k+3}\left(\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{k\pi}{4}\right)-\sin\left(\frac{\pi}{4}\right)\sin\left(\frac{k\pi}{4}\right)\right)$ $T_{k+1}=\left(\sqrt{2}\right)^{k+3}\left(\cos\left(\frac{k\pi}{4}+\frac{\pi}{4}\right)\right)$ $T_{k+1}=\left(\sqrt{2}\right)^{k+3}\left(\cos\frac{(k+1)\pi}{4}\right)$ $\therefore$ True by mathematical induction for all positive integers n.	
Q15 a)	$A=\frac{1}{2}\times 2y\times 2y\times\sin 60°$ $A=\sqrt{3}y^2$  $\Delta V=\lim_{\Delta x\to 0}\sum_{x=-2}^{2}\sqrt{3}y^2\Delta x$ $V=\int_{-2}^{2}\sqrt{3}y^2\,dx$ $V=2\sqrt{3}\int_{0}^{2}(4-x^2)\,dx$ $V=2\sqrt{3}\times\left[4x-\frac{x^3}{3}\right]_0^2$ $V=2\sqrt{3}\times\left(4\times 2-\frac{2^3}{3}\right)$ $V=\frac{32\sqrt{3}}{3}\ cm^3$ $\therefore$ The total volume of 30 chocolates is $\frac{32\sqrt{3}}{3}\times 30=320\sqrt{3}\ cm^3$	3 Marks Correct solution 2 Marks Makes significant progress in finding the volume 1 Mark Finds the cross-sectional area
Q15 b) i)	$LHS=\frac{(1-t)^n}{t}$ $LHS==\frac{\left[t\left(\frac{1}{t}-1\right)\right]^n}{t}$ $LHS=\frac{t^n\left(\frac{1}{t}-1\right)^n}{t}$ $LHS=\frac{t^n}{t^1}\left(\frac{1}{t}-1\right)^n$ $\therefore\frac{(1-t)^n}{t}=t^{n-1}\left(\frac{1}{t}-1\right)^n$	1 Mark Correct solution

Q15 b) ii)	$I_n = \int_1^x \frac{(1-t)^n}{t}\,dt$ $I_n = \int_1^x t^{n-1}\left(\frac{1}{t}-1\right)^n dt$ $u = (t^{-1}-1)^n$ $\quad\quad v' = t^{n-1}$ $u' = n\left(\frac{1}{t}-1\right)^{n-1} \times (-t^{-2})$ $\quad\quad v = \frac{t^n}{n}$ $I_n = \left[\frac{t^n}{n}\left(\frac{1}{t}-1\right)^n\right]_1^x - \int_1^x n\left(\frac{1}{t}-1\right)^{n-1} \times (-t^2) \times \frac{t^n}{n}dt$ $I_n = \left[\frac{x^n}{n}\left(\frac{1}{x}-1\right)^n - \frac{1^n}{n}\left(\frac{1}{1}-1\right)^n\right] + \int_1^x t^{n-2}\left(\frac{1}{t}-1\right)^{n-1} dt$ $I_n = \frac{x^n}{n}\left(\frac{1}{x}-1\right)^n + I_{n-1}$ $I_n = \frac{x}{n} \times x^{n-1}\left(\frac{1}{x}-1\right)^n + I_{n-1}$ $I_n = \frac{x}{n} \times \frac{(1-x)^n}{x} + I_{n-1}$ $I_n = \frac{(1-x)^n}{n} + I_{n-1}$	3 Marks Correct solution 2 Marks Correct substitution and establishes reduction formula 1 Mark Correct integration by parts
Q15 c)	$ax^3 + bx^2 + cx + d = 0$, let the roots be, $\alpha, \frac{1}{\alpha}, \beta$ $\alpha + \frac{1}{\alpha} + \beta = -\frac{b}{a} \quad (1)$ $\alpha \times \frac{1}{\alpha} + \alpha \times \beta + \frac{1}{\alpha} \times \beta = \frac{c}{a}$ $1 + \beta\left(\alpha + \frac{1}{\alpha}\right) = \frac{c}{a} \quad (2)$ $\alpha \times \frac{1}{\alpha} \times \beta = -\frac{d}{a}$ $\beta = -\frac{d}{a} \quad (3)$ Sub (3) into (1) $\alpha + \frac{1}{\alpha} - \frac{d}{a} = -\frac{b}{a}$ $\alpha + \frac{1}{\alpha} = \frac{d-b}{a} \quad (4)$ Sub (3) and (4) into (2) $1 - \frac{d}{a}\left(\frac{d-b}{a}\right) = \frac{c}{a}$ $a^2 - d^2 + bd = ac$ $a^2 - d^2 = ac - bd$	3 Marks Correct solution 2 Marks Obtains $\alpha + \frac{1}{\alpha} = \frac{d-b}{a}$ 1 Mark Obtains $\beta = -\frac{d}{a}$
Q15 d) i)	In ΔEFP and ΔPFT, $\angle EFP = \angle PFT$ (common) $\angle PTF = \angle TAE$ (alternate segment with PT tangent) $\angle TAE = \angle EPF$ (alternate angles, $TA \parallel PS$) $\therefore \angle PTF = \angle EPF$ $\therefore \Delta EFP$ is similar to ΔPFT (equiangular)	2 Marks Correct solution 1 Mark Proves $\angle PTF = \angle TAE$ with correct geometric reasoning

Q15 d) ii)	Since ΔEFP is similar to ΔPFT, then the corresponding sides are in the same ratio. $\frac{PF}{FT} = \frac{EF}{PF}$ $\therefore PF^2 = EF \times FT$	1 Mark Correct solution
Q15 d) iii)	$FS^2 = EF \times FT$ (product of two intervals from the points to the circle on the secant is equal to the square of the tangent) $PF^2 = EF \times FT$ (shown above) $FS^2 = PF^2$ $FS = PF$ $\therefore F$ is the midpoint of PS	2 Marks Correct solution 1 Mark Proves $FS^2 = EF \times FT$ with the correct geometric reasoning
Q16 a)	The letters 'E' occurs twice in TELEGRAPH Case 1: No 'E', consider the letters TLGRAPH Number of selections is ${}^7C_4$ and the possible arrangements of this selection is 4! Number of arrangement with no 'E' is ${}^7C_4 \times 4! = 840$ Case 2: 1 'E' so 3 other letters from the remaining 7 letters is ${}^7C_3$ and the arrangements of this selection is 4! Number of arrangement with 1 'E' is ${}^7C_3 \times 4! = 840$ Case 3: 2 'E's so 2 other letters from the remaining 7 letters is ${}^7C_2$ and the arrangements of this selection is $\frac{4!}{2!}$ Number of arrangement with 2 'E's is ${}^7C_2 \times \frac{4!}{2!} = 252$ Total number of distinct arrangements $= 840 + 840 + 252 = 1932$	2 Marks Correct solution 1 Mark Makes some progress
Q16 b) i)	$(m-n)^2 \geq 0$ $m^2 - 2mn + n^2 \geq 0$ $m^2 + 2mn + n^2 \geq 2mn + 2mn$ $(m+n)^2 \geq 4mn$ $m + n \geq 2\sqrt{mn} \quad (\because m > 0, n > 0)$	1 Mark Correct solution
Q16 b) ii)	$m + n \geq 2\sqrt{mn}$ $n + p \geq 2\sqrt{np}$ $p + q \geq 2\sqrt{pq}$ $(m+n)(n+p)(m+p) \geq 2\sqrt{mn} \times 2\sqrt{np} \times 2\sqrt{pq}$ $(m+n)(n+p)(m+p) \geq 8\sqrt{m^2n^2p^2}$ $(m+n)(n+p)(m+p) \geq 8mnp$	2 Marks Correct solution 1 Mark Makes significant progress
Q16 b) iii)	$\left(\frac{m}{n} + \frac{n}{p}\right) + \left(\frac{p}{q} + \frac{q}{m}\right) \geq 2\sqrt{\frac{m}{n} \times \frac{n}{p}} + 2\sqrt{\frac{p}{q} \times \frac{q}{m}}$ $\left(\frac{m}{n} + \frac{n}{p}\right) + \left(\frac{p}{q} + \frac{q}{m}\right) \geq 2\sqrt{\frac{m}{p}} + 2\sqrt{\frac{p}{m}}$	2 Marks Correct solution 1 Mark Makes significant progress

	$\left(\frac{m}{n}+\frac{n}{p}\right)+\left(\frac{p}{q}+\frac{q}{m}\right)\geq 2\left(\sqrt{\frac{m}{p}}+\sqrt{\frac{p}{m}}\right)$ $\left(\frac{m}{n}+\frac{n}{p}\right)+\left(\frac{p}{q}+\frac{q}{m}\right)\geq 2\left(2\sqrt{\sqrt{\frac{m}{p}}\times\sqrt{\frac{p}{m}}}\right)$ $\left(\frac{m}{n}+\frac{n}{p}\right)+\left(\frac{p}{q}+\frac{q}{m}\right)\geq 2\left(2\sqrt{\sqrt{\frac{mp}{mp}}}\right)$ $\left(\frac{m}{n}+\frac{n}{p}\right)+\left(\frac{p}{q}+\frac{q}{m}\right)\geq 4(\sqrt{1})$ $\left(\frac{m}{n}+\frac{n}{p}\right)+\left(\frac{p}{q}+\frac{q}{m}\right)\geq 4$	
Q16 c)	$f(x)=x^{x^2}$ $f(x)=\left(e^{\log_e x}\right)^{x^2}$ $f(x)=e^{x^2\log_e x}$ Let $g(x)=x^2\log_e x$ $f(x)=e^{g(x)}$ $f'(x)=g'(x)e^{g(x)}$ $g'(x)=2x\log_e x+x^2\times\frac{1}{x}$ $g'(x)=2x\log_e x+x$ $f'(x)=(2x\log_e x+x)e^{x^2\log_e x}$ $f'(x)=x(2\log_e x+1)x^{x^2}$ $f'(x)=(2\log_e x+1)x^{x^2+1}$	2 Marks Correct solution 1 Mark Makes significant progress
Q16 d) i)	$\frac{A^{n+1}-A^nB+B^{n+1}-B^nA}{A-B}$ $=\frac{A^n(A-B)+B^n(B-A)}{A-B}$ $=\frac{A^n(A-B)-B^n(A-B)}{A-B}$ $=\frac{(A-B)(A^n-B^n)}{A-B}$ $=A^n-B^n$	1 Mark Correct solution
Q16 c) ii)	$A^{n+1}+B^{n+1}=(A-B)(A^n-B^n)+A^nB+B^nA$ Given that $A>0, B>0$, If $A\geq B$, then $A-B\geq 0$ and $A^n-B^n\geq 0$, So $(A-B)(A^n-B^n)\geq 0$ If $A<B$, then $A-B<0$ and $A^n-B^n<0$, So $(A-B)(A^n-B^n)>0$ $\therefore (A-B)(A^n-B^n)\geq 0$ Hence $A^{n+1}+B^{n+1}\geq 0+A^nB+B^nA$ $\therefore A^{n+1}+B^{n+1}\geq A^nB+B^nA$	2 Marks Correct solution 1 Mark Recognises and establishes relationship from part i)
Q16 c) iii)	1. Prove statement is true for $n=1$. $LHS=\left(\frac{A+B}{2}\right)^1$ $LHS=\frac{A+B}{2}$	3 Marks Correct solution

	$RHS = \frac{A^1 + B^1}{2}$ $RHS = \frac{A + B}{2}$ $LHS = RHS$ $\therefore$ statement is true for $n = 1$ 2. Assume statement is true for $n = k$ (k some positive integer) i.e. $\left(\frac{A+B}{2}\right)^k \le \frac{A^k + B^k}{2}$ 3. Prove statement is true for $n = k + 1$ i.e. $\left(\frac{A+B}{2}\right)^{k+1} \le \frac{A^{k+1} + B^{k+1}}{2}$ $\left(\frac{A+B}{2}\right)^{k+1} = \left(\frac{A+B}{2}\right)^k \times \left(\frac{A+B}{2}\right)$ From step 2 $\left(\frac{A+B}{2}\right)^{k+1} \le \frac{A^k + B^k}{2} \times \left(\frac{A+B}{2}\right)$ $\left(\frac{A+B}{2}\right)^{k+1} \le \frac{A^{k+1} + A^kB + B^kA + B^{k+1}}{4}$ $\left(\frac{A+B}{2}\right)^{k+1} \le \frac{A^{k+1} + B^{k+1}}{4} + \frac{A^kB + B^kA}{4}$ From part ii) $\left(\frac{A+B}{2}\right)^{k+1} \le \frac{A^{k+1} + B^{k+1}}{4} + \frac{A^{k+1} + B^{k+1}}{4}$ $\left(\frac{A+B}{2}\right)^{k+1} \le \frac{2\left(A^{k+1} + B^{k+1}\right)}{4}$ $\left(\frac{A+B}{2}\right)^{k+1} \le \frac{A^{k+1} + B^{k+1}}{2}$ $\therefore$ Statement is true by mathematical induction for all positive integer n.	2 Marks Makes significant progress 1 Mark Prove statement is true for $n = 1$.