Student Name: Tiffany Ho 13609403

2003
TRIAL HIGHER SCHOOL CERTIFICATE

MATHEMATICS
Extension 2

CARINGBAH HIGH

OMNIA VINCIT DILIGENTIA

General Instructions

Reading Time: 5 minutes
Working Time: 3 hours

- Attempt all questions
- Start each question on a new page
- Each question is of equal value
- Show all necessary working.
- Marks may be deducted for careless work or incomplete solutions
- Standard integrals are printed on the last page
- Board-approved calculators may be used
- This examination paper must not be removed from the examination room

Question 1. (15 marks) Start a new page. **Marks**

a) Find $\int \sec^2 x\,(\tan^2 x + 2)dx$. 2

b) Find $\int \frac{5}{x^2 + 6x + 13}dx$. 2

c) Use $t = \tan\left(\frac{x}{2}\right)$ to find $\int \frac{dx}{1 + \sin x + \cos x}$. 3

d) Find $\int \frac{e^{2x}}{\left(e^x + 1\right)^2}dx$ using the substitution $u = e^x + 1$ 2

e) Find $\int 3^x dx$. 1

f) i) Let $I_n = \int_0^1 x^n e^x dx$ where $n \geq 0$. Show that 3

$I_n = e - nI_{n-1}$ for $n \geq 1$.

ii) Hence evaluate $\int_0^1 x^3 e^x dx$. 2

Question 2. **(15 marks) Start a new page** **Marks**

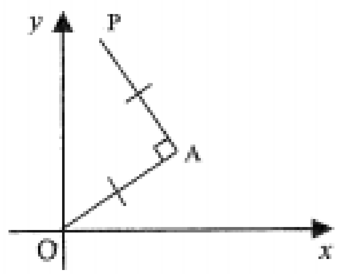
a) Let $z = 3 - 4i$ and $w = 2 + 5i$. Express the following in the form $x + iy$, where x and y are real numbers.

i) z^2 1

ii) $i^3 \frac{z}{w}$ 2

b) Find all the complex numbers $z = a + ib$, where a and b are real, such that $\left|z^2\right| + i\,\overline{z} = 11 + 3i$ 3

c)



The point A in the complex plane corresponds to the complex number z. The triangle OAP is a right angled isosceles triangle.

i) Find in terms of z the complex number corresponding to the point P. 1

ii) Let M be the midpoint of OP. What complex number corresponds to M ? 1

d) i) Express $3 - 3i$ in modulus-argument form. 1

ii) Hence evaluate $(3 - 3i)^7$, expressing it in the form $a + ib$ where a and b are real numbers. 2

e) i) On the same diagram, draw a neat sketch of the locus specified by: 2

α) $\left|z - (5 + 4i)\right| = 4$

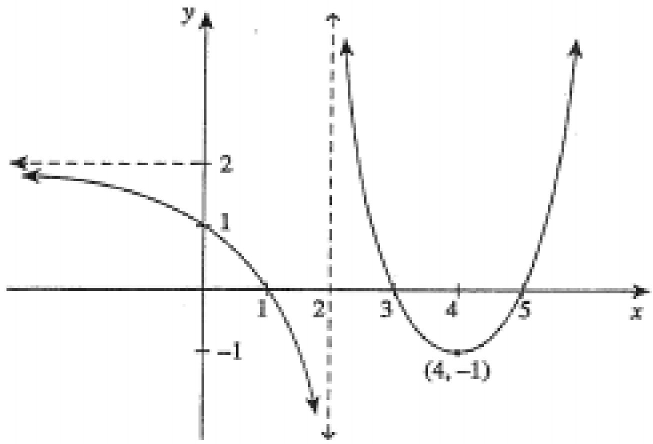
β) $\left|z + 4\right| = \left|z - 6\right|$

ii) Hence write down the value of z which simultaneously satisfies $\left|z - (5 + 4i)\right| = 4$ and $\left|z + 4\right| = \left|z - 6\right|$ 1

iii) Use your diagram in (i) to determine the value(s) of k for which the simultaneous equations $\left|z - (5 + 4i)\right| = 4$ and $\left|z - 4i\right| = k$ have exactly one solution for z. 1

Question 3. (15 marks) Start a new page. **Marks**

a) The graph of $y = f(x)$ is drawn below.



As $x \to -\infty, f(x) \to 2$. The line $x = 2$ is a vertical asymptote. The y-intercept is $y = 1$ and the x-intercepts are $x = 1, x = 3$ and $x = 5$.
Draw **separate** ***half-page*** sketches of the graphs of the following:

i) $y = |f(x)|$ 2

ii) $y = f(|x|)$ 2

iii) $y = \dfrac{1}{f(x)}$ 2

iv) $y = \tan^{-1}[f(x)]$ 2

b) i) Find the coordinates and the nature of the stationary points on the curve $y = x^3 + 6x^2 + 9x + k$ where k is real. 2

ii) Hence find the set of values of k for which the equation $x^3 + 6x^2 + 9x + k = 0$ has three real and different roots. 2

c) i) Find the domain and range of the function $f(x) = \tan^{-1}(e^x)$. 1

ii) Sketch the curve $f(x) = \tan^{-1}(e^x)$ showing any intercepts on the coordinate axes and the equations of any asymptotes. 2

Question 4. (15 marks) Start a new page. **Marks**

a) The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a>b>0$, has eccentricity $e=\frac{1}{2}$.
The point $P(2,3)$ lies on the ellipse.

i) Find the values of a and b. 3

ii) Sketch the graph of the ellipse showing clearly the intercepts on the axes and the coordinates of the foci. 2

b) The normal at the point $P\left(cp,\frac{c}{p}\right)$ on the hyperbola $xy=c^2$ meets the x-axis at Q. Also let M be the midpoint of PQ.

i) Show that the normal at P has the equation $p^3x-py=c\left(p^4-1\right)$ 2

ii) Show that M has coordinates $\left(\frac{c\left(2p^4-1\right)}{2p^3},\frac{c}{2p}\right)$ 2

iii) Hence or otherwise, find the equation of the locus of M. 3

c) The polynomial $P(z)$ is defined by $P(z)=z^4-2z^3-z^2+2z+10$.

i) Given that $z=2-i$ is a root of $P(z)$ write down another root giving a reason for your answer. 1

ii) Hence, express $P(z)$ as a product of real quadratic factors. 2

Question 5. (15 marks) Start a new page. **Marks**

a) i) Suppose that the polynomial $P(x)$ has a double zero at $x = \alpha$. Prove that $P'(x)$ also has a zero at $x = \alpha$. 2

ii) The polynomial $P(x) = x^4 + ax^3 + bx + 21$ has a double zero at $x = 1$. Find the values of a and b. 2

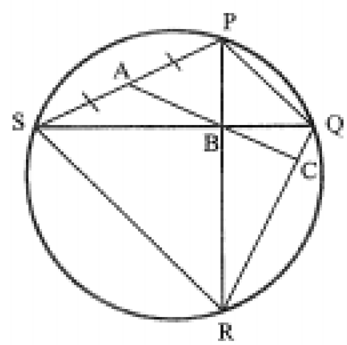
b) i) The equation $x^3 + px^2 + qx + r = 0$ (where p, q, r are non zero) has roots α, β, γ such that $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ are consecutive terms in an arithmetic sequence. 3

Show that $\beta = \frac{-3r}{q}$.

ii) The equation $x^3 - 26x^2 + 216x - 576 = 0$ has roots α, β, γ such that $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ are consecutive terms in an arithmetic sequence. 3

Find the values of α, β, γ.

c)



PQRS is a cyclic quadrilateral. The diagonals PR and SQ intersect at right angles at B. A is the midpoint of PS. AB produced meets QR at C.

Let $\angle ABP = \alpha$. Using the larger diagram provided to indicate angles, show that

i) B, P and S are concyclic points. 1

ii) $\angle APB = \angle ABP$. 1

iii) AC is perpendicular to QR. 3

Question 6. (15 marks) Start a new page. **Marks**

a) If $\bar{z}_1 + \bar{z}_2 = 5 + 2i$, find $z_1 + z_2$ 1

b) The arc of the curve $y = x(2 - x^2)$ from $x = 0$ to $x = 1$ is rotated about y axis. Find by using **cylindrical shells** the volume of the solid formed. 4

c) i) Show that $a^2 + b^2 > 2ab$, where a and b are distinct positive real numbers. 1

ii) Hence show that $a^2 + b^2 + c^2 > ab + bc + ca$, where a, b and c are distinct positive real numbers. 2

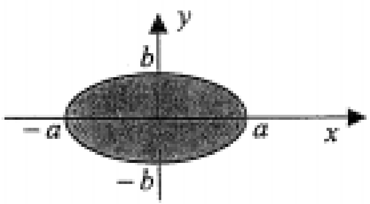
iii) Hence or otherwise, prove that $\dfrac{a^2b^2 + b^2c^2 + c^2a^2}{a+b+c} > abc.$ 2

d) i) Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ using the substitution $u = a - x$ 2

ii) Hence evaluate $\int_0^2 x^2\sqrt{2-x}\,dx$, writing your answer in the form $a\sqrt{b}$. 3

Question 7. (15 marks) Start a new page. **Marks**

a)

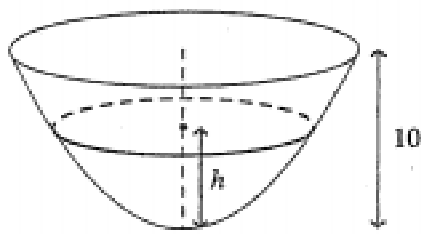


The diagram shows the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with major diameter $2a$ and minor diameter $2b$.

i) Show that the shaded area of the ellipse is given by $\frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$. **2**

ii) Hence show that the shaded area is πab square units. **2**

iii)



The diagram above shows a solid of height 10 cm. At height h cm above the vertex, the cross-section of the solid is an ellipse with major diameter $10\sqrt{h}$ cm and minor diameter $8\sqrt{h}$ cm.

α) Show that the cross-section at height h cm above the vertex has area $20\pi h \text{ cm}^2$. **2**

β) Find the volume of the solid in exact form. **2**

b) If α, β, γ are the roots of the equation $2x^3-7x^2+5x-3=0$,

i) Show that the equation with roots $\alpha^2, \beta^2, \gamma^2$ is given by $4x^3-29x^2-17x-9=0$ **2**

ii) Hence evaluate $\alpha^3+\beta^3+\gamma^3$. **1**

c) i) Expand $(\cos\theta+i\sin\theta)^3$ into powers of $\cos\theta$ and $\sin\theta$. **1**

ii) By using De Moivres Theorem show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. **2**

iii) Hence find the exact value of $4\cos^3\left(\frac{\pi}{12}\right)-3\cos\left(\frac{\pi}{12}\right)$. **1**

Question 8. (15 marks) Start a new page. **Marks**

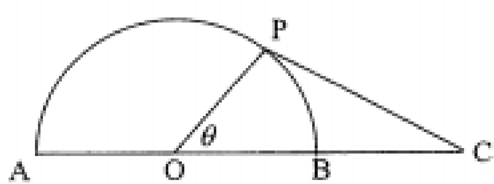
a) Find *all* solutions in radians of the equation 3

$$\frac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta}=\frac{3}{4}$$

b) **For this question assume that tidal motion is simple harmonic.**

On a certain day, the depth of water in a harbour at high tide at 5 am is 9 metres. At the following low tide at 11:20 am the depth is 3 metres. Find the latest time before noon that a ship can enter the harbour if a minimum depth of 7.5 metres is required. (**Show all reasoning**). 4

c)



In the diagram above the fixed points A, O, B and C are on a straight line such that AO = OB = BC = 1unit. The points A and B are also joined by a semicircle and P is a variable point on this semicircle such that $\angle$ POC $=\theta$.

R is the region bounded by the arc AP of the semicircle and the straight lines AC and PC.

i) Show that the area S of R is given by: $S=\dfrac{\pi}{2}-\dfrac{\theta}{2}+\sin\theta$. 1

ii) Find the value of θ for which S is a maximum. 2

iii) Show that the perimeter L of R is given by: 2

$L=3+\pi-\theta+\sqrt{5-4\cos\theta}$.

iv) Show that L has just one stationary point and that it occurs at the same value of θ for which S is a maximum. 2

v) Hence find the greatest value of L in the interval $0\le\theta\le\pi$. 1

END OF PAPER

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

CARINGBAH HIGH EXT 2 TRIAL SOLN'S 2003

1 a) $I = \int \sec^2 x \tan^2 x\,dx + 2\int \sec^2 x\,dx$

$= \int u^2\,du + 2\tan x \quad \{\text{where } u = \tan x\}$

$= \frac{1}{3}\tan^3 x + 2\tan x + c$

b) $I = \int \frac{5}{(x+3)^2+4}\,dx$

$= 5/2 \tan^{-1}\left(\frac{x+3}{2}\right) + c$

c) $t = \tan\frac{x}{2} \Rightarrow dt = \frac{1}{2}\sec^2 x/2\,dx$

$\therefore dx = \frac{2\,dt}{1+t^2}$

$I = \int \frac{\frac{2}{1+t^2}}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}\,dt$

$= \int \frac{1}{1+t}\,dt$

$= \ln\left(1+\tan\frac{x}{2}\right) + c$

d) $u = e^x+1 \rightarrow du = e^x dx$

$I = \int \frac{e^x}{(e^x+1)^2} \cdot e^x\,dx$

$= \int \frac{u-1}{u^2}\,du$

$= \int \frac{1}{u} - \frac{1}{u^2}\,du$

$= \ln(e^x+1) + \frac{1}{e^x+1} + c$

e) $I = \frac{1}{\ln 3} \cdot 3^x + c$

f) i) Using parts with

$u = x^n \qquad v' = e^x$

$u' = n x^{n-1} \qquad v = e^x$

$I_n = \left.x^n e^x\right]_0^1 - \int_0^1 (n x^{n-1}) e^x\,dx$

$= e - 0 - n\int x^{n-1} e^x\,dx$

$= e - n I_{n-1}$

ii) $I_3 = e - 3I_2$

$= e - 3[e - 2I_1]$

$= e - 3e + 6I_1$

$= -2e + 6[e - I_0]$

where $I_0 = \int_0^1 e^x\,dx = e - 1$

$\therefore I_3 = -2e + 6[e - (e-1)]$

$= 6 - 2e$

2a) i) $(3-4i)^2 = -7-24i$

ii) $i^3 \times \frac{3-4i}{2+5i} \times \frac{2-5i}{2-5i}$

$= i^3 \left[\frac{-14-23i}{29}\right]$

$= \frac{-23}{29} + \frac{14i}{29}$

b) $a^2+b^2+i(a-b) = 11+3i$

$\therefore a^2+b^2+b = 11$ and $a = 3$ } equating Real & Imag parts

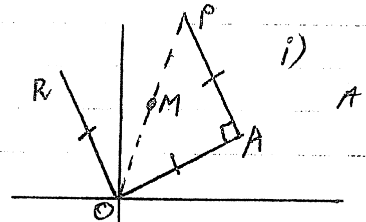
$\therefore b^2+b-2=0$

$(b+2)(b-1)=0$

$b=-2,\ b=1$

$\therefore z_1 = 3+i \qquad z_2 = 3-2i$

c)



i) $\vec{OR} = i\vec{OA} = iz$

Also $\vec{OR} = \vec{AP}$

$\vec{OP} = \vec{OA} + \vec{AP}$

$= z + iz$

$= (1+i)z$

ii) $\vec{OM} = \frac{1}{2}\vec{OP}$

$= \frac{1+i}{2}z$

d) i) $|3-3i| = 3\sqrt{2} \qquad \arg(3-3i) = -\frac{\pi}{4}$

$\therefore 3-3i = 3\sqrt{2}(\cos(-\pi/4) + i\sin(-\pi/4))$

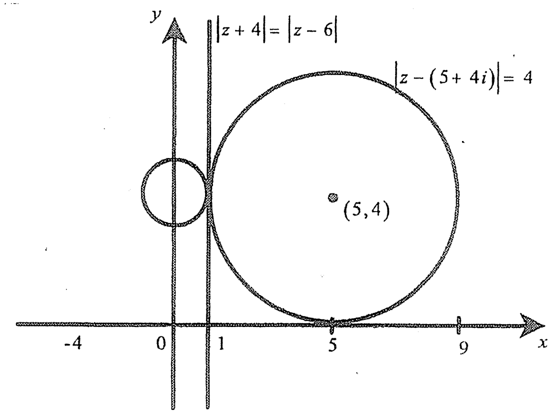
$= 3\sqrt{2}(\cos(\pi/4) - i\sin(\pi/4))$

ii) $\therefore (3-3i)^7 = (3\sqrt{2})^7\left(\cos\frac{7\pi}{4} - i\sin\left(\frac{7\pi}{4}\right)\right)$

$= 17496\sqrt{2}\left[\frac{1}{\sqrt{2}} + i \times \frac{1}{\sqrt{2}}\right]$

$= 17496 + 17496i$

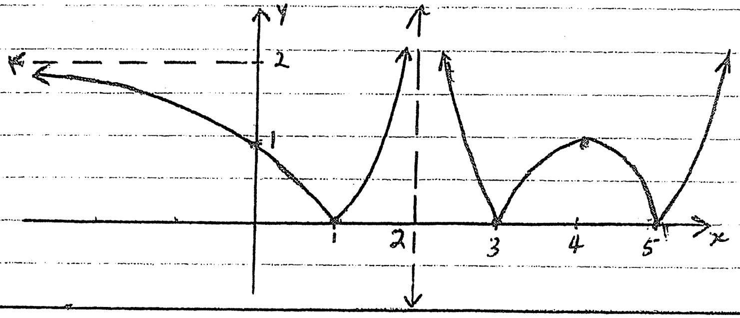
e) i)



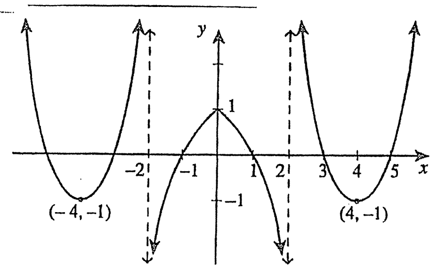
ii) $z = 1+4i$

iii) $k = 1$

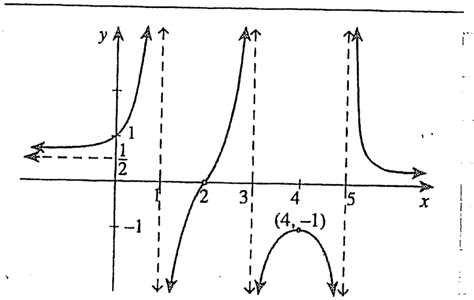
3a) i)



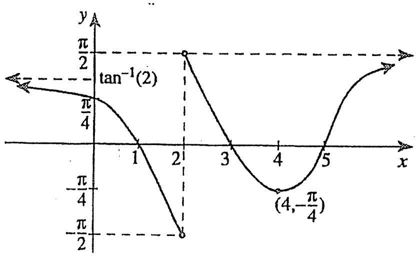
ii)



iii)



iv)



3b) i) $y = x^3 + 6x^2 + 9x + k$

$y' = 3x^2 + 12x + 9$

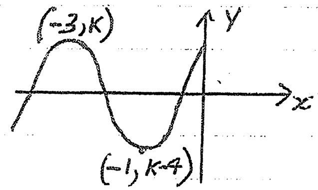
$y'' = 6x + 12$

$y' = 0 \rightarrow x = -1, -3$

$x = -1,\ y = k - 4,\ y'' = 6 > 0 \therefore (-1, k-4)$ is a minimum.

$x = -3,\ y = k,\ y'' = -6 < 0 \therefore (-3, k)$ is a maximum.

ii)



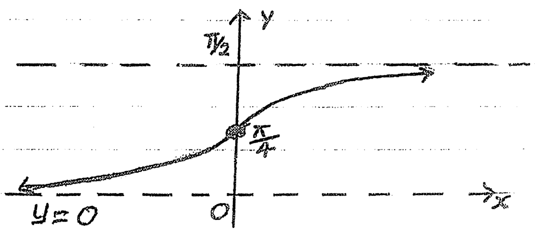
To have 3 real and different roots the curve must cut the x-axis in 3 distinct points. Hence K, K−4 must have opposite signs

$\therefore\ 0 < K < 4$

c) $f(x) = \tan^{-1}(e^x)$

D: all real x

R: $0 < y < \pi/2$



4 a) i) $e = \frac{1}{2} \quad \therefore b^2 = a^2\left(1 - \frac{1}{4}\right)$

$$\therefore b^2 = \frac{3}{4}a^2$$

Since $P(2,3)$ lies on E, then

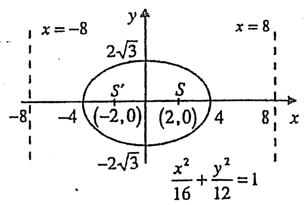
$$\frac{4}{a^2} + \frac{9}{b^2} = 1$$

$$\therefore \frac{4}{a^2} + \frac{12}{a^2} = 1$$

$\therefore a^2 = 16 \quad \& \quad b^2 = 12$

$\therefore a = 4 \quad\quad b = 2\sqrt{3}$

ii)



b) i) $y = c^2/x \Rightarrow \frac{dy}{dx} = \frac{-c^2}{x^2}$

$\therefore$ At $x = cp$ grad. of Tangent $= -1/p^2$

$\therefore$ grad. of Normal $= p^2$

$\therefore$ eq^n of N: $y - \frac{c}{p} = p^2(x - cp)$

$$py - c = p^3x - cp^4$$

$$\therefore p^3x - py = c(p^4 - 1)$$

ii) At Q $y = 0 \Rightarrow p^3x = c(p^4 - 1)$

$$x = \frac{c(p^4-1)}{p^3}$$

$$\therefore M = \left\{ \frac{\frac{c(p^4-1)}{p^3} + cp}{2}, \frac{\frac{c}{p} + 0}{2} \right\}$$

$$= \left\{ \frac{c(2p^4-1)}{2p^3}, \frac{c}{2p} \right\}$$

iii) $y = c/2p \;①\!\rightarrow p = c/2y \rightarrow p^3 = \frac{c^3}{8y^3} \;\&\; p^4 = \frac{c^4}{16y^4}$

$$\therefore x = \frac{\frac{c}{2}\left[\frac{2 \times c^4}{16y^4} - 1\right]}{c^3/8y^3}$$

$$= \frac{c}{2}\left[\frac{2c^4 - 16y^4}{16y^4}\right] \times \frac{8y^3}{c^3}$$

$$= \frac{c^4 - 8y^4}{2c^2y}$$

$\therefore$ Locus is $2c^2yx = c^4 - 8y^4$

c) i) $z = 2 + i$ { roots occur in complex conjugate pairs when coeff's are real.

ii) $P(z) = [z - (2+i)][z - (2-i)].Q(z)$

$$= (z^2 - 4z + 5).Q(z)$$

$$= (z^2 - 4z + 5)(z^2 + 2z + 2)$$

5a)

i) $P(x)=(x-\alpha)^2 Q(x)$ (where $Q(x)$ is a polynomial)

$P'(x)=2(x-\alpha)Q(x)+(x-\alpha)^2 Q'(x)$

$=(x-\alpha)\left[2Q(x)+(x-\alpha)Q'(x)\right]$

$\therefore P'(\alpha)=0$ i.e. $x=\alpha$ is a root of $P'(x)=0$

ii) $P(x)=x^4+ax^3+bx+21$

$P'(x)=4x^3+3ax^2+b$

$P(1)=1+a+b+21=0$

$P'(1)=4+3a+b=0$

$a+b=-22$

$3a+b=-4$

Solving simultaneously $a=9$, $b=-31$.

b)

(i) $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$ in AP $\Rightarrow \frac{1}{\beta}-\frac{1}{\alpha}=\frac{1}{\gamma}-\frac{1}{\beta} \Rightarrow \frac{2}{\beta}=\frac{1}{\alpha}+\frac{1}{\gamma}$. Then $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{3}{\beta}$

But $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\beta\gamma+\gamma\alpha+\alpha\beta}{\alpha\beta\gamma}=-\frac{q}{r}$. Hence $\frac{3}{\beta}=-\frac{q}{r}$. $\therefore \beta=\frac{-3r}{q}$

(ii) $x^3-26x^2+216x-576=0$ such that $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$ in AP $\Rightarrow \beta=\frac{-3r}{q}=\frac{3\times 576}{216}=8$

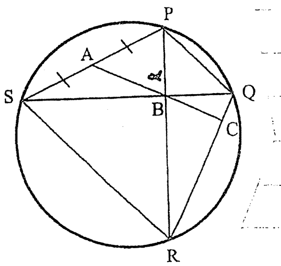
Then $\alpha+\gamma=26-8=18$ and $\alpha\gamma=576\div 8=72$.

Hence α, γ zeros of $x^2-18x+72=(x-12)(x-6)$.

α, β, γ are 6, 8, 12 or 12, 8, 6 respectively.

c) i) $\angle PBS=90°$ {diagonals intersect at right ∠s.}

∴ ∠ in a semi-circle with PS the diameter.



ii) since A is the midpt of diameter PS, then AP = AB {= radii} → △ APB isos

Hence ∠APB = ∠ABP

iii) ∠ABS = 90 − α → ∠QBC = 90 − α {vertically opp.}

∠SPR = ∠SQR = α {∠ in same segment

∴ ∠BCQ = 180 − (α) − (90 − α)

= 90°

∴ AC ⊥ QR

6) a) $\overline{z_1}+\overline{z_2}=\overline{z_1+z_2} \Rightarrow z_1+z_2=5-2i$

b)

$\Delta V=2\pi xy\Delta x$

$V=\lim_{\Delta x\to 0}\sum_{x=0}^{1}2\pi xy\Delta x$

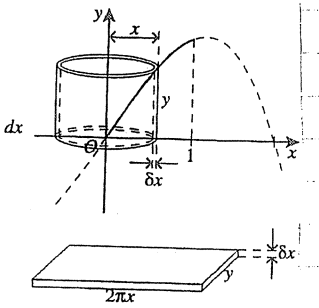
$=\int_0^1(2\pi x)x(2-x^2)\,dx$

$=2\pi\left[\frac{2x^3}{3}-\frac{x^5}{5}\right]_0^1$

$=2\pi\left[\frac{2}{3}-\frac{1}{5}\right]$

$=2\pi\left[\frac{7}{15}\right]$

$=\frac{14\pi}{15}$ cubic units.



c) i) Since $a\neq b$, $a-b>0$, so $(a-b)^2>0$.

Hence $a^2-2ab+b^2>0$

$a^2+b^2>2ab$.

ii) From (i), $a^2+b^2>2ab$,

$b^2+c^2>2bc$ and

$a^2+c^2>2ac$.

Adding, $2(a^2+b^2+c^2)>2(ab+ac+bc)$

$a^2+b^2+c^2>ab+ac+bc$.

iii) Let $A=ab$, $B=bc$ and $C=ac$.

Then A, B and C are distinct positive numbers, and from (ii),

$A^2+B^2+C^2>AB+AC+BC$.

Substituting,

$a^2b^2+b^2c^2+a^2c^2>(ab)(bc)+(ab)(ac)+(bc)(ac)$.

Now $(ab)(bc)+(ab)(ac)+(bc)(ac)=abc(a+b+c)$.

Hence $\frac{a^2b^2+b^2c^2+a^2c^2}{a+b+c}>abc$.

6d) i) $u = a - x$

$\therefore du = -dx$

when $x = 0 \rightarrow u = a$

$x = a \rightarrow u = 0$

$\therefore LHS = \int_0^a f(x)\,dx$

$= -\int_a^0 f(a-u)\,du$

$= \int_0^a f(a-u)\,du$

$= RHS$

ii) $\int_0^2 x^2 \sqrt{2-x}\,dx$

$= \int_0^2 (2-x)^2 . \sqrt{x}\,dx$

$= \int_0^2 4x^{1/2} - 4x^{3/2} + x^{5/2}\,dx$

$= \left[\frac{8}{3} x^{3/2} - \frac{8}{5} x^{5/2} + \frac{2}{7} x^{7/2}\right]_0^2$

$= \frac{8}{3}(\sqrt{2})^3 - \frac{8}{5}(\sqrt{2})^5 + \frac{2}{7}(\sqrt{2})^7 - 0$

$= \frac{128\sqrt{2}}{105}$

Q7 a) i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$

$= \pm\frac{b}{a}\sqrt{a^2 - x^2}.$

Area of the ellipse in the first quadrant $= \int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx.$

$\therefore$ Area of the ellipse $= \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx.$

ii) $y = \sqrt{a^2-x^2}$ is the equation of a circle with centre at the origin and radius a.

The expression $\int_0^a \sqrt{a^2-x^2}\,dx$ gives the area of the first quadrant of this circle, which is equal to $\frac{\pi a^2}{4}$.

Hence the area of the ellipse is $\frac{4b}{a}\frac{\pi a^2}{4} = \pi ab.$

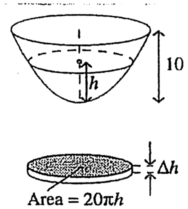
iii) α. $a = 4\sqrt{h}$ and $b = 5\sqrt{h}$.

From (ii),

area $= \pi ab$

$= \pi(4\sqrt{h})(5\sqrt{h})$

$= 20\pi h.$



β. Volume $= \int_0^{10} 20\pi h\,dh$

$= \left[10\pi h^2\right]_0^{10}$

$= 1000\pi\ \text{cm}^3$

7b) i) Let $x=\alpha^2 \to \alpha=\sqrt{x}$

∴ $2(\sqrt{x})^3-7(\sqrt{x})^2+5\sqrt{x}-3=0$

$2x\sqrt{x}-7x+5\sqrt{x}-3=0$

$\sqrt{x}(2x+5)=7x+3$

$x(4x^2+20x+25)=49x^2+42x+9$

∴ $4x^3-29x^2-17x-9=0$

ii) α,β,γ are the roots of:

$2x^3-7x^2+5x-3=0$

∴ $2\alpha^3=7\alpha^2-5\alpha+3$

$2\beta^3=7\beta^2-5\beta+3$

$2\gamma^3=7\gamma^2-5\gamma+3$

∴ $2(\alpha^3+\beta^3+\gamma^3)=7(\alpha^2+\beta^2+\gamma^2)-5(\alpha+\beta+\gamma)+9$

∴ $\alpha^3+\beta^3+\gamma^3=\frac{1}{2}\left[7\times\frac{29}{4}-5\times\frac{7}{2}+9\right]$

$=21\frac{1}{8}$ or $\frac{169}{8}$.

c) i) $(\cos\theta+i\sin\theta)^3=c^3+3c^2(is)+3c(is)^2+(is)^3$

$=c^3-3cs^2+i(3c^2s-s^3)$ —①

ii) $(\cos\theta+i\sin\theta)^3=\cos3\theta+i\sin3\theta$ —②

∴ equating real parts of ① & ②

$\cos3\theta=\cos^3\theta-3\cos\theta.\sin^2\theta$

$=\cos^3\theta-3\cos\theta(1-\cos^2\theta)$

$=4\cos^3\theta-3\cos\theta$

iii) $\cos\left(3\times\frac{\pi}{12}\right)=\frac{1}{\sqrt{2}}$

8a) $\frac{s^3+c^3}{s+c}=\frac{(s+c)(s^2-sc+c^2)}{s+c}$

$=1-sc$

∴ $1-\sin\theta\cos\theta=\frac{3}{4}$

$\sin\theta\cos\theta=\frac{1}{4}$

$\sin2\theta=\frac{1}{2}$

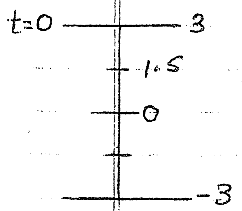
$2\theta=n\pi+(-1)^n(\pi/6)$

∴ $\theta=\frac{n\pi}{2}+(-1)^n\frac{\pi}{12}$

b) High Tide = 9m at 5am. // Low T = 3m at 11:20am

∴ amplitude $=\frac{9-3}{2}=3$m

$P=\frac{2\pi}{n}$ ∴ $760=2\pi/n \to n=\frac{\pi}{380}$



Since the motion is SHM & periodic then $\ddot{x}=-n^2x$ which has solution

$x=a\cos(nt+\alpha)$

∴ $x=3\cos\left(\frac{\pi t}{380}+\alpha\right)$

when $x=3$, $t=0 \Rightarrow \alpha=0$

∴ $x=3\cos\left(\frac{\pi t}{180}\right)$

When $x=1.5 \Rightarrow 0.5=\cos\frac{\pi t}{380}$

∴ $t=\frac{380}{\pi}\times\cos^{-1}(0.5) \doteqdot 126$ min (2h 6m

∴ the latest time before noon is 7:06 am.

8c

i) Area of $\Delta OPC = \frac{1}{2} \times 1 \times 2 \times \sin\theta = \sin\theta$

Area of sector $OPB = \frac{1}{2} \times 1 \times 1 \times \theta = \theta/2$

Area of semi-circle $= \frac{1}{2} \times \pi \times 1^2 = \pi/2$

$\therefore$ Area of S = Area of ΔOPC + Semi-circle − Sect OPB

$= \pi/2 - \theta/2 + \sin\theta$

ii) $S' = \cos\theta - 1/2$ & $S'' = -\sin\theta$

$S' = 0 \rightarrow \theta = \pi/3$ + $S''(\pi/3) = -\sqrt{3}/2 < 0$

$\therefore$ max S at $\theta = \pi/3$.

iii) $L = AP + AC + PC = (AB - PB) + AC + PC$

Now $PC^2 = 1^2 + 2^2 - 2 \times 1 \times 2 \times \cos\theta = 5 - 4\cos\theta$

$\therefore PC = \sqrt{5 - 4\cos\theta}$

$AP = \frac{2\pi \times r}{2} - r\theta = \pi - \theta$

$AC = 3$.

$\therefore L = 3 + \pi - \theta + \sqrt{5 - 4\cos\theta}$

iv) $L' = -1 + \frac{1}{2}(5 - 4\cos\theta)^{-1/2} \times 4\sin\theta$

$= -1 + \frac{2\sin\theta}{\sqrt{5 - 4\cos\theta}}$

$\therefore L' = 0 \Rightarrow \sqrt{5 - 4\cos\theta} = 2\sin\theta$

$5 - 4\cos\theta = 4\sin^2\theta$

$5 - 4\cos\theta = 4(1 - \cos^2\theta)$

$\therefore (2\cos\theta - 1)^2 = 0 \Rightarrow \cos\theta = 1/2 \Rightarrow \theta = \pi/3$.

Hence only 1 stat pt at $\theta = \pi/3$.

v) TESTING L' either side of $\theta = \pi/3$

θ	$\pi/6$	$\frac{\pi}{3}$	$\pi/2$
L'	−	0	−

gives a Horizontal Point of inflexion + shows the curve is a decreasing function

$\therefore$ greatest value occurs when $\theta = 0$

$\therefore L_{MAX} = 4 + \pi$.