2009

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

MATHEMATICS EXTENSION 2

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using blue or black pen
- Board approved calculators may be used
- Write your name on each page
- Each question is to be started on a new page.
- This examination paper must NOT be removed from the examination room

- There is a total of eight questions.
- Each question is worth 15 marks.
- Marks may be deducted for careless or badly arranged work.

Question 1 (15 marks) Start a NEW page. **Marks**

a) Find $\int \cos x \sin^4 x \, dx$. 1

b) Find $\int \frac{dx}{x^2 - 4x + 8}$. 2

c) Use the substitution $u = x - 2$ to find the exact value of $\int_1^3 x(x-2)^5 \, dx$. 3

d) i) Find the values of A, B and C so that 2

$$\frac{5}{(x^2+4)(x+1)} \equiv \frac{Ax+B}{x^2+4} + \frac{C}{x+1}.$$

ii) Hence find $\int \frac{5}{(x^2+4)(x+1)} \, dx$. 3

e) i) If $I_n = \int_1^e x(\ln x)^n \, dx$ for $n = 0,1,2,3,\ldots$ use integration by parts 2

to show that $I_n = \frac{e^2}{2} - \frac{n}{2} I_{n-1}$ for $n = 1,2,3,\ldots$

ii) Hence find the value of I_2. 2

Question 2 (15 marks) Start a NEW page. **Marks**

a) If $z = 1 - i$, find

i) $\bar{z}$ 1

ii) $|z|$ 1

iii) $\arg z$ 1

iv) $\arg iz$ 1

v) z^6 in $a + ib$ form. 2

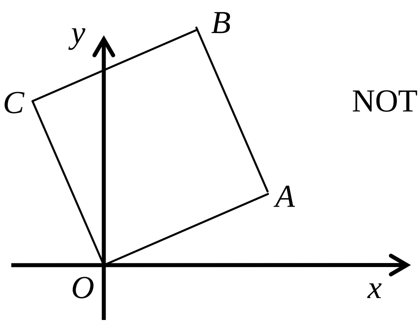
b) Express $\dfrac{i^5(1-i)}{2+i}$ in the form $a + ib$ where a and b are real. 3

c) Graph the region in the Argand diagram which simultaneously satisfies 3

$1 \le |z - i| \le 2$ and $\text{Im} z \ge 0$.

d) In the Argand diagram A represents the point $z_1 = \sqrt{3} + i$, and O is the origin.

Given that *OABC* is a square:



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i) Find the complex number (z_3) represented by C. 1

ii) Find the complex number (z_2) represented by B. 2

Question 3 (15 marks) Start a NEW page. **Marks**

a) i) Without using calculus, draw a good size neat sketch of $y=(x+1)^2(1-x)$. 2

ii) On a separate diagram using the same scale as above, and also without calculus, sketch $y^2=(x+1)^2(1-x)$, paying close attention to the shape of the curve as '*y*' approaches zero. 2

b) Neatly sketch each of the following on separate axes for $0\le x\le 2\pi$.

i) $y=\sin^2 x$. 1

ii) $y=|\sin x|$. 1

iii) $y=\sqrt{\sin x}$. 1

iv) $y=\dfrac{1}{\sin x}$. 1

v) $y=\dfrac{|\sin x|}{\sin x}$. 1

vi) $y=e^{\sin x}$. 2

c) A plane curve is defined by the equation $x^2+2xy+y^5=4$. 4

The curve has a horizontal tangent at the point $P(X,Y)$.

By using implicit differentiation or otherwise, show that X is the unique solution to the equation $X^5+X^2+4=0$.

[Do not solve this equation]

Question 4 (15 marks) Start a NEW page. **Marks**

a) Given the hyperbola $16x^2 - 9y^2 = 144$, find:

i) the length of the transverse axis 1

ii) the eccentricity 1

iii) the coordinates of the foci 1

iv) the equations of the directrices 1

v) the equations of the asymptotes. 1

b) If α, β and γ are the roots of the equation $x^3 + 6x + 1 = 0$, find the polynomial equation whose roots are $\alpha\beta, \beta\gamma$ and $\alpha\gamma$. 3

c) Consider the equation $x^4 - 5x^3 + 7x^2 + 3x - 10 = 0$.

i) Given that $2 - i$ is a root of the equation explain why $2 + i$ is also a root. 1

ii) Find the other roots of the equation. 3

d) The area enclosed between the curves $y = \sqrt{x}$ and $y = x^2$ is rotated about $y-$axis through one complete revolution. Use the method of cylindrical shells to find the volume of the solid that is generated. 3

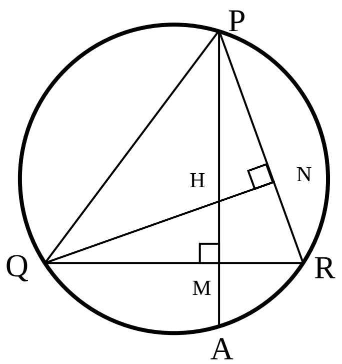
Question 5 (15 marks) Start a NEW page. **Marks**

a) Solve for x: $2^{3x+1} = 5^{x+1}$, correct to 3 significant figures. 2

b) i) Prove the result $\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} \left[f(x) + f(-x) \right] dx$ 2

ii) Hence use the above result to evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1+\sin x}\,dx$ 3

c) The altitudes PM and QN of an acute angled triangle PQR meet at H. PM produced cuts the circle PQR at A. [A larger diagram is included, use it and submit it with your solutions]



i) Explain why PQMN is a cyclic quadrilateral. 1

ii) Prove that HM = MA. 3

d) The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The tangent at P cuts the y- axis at B and M is the foot of the perpendicular from P to the y-axis.

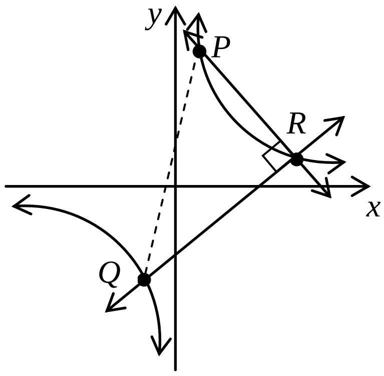
i) Show that the equation of the tangent to the ellipse at the point P is given by $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ 2

ii) Show that $OM \bullet OB = b^2$, where O is the origin. 2

Question 6 (15 marks) Start a NEW page. **Marks**

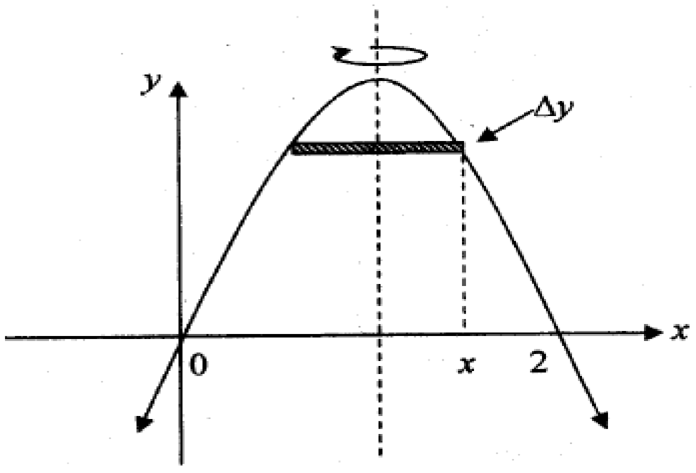
a) The points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ lie on the rectangular hyperbola $xy = c^2$. 4

The chord PQ subtends a right angle at another point $R\left(cr, \frac{c}{r}\right)$ on the hyperbola.



Show that the normal at R is parallel to PQ.

b) The area bounded by the curve $y = 2x - x^2$ and the x-axis is rotated through $180°$ about the line $x = 1$.



i) Show that the volume, ΔV, of a representative horizontal slice of width Δy is given by $\boldsymbol{\Delta V = \pi(x-1)^2\Delta y}$. 2

ii) Hence show that the volume of the solid of revolution is given by 2

$$\boldsymbol{V = \lim_{\Delta y \to 0} \sum_{y=0}^{1} \pi(1-y)\Delta y}$$

iii) Hence, find the volume of the solid of revolution. 2

Question 6 continued on page 8

Question 6 continued **Marks**

c) i) Show that $a^2 + b^2 > 2ab$, where a and b are distinct positive real numbers. 1

ii) Hence show that $a^2 + b^2 + c^2 > ab + bc + ca$, where a , b and c are distinct positive real numbers. 2

iii) Hence, or otherwise prove that $\dfrac{a^2b^2 + b^2c^2 + c^2a^2}{a+b+c} > abc$, where a , b and c are distinct positive real numbers. 2

End of Question 6

Question 7 (15 marks) Start a NEW page. **Marks**

a) Given that $P(x) = 3x^3 - 11x^2 + 8x + 4$ has a double root, fully factorise $P(x)$. 3

b) Show that $\tan^{-1}x > x - \frac{1}{3}x^3$ for all values of $x > 0$. 3

c) The acceleration of a particle which is moving along the x-axis is given by

$$\frac{d^2x}{dt^2} = 2x^3 - 10x.$$

i) If the particle starts at the origin with velocity u show that its velocity v is given by $v^2 - u^2 = x^4 - 10x^2$. 2

ii) If $u = 3$ show that the particle oscillates within the interval $-1 \leq x \leq 1$. 4

iii) Is the motion referred to in (ii) an example of simple harmonic motion? Give a clear reason for your answer. 1

iv) If $u = 6$, carefully describe the motion. 2

Question 8 (15 marks) Start a NEW page. **Marks**

a) Let ω be one of the non-real cube roots of unity.

i) Show that $1 + \omega + \omega^2 = 0$. 1

ii) Hence find the value of $(2 - \omega)(2 - \omega^2)(2 - \omega^4)(2 - \omega^5)$. 2

b) i) By using the expansion for $\cos(A + B)$, show that 2

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

ii) By using the substitution $x = 2\cos\theta$, solve the equation $x^3 - 3x = \sqrt{2}$. 3

iii) Hence explain why $\cos\left(\frac{\pi}{12}\right) + \cos\left(\frac{7\pi}{12}\right) + \cos\left(\frac{9\pi}{12}\right) = 0$ 1

c) The equation $x^3 - 3px + q = 0$, where $p > 0, q \neq 0$ are both real, has three distinct, non- zero real roots.

i) Show that the graph of $y = x^3 - 3px + q$ has a relative maximum value of $q + 2p\sqrt{p}$ and a relative minimum of $q - 2p\sqrt{p}$. 3

ii) Hence show giving reasons that $q^2 < 4p^3$. 3

END OF EXAM

YEAR 12 EXTENSION 2 CARINGBAH HIGH SCHOOL TRIAL HSC SOLUTIONS 2009

1a) $\int \cos x \sin^4 x \, dx = \frac{\sin^5 x}{5} + C$ ☑

b) $\int \frac{dx}{x^2 - 4x + 8} = \int \frac{dx}{(x-2)^2 + 2^2}$ ☑

$= \frac{1}{2}\tan^{-1}\left(\frac{x-2}{2}\right) + C$ ☑

c) Let $u = x - 2 \rightarrow x = u + 2 \rightarrow du = dx$
When $x = 1, u = -1; x = 3, u = 1$

$\therefore \ I = \int_{-1}^{1} (u+2).u^5 \, du$ ☑

$= \int_{-1}^{1} u^6 + 2u^5 \, du$

$= \left[\frac{u^7}{7} + \frac{2u^6}{6}\right]_{-1}^{1}$ ☑

$= \left(\frac{1}{7} + \frac{1}{3}\right) - \left(-\frac{1}{7} + \frac{1}{3}\right) = \frac{2}{7}$ ☑

d) i) $5 \equiv (Ax + B)(x + 1) + C(x^2 + 4)$

Let $x = -1 : 5 = 5C \rightarrow C = 1$ ☑

Let $x = 0 : \ 5 = B + 4C \rightarrow B = 1$
Let $x = 1 : \ 5 = 2(A + B) + 5C \rightarrow A = -1$ ☑

ii) $I = \int \frac{-x+1}{x^2+4} + \frac{1}{x+1} \, dx$

$= \int \frac{-x}{x^2+4} + \frac{1}{x^2+4} + \frac{1}{x+1} \, dx$ ☑

$= -\frac{1}{2} ln(x^2 + 4) + \frac{1}{2} tan^{-1}\left(\frac{x}{2}\right) + ln(x + 1) + c$ ☑☑

e) $I_n = \int_1^e x(lnx)^n \, dx$

Let $u = (lnx)^n, \quad v' = x$

$u' = \frac{n}{x}(lnx)^{n-1}, \quad v = \frac{x^2}{2}$ ☑

$\therefore \ I_n = \left[\frac{x^2}{2}(lnx)^n\right]_1^e - \int_1^e \frac{x^2}{2}.\frac{n}{x}(lnx)^{n-1} \, dx$

$= \frac{e^2}{2} - \frac{1^2}{2}.0^n - \frac{n}{2}I_{n-1} = \frac{e^2}{2} - \frac{n}{2}I_{n-1}$ ☑

e)ii) $I_2 = \frac{e^2}{2} - \frac{2}{2}I_1$

$= \frac{e^2}{2} - \left[\frac{e^2}{2} - \frac{1}{2}I_0\right]$ ☑

$= \frac{1}{2}\int_1^e x \, dx = \frac{e^2-1}{4}$ ☑

2a) i) $\bar{z} = 1 + i$ ☑

ii) $|z| = \sqrt{2}$ ☑

iii) $\arg z = -\frac{\pi}{4}$ ☑

iv) $\arg iz = \arg(i) + \arg z = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$ ☑

v) $z^6 = \left(\sqrt{2}cis\left(-\frac{\pi}{4}\right)\right)^6$ ☑

$= \sqrt{2}^6.cis\left(-\frac{6\pi}{4}\right)$ (*DMT*)

$= 8 \times i = 0 + 8i$ ☑

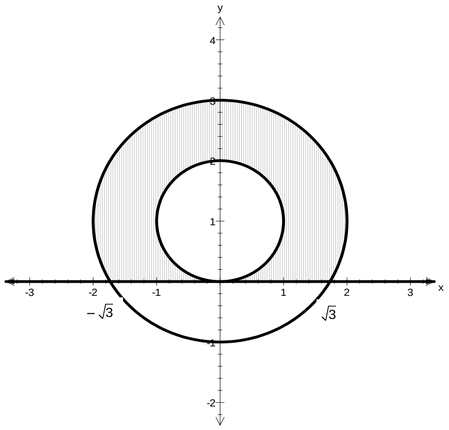
b) $\frac{i^5(1-i)}{2+i} = \frac{i.(i^2)^2(1-i)}{2+i} \times \frac{2-i}{2-i}$ ☑

$= \frac{i(1-i)(2-i)}{1+4}$ ☑

$= \frac{i(2-i-2i-1)}{5}$

$= \frac{3}{5} + \frac{1}{5}i$ ☑

2c)



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d) i) C is $z_3 = i(\sqrt{3} + i) = -1 + i\sqrt{3}$ ☑

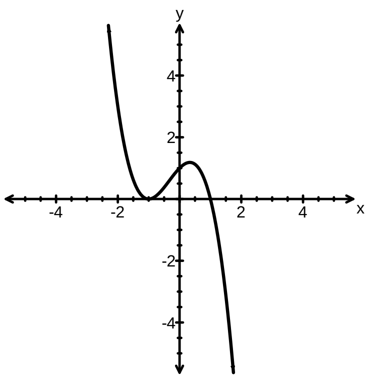
ii) B is $z_2 = \overrightarrow{OA} + \overrightarrow{AB}$

$= \overrightarrow{OA} + \overrightarrow{OC}$ ☑

$= z_1 + z_3$

$= (\sqrt{3} - 1) + i(1 + \sqrt{3})$ ☑

3a) i)



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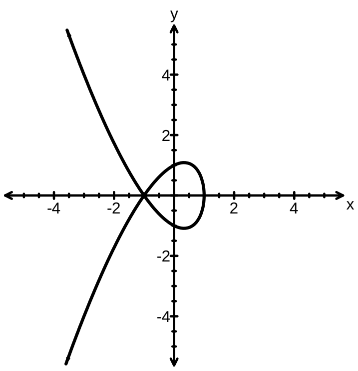
ii)



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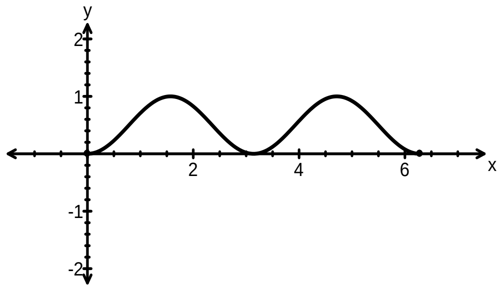
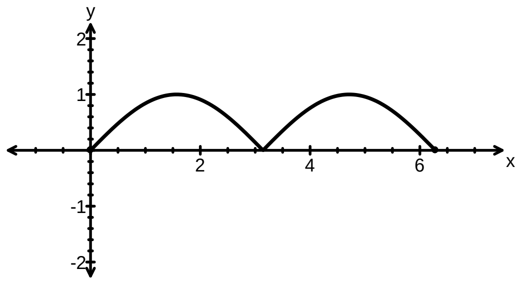
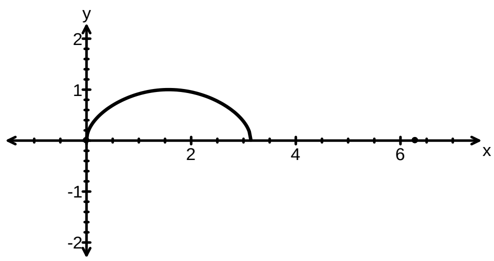
3b) i) ☑



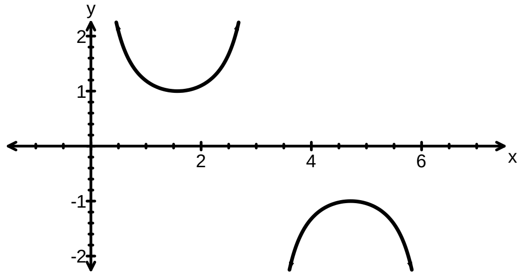
ii) ☑



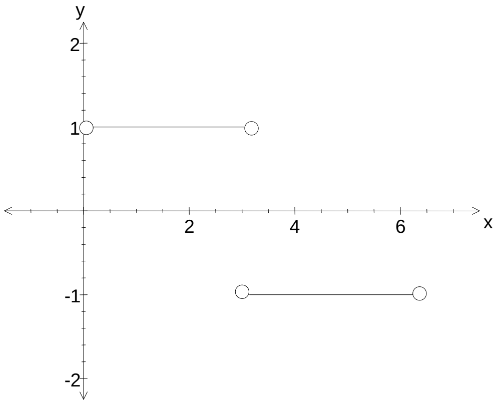
iii) ☑



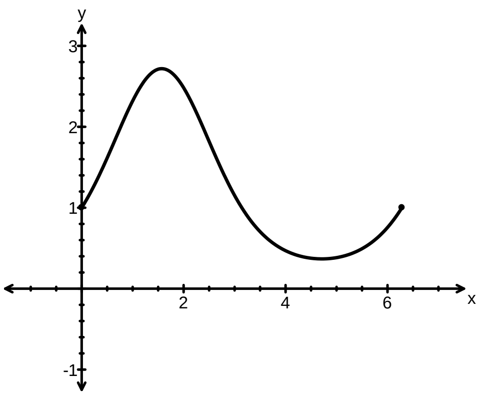
iv) ☑



3b) v) ☑



vi) ☑☑



c) Horizontal tangent at $x = X$ $\rightarrow$ $\frac{dy}{dx} = 0$.

$$\frac{d}{dx}\left(x^2 + 2xy + y^5\right) = \frac{d}{dx}(4)$$

$$2x + 2\left(y \quad \frac{dy}{dx}\right) \quad \frac{dy}{dx}$$ ☑

$$\left(2x + 5y^4\right)\frac{dy}{dx} = -2(x+y)$$

$$\therefore \quad \frac{dy}{dx} = \frac{-2(x+y)}{\left(2x+5y^4\right)}$$ ☑

$\therefore\ -2(x+y) = 0 \ \rightarrow y = -x$ where $x = X$ ☑

$$\therefore\ X^2 + 2X(-X) + (-X)^5 = 4$$

$$X^2 - 2X^2 - X^5 = 4$$ ☑

$$\therefore\ X^5 + X^2 + 4 = 0$$

4a) $\frac{x^2}{9} - \frac{y^2}{16} = 1 \ \rightarrow \ a = 3,\ b = 4$

i) transverse axis $= 6$ ☑

ii) $e^2 = 1 + \frac{b^2}{a^2} = \frac{25}{9}$

$\therefore \ e = \frac{5}{3}$ ☑

iii) $ae = 3 \times \frac{5}{3} = 5$

$\therefore$ Foci are $S(5,0)$ and $S'(-5,0)$ ☑

iv) $\frac{a}{e} = \frac{3}{5/3} = \frac{9}{5}$

$\therefore$ directrices are $x = \pm\frac{9}{5}$ ☑

v) $\frac{b}{a} = \frac{4}{3} \ \rightarrow$ asymptotes $y = \pm\frac{4}{3}x$ ☑

b) Let the roots be: $\alpha\beta, \beta\gamma, \alpha\gamma$

which is the equivalent of: $\frac{\alpha\beta\gamma}{\gamma}, \frac{\beta\gamma\alpha}{\alpha}, \frac{\alpha\gamma\beta}{\beta}$

and since $\alpha\beta\gamma = -\frac{d}{a} = -1$

then the roots are $-\frac{1}{\alpha}, -\frac{1}{\beta}, -\frac{1}{\gamma}$. ☑

Let $x = -\frac{1}{\alpha} \ \rightarrow \ \alpha = -\frac{1}{x}$

and since α satisfies the original equation then we get

$\left(-\frac{1}{x}\right)^3 + 6\left(-\frac{1}{x}\right) + 1 = 0$ ☑

which gives $x^3 - 6x^2 - 1 = 0$. ☑

c) i) Since the coefficients are real, the complex roots occur in conjugate pairs. ☑

ii) Let $P(x) = \big(x - (2 - i)\big)\big(x - (2 + i)\big)Q(x)$

$= (x^2 - 4x + 5)(x^2 + ax + b)$ ☑

$\therefore \ ax^3 - 4x^3 = -5x^3 \ \rightarrow \ a = -1$

also $5b = -10 \ \rightarrow \ b = -2$ ☑

$\therefore \ P(x) = (x^2 - 4x + 5)(x^2 - x - 2)$

$= (x^2 - 4x + 5)(x - 2)(x + 1)$

Hence the other roots are $x = 2$ and $x = -1$. ☑

4d) $V_s = 2\pi xy\delta x$

$\therefore \quad V = \lim_{\delta x \to 0} \sum_{x=0}^{1} 2\pi x y \delta x$ ☑

$= 2\pi \int_0^1 x(\sqrt{x} - x^2)\, dx$ ☑

$= 2\pi \int_0^1 x^{\frac{3}{2}} - x^3\, dx$

$= 2\pi \left[\frac{2x^{\frac{5}{2}}}{5} - \frac{x^4}{4}\right]_0^1$

$= \frac{3\pi}{10}\ u^3$ ☑

5a) $\ln\left(2^{3x+1}\right) = \ln\left(5^{x+1}\right)$

$\therefore \quad (3x+1)\ln 2 = (x+1)\ln 5$ ☑

$3x\ln 2 + \ln 2 = x\ln 5 + \ln 5$

$x(3\ln 2 - \ln 5) = \ln 5 - \ln 2$

$x = \dfrac{\ln 5 - \ln 2}{3\ln 2 - \ln 5} \approx 1.95$ ☑

b) i) $I = \int_{-a}^{a} f(x)\, dx$

$= \int_{-a}^{0} f(x)\, dx + \int_0^a f(x)\, dx$

Let $x = -u \rightarrow \quad dx = -du$

When $x = -a, u = a; x = 0, u = 0$ ☑

$\therefore \quad \mathrm{I} = -\int_a^0 f(-u)\, du + \int_0^a f(x)\, dx$

$= \int_0^a f(-u)\, du + \int_0^a f(x)\, dx$

$= \int_0^a f(-x)\, dx + \int_0^a f(x)\, dx$ ☑

$= \int_0^a [f(x) + f(-x)]\, dx$

ii)

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1+\sin x}\, dx = \int_0^{\frac{\pi}{4}} \frac{1}{1+\sin x} + \frac{1}{1+\sin(-x)}\, dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{1+\sin x} + \frac{1}{1-\sin x}\, dx$$ ☑

$$= \int_0^{\frac{\pi}{4}} \frac{1-\sin x+1+\sin x}{1-\sin^2 x}\, dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{2}{\cos^2 x}\, dx$$ ☑

$$= 2\int_0^{\frac{\pi}{4}} \sec^2 x\, dx$$

$$= 2\left[\tan x\right]_0^{\frac{\pi}{4}} = 2$$ ☑

c) i) $\angle PNQ = \angle PMQ = 90^\circ$ [$\angle' s$ in same segment] ☑

ii) Join QA and AR.
Let ∠RPA = α = ∠RQA
[∠′s in same segment in cyclic quad PQAR] ☑
Also ∠NPM = α = ∠NQM
[∠′s in same segment in cyclic quad PQMN] ☑

Also since ∠HMQ = ∠AMQ = 90°
and since QM is a common side
then ΔQHM ≡ ΔQAM [AAS] ☑
Hence HM = MA [corres sides in cong Δ′s]

d) i) $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{1}{\frac{dx}{d\theta}}$

$\therefore \quad m = \frac{dy}{dx} = b\cos\theta \times \frac{1}{-a\sin\theta} = -\frac{b\cos\theta}{a\sin\theta}$ ☑

$\therefore \quad y - b\sin\theta = -\frac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

$ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$

$\therefore \quad bx\cos\theta + ay\sin\theta = ab(\sin^2\theta + \cos^2\theta)$

$bx\cos\theta + ay\sin\theta = ab$ $[\div ab]$ to get ☑

$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1.$

ii) M has coordinates $(0, b\sin\theta)$
In the equation in (i) let $x = 0$ to give
the coordinates of B $\left(0, \frac{b}{\sin\theta}\right)$. ☑

$\therefore OM \cdot OB = |b\sin\theta| \times \frac{b}{|\sin\theta|} = b^2$ ☑

6a) $m_{PQ} = \dfrac{\frac{c}{q} - \frac{c}{p}}{cq - cp} = \dfrac{c(p-q)}{pq} \times \dfrac{1}{c(q-p)} = \dfrac{-1}{pq}$ ☑

Since $\angle PRQ = 90^\circ$ then $m_{PR} \times m_{QR} = -1$

$\therefore \frac{-1}{pr} \times \frac{-1}{qr} = -1 \quad \rightarrow \quad r^2 = -\frac{1}{pq}$ ☑

Now $xy = c^2 \quad \rightarrow \quad y' = -\frac{c^2}{x^2}$

Hence at R the gradient of the tangent $= \frac{-1}{r^2}$

$\therefore$ at R the gradient of the normal $= r^2$ ☑

but $r^2 = -\frac{1}{pq} = m_{PQ}$ ☑

$\therefore$ PQ parallel to the normal at R.

bi) Let r be the radius of a typical slice

$\therefore \; r + 1 = x \quad \rightarrow \quad r = x - 1$ ☑

Now $\Delta V = \pi r^2 h = \pi(x-1)^2 \Delta y$ ☑

ii) When $x = 1, y = 2 - 1 = 1$ ☑

$\therefore \; V = \lim_{\Delta y \to 0} \sum_{y=0}^{1} \pi(x-1)^2 \Delta y$

Now $\quad -y = x^2 - 2x$

$\therefore \quad 1 - y = x^2 - 2x + 1$ ☑

$\therefore \quad 1 - y = (x-1)^2$

hence $\therefore \; V = \lim_{\Delta y \to 0} \sum_{y=0}^{1} \pi(1-y)\Delta y$

iii) $\quad V = \pi \int_0^1 1 - y \; dy$

$= \pi\left[y - \frac{y^2}{2}\right]_0^1$ ☑

$= \frac{\pi}{2} \quad u^3$ ☑

c) i) $(a-b)^2 > 0 \quad (a \neq b)$

$\therefore \quad a^2 - 2ab + b^2 > 0$ ☑

$\therefore \quad a^2 + b^2 > 2ab$ -------$\boxed{1}$

ii) Similarly $\; b^2 + c^2 > 2bc$ -------$\boxed{2}$

and $\quad c^2 + a^2 > 2ca$ -------$\boxed{3}$ ☑

Now $\boxed{1} + \boxed{2} + \boxed{3}$ gives

$2(a^2 + b^2 + c^2) > 2(ab + bc + ca)$ ☑

$\therefore \quad a^2 + b^2 + c^2 > ab + bc + ca$

iii) Using the result in (ii)

Let $a \rightarrow ab; \;\; b \rightarrow bc; \;\; c \rightarrow ca.$

$\therefore \quad a^2b^2 + b^2c^2 + c^2a^2 > ab.bc + bc.ca + ca.ab$ ☑

$a^2b^2 + b^2c^2 + c^2a^2 > abc(a + b + c)$ ☑

$\therefore \quad \frac{a^2b^2 + b^2c^2 + c^2a^2}{a+b+c} > abc$

7a) $\quad P(x) = 3x^3 - 11x^2 + 8x + 4$

$P'(x) = 9x^2 - 22x + 8$

$P'(x) = 0$ for stationary points.

$\therefore (9x-4)(x-2) = 0 \quad \rightarrow \quad x = 2$ or $x = \frac{9}{4}$. ☑

Since $\;P(2) = 24 - 44 + 16 + 4 = 0$

then $\; x = 2 \;$ is the double root. ☑

$\therefore \; P(x) = (x-2)^2(3x+1)$ ☑

b) Let $P(x) = \tan^{-1}x - x + \frac{1}{3}x^3$

$\therefore \quad P'(x) = \frac{1}{1+x^2} - 1 + x^2$ ☑

$= \frac{1-(1+x^2)+x^2(1+x^2)}{1+x^2}$

$= \frac{x^4}{1+x^2} > 0$ for $x > 0$ ☑

$\therefore P(x)$ is an increasing function for $x > 0$

and since $P(0) = 0$ then $\tan^{-1}x > x - \frac{1}{3}x^3$ ☑

c) i) $\quad \frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x^3 - 10x$

$\therefore \quad \frac{1}{2}v^2 = \frac{x^4}{2} - 5x^2 + c$ ☑

when $x = 0, \; v = u \quad \rightarrow \quad c = \frac{1}{2}u^2$

$\therefore \quad \frac{1}{2}v^2 = \frac{x^4}{2} - 5x^2 + \frac{1}{2}u^2$ ☑

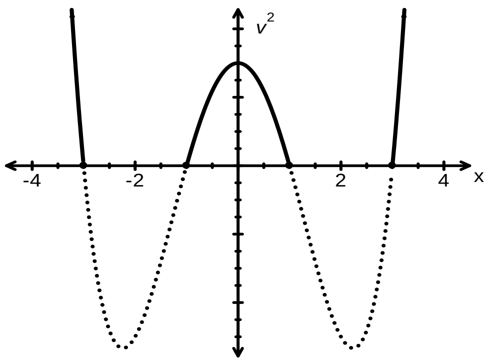
$\therefore \quad v^2 - u^2 = x^4 - 10x^2$

7cii) If $u = 3$ then $v^2 - 9 = x^4 - 10x^2$

$\therefore \quad v^2 = x^4 - 10x^2 + 9$

$= (x^2 - 1)(x^2 - 9)$

$= (x-1)(x+1)(x-3)(x+3)$ ☑



Since $v^2 \geq 0$ for motion to exist then $(x-1)(x+1)(x-3)(x+3) \geq 0$ and since the particle starts at $x = 0$ with $v = 3$ it is moving to the right. ☑

At $x = 1, v = 0$ and $\ddot{x} = -8$ and so the particle will move to the left until it reaches $x = -1$ where $v = 0$ and $\ddot{x} = 8$. This means the particle will then move to the right until $v = 0$ again at $x = 1$. ☑☑

Thus it oscillates in the interval $-1 \leq x \leq 1$.

iii) Not SHM since $\ddot{x} \neq -n^2(x - b)$ ☑

iv) When $u = 6$, $\quad v^2 = x^4 - 10x^2 + 36$

$= (x^2 - 5)^2 + 11$

$\therefore \quad v^2 > 0$ for all x in the domain, hence the particle will never stop. ☑

It moves to the right with decreasing velocity until it passes $x = \sqrt{5}$ $\left(\text{with } v = \sqrt{11}\right)$ and continues to the right with increasing velocity. ☑

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8 a) i) Let ω be a solution *to* $z^3 = 1$

$\therefore \quad z^3 - 1 = 0 \;\rightarrow\; (z-1)(z^2 + z + 1) = 0$

but as ω is a non real root, then

$\omega^2 + \omega + 1 = 0.$ ☑

ii) Now $\omega^3 = 1$, hence

$(2-\omega)(2-\omega^2)(2-\omega^4)(2-\omega^5)$ reduces to

$(2-\omega)(2-\omega^2)(2-\omega)(2-\omega^2)$ ☑

$=(2-\omega)^2(2-\omega^2)^2$

$= [(2-\omega)(2-\omega^2)]^2$

$= [4 - 2\omega^2 - 2\omega + \omega^3]^2$

$= [5 - 2(\omega^2 + \omega)]^2$

$= [5 - 2 \times -1]^2 = 49$ ☑

8 b) i) $\cos(3\theta) = \cos(\theta + 2\theta)$

$= \cos\theta\cos 2\theta - \sin\theta\sin 2\theta$

$= \cos\theta\,(2\cos^2\theta - 1) - \sin\theta\, 2\sin\theta\cos\theta$ ☑

$= 2\cos^3\theta - \cos\theta - 2\cos\theta\,(1 - \cos^2\theta)$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$ ☑

$= 4\cos^3\theta - 3\cos\theta.$

ii) $x^3 - 3x = \sqrt{2}$ and let $x = 2\cos\theta$ to obtain

$8\cos^3\theta - 6\cos\theta = \sqrt{2}$

$\therefore 4\cos^3\theta - 3\cos\theta = \frac{1}{\sqrt{2}}$

$\therefore \cos 3\theta = \frac{1}{\sqrt{2}}$ ☑

$\therefore \quad 3\theta = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \ldots\ldots$

$\therefore \quad \theta = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{9\pi}{12}, \ldots\ldots$ ☑

hence $x = 2\cos\frac{\pi}{12}, 2\cos\frac{7\pi}{12}, 2\cos\frac{9\pi}{12}$ ☑

iii) The sum of the roots of $x^3 - 3x - \sqrt{2} = 0$ is given by $-\frac{b}{a} = -\frac{0}{1} = 0.$

$\therefore \quad 2\cos\frac{\pi}{12} + 2\cos\frac{7\pi}{12} + 2\cos\frac{9\pi}{12} = 0$ ☑

$\therefore \quad \cos\frac{\pi}{12} + \cos\frac{7\pi}{12} + \cos\frac{9\pi}{12} = 0$

c) i) $y = x^3 - 3px + q,\ y' = 3x^2 - 3p,\ y'' = 6x$

For stat pts $y' = 0 \;\rightarrow\; 3(x^2 - p) = 0$

$\therefore \quad x = \pm\sqrt{p}$ ☑

When $x = \sqrt{p}, y = p\sqrt{p} - 3p\sqrt{p} + q = q - 2p\sqrt{p}$

When $x = \sqrt{p}, y'' = 6\sqrt{p} > 0 \;\rightarrow$ minimum. ☑

When $x = -\sqrt{p}, y = -p\sqrt{p} + 3p\sqrt{p} + q$

$= q + 2p\sqrt{p}$

When $x = -\sqrt{p}, y'' = -6\sqrt{p} < 0 \;\rightarrow$ maximum. ☑

ii) Since $x^3 - 3px + q = 0$ has 3 distinct non zero real roots then the turning points must be on either side of the $x -$ axis. ☑

$\therefore \quad$ minimum $\times$ maximum < 0 ☑

$\therefore \quad \left(q + 2p\sqrt{p}\right)\left(q - 2p\sqrt{p}\right) < 0$

$q^2 - 4p^2 \cdot p < 0$ ☑

hence $q^2 < 4p^3$