# CARINGBAH HIGH SCHOOL

# 2011

# TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

# Mathematics Extension 2

**General Instructions**

| | |
|---|---|
| Reading time - 5 minutes | **Total marks - 120** |
| Working time - 3 hours | Attempt Questions 1 - 8 |
| Write using black or blue pen. | All questions of equal value. |
| Board-approved calculators may be used. | All necessary working should be shown in every question. |
| A table of standard integrals is provided at the back of this paper. | |

**Question 1** (15 marks) **Marks**

(a) Find $\int \frac{dx}{(2x+1)^3}$ **2**

(b) Using integration by parts find the exact value of $\int_0^{\frac{1}{2}} \cos^{-1}x \, dx$. **3**

(c) Use the substitution $u = x - 1$ to find $\int \frac{x}{\sqrt{x-1}} dx$ **3**

(d) Use the substitution $t = \tan\frac{x}{2}$ to find the exact value of **4**

$$\int_0^{\frac{\pi}{2}} \frac{1}{\sin x + 2} dx$$

(e) Evaluate $\int_0^1 \frac{5}{(2x+1)(2-x)} dx$ **3**

**Question 2** (15 marks) Start a new page.

(a) Find the complex square roots of $7+6\sqrt{2}\,i$ giving your answer in the form $x+iy$ where $x$ and $y$ are real. **2**

(b) If $z=3+i$ find $\frac{i}{z}$ in the form $x+iy$. **2**

(c) Let $z_1=3+6i$ and $z_2=-3-6i$.

Show that the locus specified by $|z-z_1| = 2|z-z_2|$ is a circle. Give its centre and radius. **2**

(d) (i) Express $-1+i$ in modulus-argument form. **1**

(ii) Express $(-1+i)^6$ in the form $x+iy$. **2**

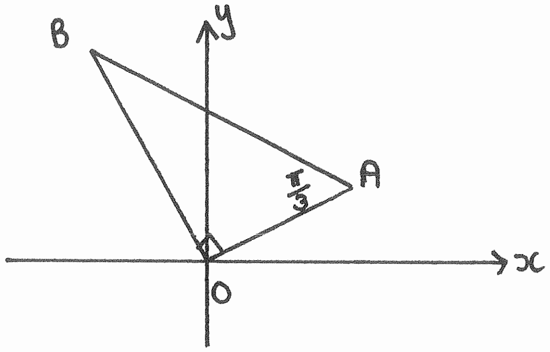
(e) Sketch the locus of $z$ satisfying:

(i) $\arg(z+2) = \frac{\pi}{4}$. **2**

(ii) $\text{Re}(z) = |z|$. **2**

**Question 2 continues on page 4**

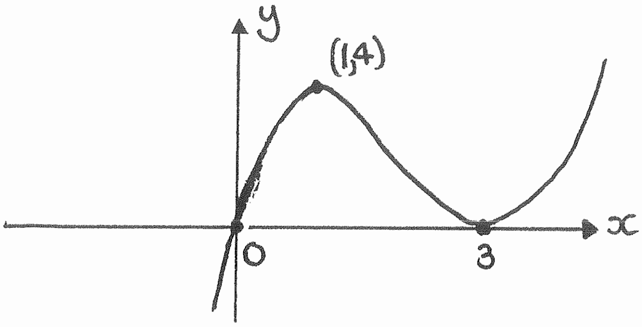
(f) In the diagram below, the points $A$ and $B$ correspond to the complex numbers $z$ and $w$ respectively. $\angle AOB$ is a right angle and $\angle BAO = \dfrac{\pi}{3}$.



Show that $3z^2 + w^2 = 0$. **2**

**Question 3** (15 marks) Start a new page.

(a) The function defined by $g(x) = x(x-3)^2$ is drawn below.



Draw separate, one-third page sketches of :

(i) $y = g(|x|)$ **1**

(ii) $y = \dfrac{1}{g(x)}$ **2**

(iii) $y = \sqrt{g(x)}$ **2**

(iv) $y = \tan^{-1}[g(x)]$ **2**

(b) For the curve $x^2 + y^2 + xy - 4 = 0$:

(i) Find the $x$ and $y$ intercepts. 2,±2 **1**

(ii) Using implicit differentiation show that $\dfrac{dy}{dx} = -\dfrac{2x+y}{x+2y}$ **2**

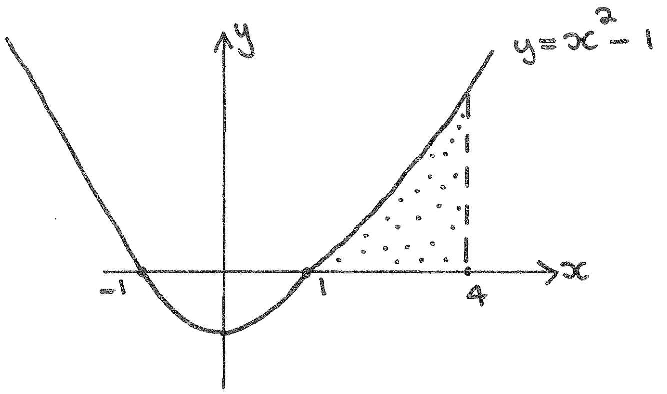
(iii) Find any stationary points on the curve. $\left(\frac{2}{\sqrt{3}}, -\frac{4}{\sqrt{3}}\right)$ $\left(-\frac{2}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right)$ **2**

(iv) Deduce that the curve has vertical tangents at the points where $x = \pm\dfrac{4}{\sqrt{3}}$. $y = 2/\sqrt{3}$ **2**

(v) Sketch the curve $x^2 + y^2 + xy - 4 = 0$. **1**

**Question 4** (15 marks) Start a new page.

(a) **4**



The area bounded by the curve $y = x^2 - 1$, the $x$-axis and the line $x = 4$, as shown in the diagram, is rotated about the $y$-axis to form a solid. Use the method of cylindrical shells to find the volume of the solid.

(b) Sketch the graph of the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$ showing the intercepts on the axes, the coordinates of the foci and the equations of the directrices. **4**

(c) The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with $a > b > 0$ has eccentricity $e$.

(i) Show that the line through the focus $S(ae, 0)$ which is perpendicular to the asymptote $y = \frac{bx}{a}$ has equation $ax + by - a^2e = 0$. **1**

(ii) Show that this line meets the asymptote at a point on the corresponding directrix. **3**

(d) Consider the polynomial $P(x) = x^3 - x^2 + x + 39$.

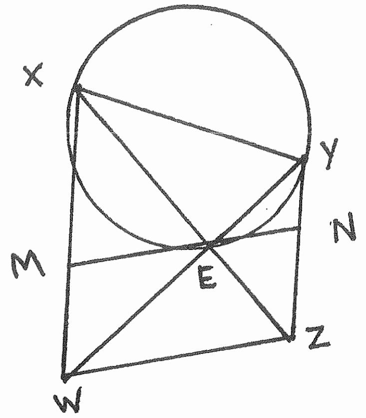
(i) Find the rational zero of $P(x)$. **1**

(ii) Find the complex zeros of $P(x)$. **2**

**Question 5** (15 marks) Start a new page.

(a) In the diagram below, $XYZW$ is a cyclic quadrilateral whose diagonals intersect at $E$. A circle is drawn through $X$, $Y$ and $E$. $MN$ is a tangent to this circle at $E$ with $M$ and $N$ lying on $XW$ and $YZ$ respectively. **3**

Copy this diagram.



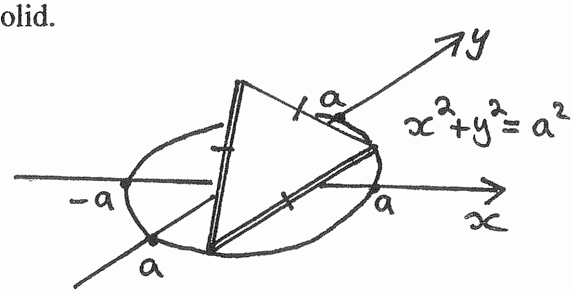
Prove that $MN$ is parallel to $WZ$.

(b) Suppose that $p$ and $q$ are real numbers.

(i) Show that $pq \leq \frac{p^2+q^2}{2}$. **2**

(ii) Hence show that for $x$ and $y$ real numbers $\frac{1}{xy} \leq \frac{x^2+y^2}{2x^2y^2}$. **2**

(c) The base of a certain solid is the circle $x^2 + y^2 = a^2$. **3**
Each cross-section of the solid is an equilateral triangle parallel to the $y$-axis with one side lying on the circle, as shown in the diagram.
Find the volume of the solid.



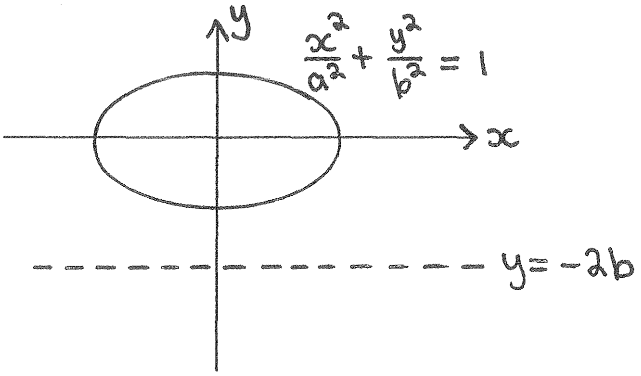
**Question 5 continues on page 8**

(d) The complex cube roots of unity $\omega$ , $\omega^2$ are two of the roots of $P(x) = x^3 + px^2 + qx + r$.
Show that $p = q = r + 1$. **3**

(e) Resolve $\dfrac{1}{(x-3)(x^2+1)}$ into partial fractions over the field of real numbers. **2**

**Question 6** (15 marks) Start a new page.

(a) The area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is rotated about the line $y=-2b$. A strip of thickness $\delta x$ perpendicular to the axis of rotation sweeps out a slice whose cross-section is an annulus.



(i) Show that this slice has a volume of $\delta V = 8\pi b y \delta x$. **2**

(ii) Hence find the volume of the solid which is formed. **3**

(b) (i) If $I_n = \int_{-1}^{0} x^n (1+x)^{\frac{1}{2}}\,dx$ show that $I_n = -\frac{2n}{2n+3} I_{n-1}$. **3**

(ii) Hence evaluate $I_3$. **2**

(c) By differentiating both sides of the formula: **3**

$1+x+x^2+x^3+...+x^n=\frac{x^{n+1}-1}{x-1}$ find an expression for:

$1+2\times2+3\times4+4\times8+...+n2^{n-1}$.

(d) Given that $1-2i$ is a zero of the polynomial $p(x)=x^3-5x^2+11x-15$ factorise $p(x)$ over the field of complex numbers. **2**

**Question 7** (15 marks) Start a new page.

(a) The normal at the point $P\left(cp, \frac{c}{p}\right)$ on the hyperbola $xy = c^2$ meets the $x$-axis at $Q$. $M$ is the midpoint of $PQ$.

(i) Show that the normal at $P$ has the equation $p^3x - py = c(p^4 - 1)$. **2**

(ii) Show that $M$ has coordinates $\left(\frac{c(2p^4 - 1)}{2p^3}, \frac{c}{2p}\right)$ **3**

(iii) Hence or otherwise, find the equation of the locus of $M$. **2**

(b) The numbers $a$, $b$ and $c$ are said to be in harmonic progression if their reciprocals $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in arithmetic progression, and $b$ is then said to be the harmonic mean of $a$ and $c$.

(i) Show that the numbers $6, 8$ and $12$ are in harmonic progression. **1**

(ii) Show that the harmonic mean of $a$ and $c$ is $\frac{2ac}{a+c}$. **2**

(iii) If $a > 0, c > 0$ show that the geometric mean $\sqrt{ac}$ is greater than or equal to the harmonic mean $\frac{2ac}{a+c}$. **2**

(c) (i) Sketch $y = x^2 - 2x - 1$ showing the $x$-intercepts. **1**

(ii) Using mathematical induction and part (i) prove that $2^n > n^2$ for all integers $n \geq 5$. **2**

**Question 8** (15 marks) Start a new page.

(a) The roots of the equation $x^3+px+m=0$ where $m \neq 0$ are $\alpha, \beta$ and $\delta$. Find an equation expressed in the form $ax^3+bx^2+cx+d=0$ whose roots are $\alpha^{-2}, \beta^{-2}$ and $\delta^{-2}$. 2

(b) The vertices of a quadrilateral *ABCD* lie on a circle radius *r*. The angles subtended at the centre of the circle by sides *AB*, *BC*, *CD* and *DA* are respectively in an arithmetic progression with first term *a* and common difference *d*. (i.e. *AB* subtends an angle of $a$).

(i) Show that $2a + 3d = \pi$ and interpret this result geometrically. 2

(ii) Show that the area of the quadrilateral *ABCD* is $2r^2 \cos d \cos\frac{d}{2}$. 3

[If required you may use the result: $\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$]

(c) Let $p = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$.
The complex number $\alpha = p + p^2 + p^4$ is a root of the quadratic equation $x^2 + ax + b = 0$, where *a* and *b* are real.

(i) Prove that $1 + p + p^2 + ... + p^6 = 0$. 2

(ii) The second root of the quadratic equation is $\beta$. Justifying your answer, express $\beta$ in terms of positive powers of *p*. 2

(iii) Find the values of the coefficients *a* and *b*. 2

(iv) Deduce that $-\sin\frac{\pi}{7} + \sin\frac{2\pi}{7} + \sin\frac{3\pi}{7} = \frac{\sqrt{7}}{2}$. 2

**End of paper**

# Ext.2 Trial HSC 2011

1. (a) $\int (2x+1)^{-3}\,dx = -\frac{1}{2}\cdot\frac{1}{2}(2x+1)^{-2}$

$= \frac{-1}{4(2x+1)^2} + c$

(b) $u = \cos^{-1}x$ $\quad \frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{-1}{\sqrt{1-x^2}}$ $\quad v = x$

[WE USE INTEGRATION BY PARTS

$\int_0^{\frac{1}{2}} \cos^{-1}x\,dx = \left[x\cos^{-1}x\right]_0^{\frac{1}{2}} - \int x \cdot \frac{-1}{\sqrt{1-x^2}}\,dx$

$= \left(\frac{1}{2}\cos^{-1}\frac{1}{2} - 0.\cos^{-1}0\right) + \int_0^{\frac{1}{2}} x(1-x^2)^{-\frac{1}{2}}\,dx$

$= \frac{1}{2}\times\frac{\pi}{3} - \left[(1-x^2)^{\frac{1}{2}}\right]_0^{\frac{1}{2}}$

$= \frac{\pi}{6} - \left(\frac{\sqrt{3}}{2} - 1\right)$

$= 1 + \frac{\pi}{6} - \frac{\sqrt{3}}{2}$

(c) $u = x-1 \Leftrightarrow x = u+1$

$du = dx$

$\int \frac{u+1}{\sqrt{u}}\,du = \int u^{\frac{1}{2}} + u^{-\frac{1}{2}}\,du$

$= \frac{2}{3}u^{3/2} + 2u^{\frac{1}{2}} + c$

$= \frac{2}{3}(x-1)^{3/2} + 2(x-1)^{\frac{1}{2}} + c$

$= \frac{2}{3}\sqrt{(x-1)^3} + 2\sqrt{x-1} + c$

(d) $t = \tan\frac{x}{2}$ $\quad x=0,\ t=0$

$x = \frac{\pi}{2},\ t = 1$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$= \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)$

$= \frac{1}{2}(1+t^2)$ $\quad \therefore\ dx = \frac{2\,dt}{1+t^2}$

Now $\sin x = \frac{2t}{1+t^2}$ :

$\int_0^{\frac{\pi}{2}} \frac{1}{\sin x + 2}\,dx$

$= \int_0^1 \frac{1}{\frac{2t}{1+t^2}+2} \times \frac{2\,dt}{1+t^2}$

$= \int_0^1 \frac{2\,dt}{2t+2(1+t^2)}$

$= \int_0^1 \frac{dt}{t^2+t+1}$

$= \int_0^1 \frac{dt}{t^2+t+\frac{1}{4}+\frac{3}{4}}$

$= \int_0^1 \frac{dt}{(t+\frac{1}{2})^2+\frac{3}{4}}$

$= \int_0^{3/2} \frac{du}{u^2+\frac{3}{4}}$

[using $u = t+\frac{1}{2}$

$t=0,\ u=\frac{1}{2}$

$t=1,\ u=\frac{3}{2}$

$du = dt$

$= \frac{1}{\frac{\sqrt{3}}{2}}\left[\tan^{-1}\frac{u}{\frac{\sqrt{3}}{2}}\right]_{\frac{1}{2}}^{3/2}$

$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{2u}{\sqrt{3}}\right]_{\frac{1}{2}}^{3/2}$

$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\sqrt{3} - \tan^{-1}\frac{1}{\sqrt{3}}\right]$

$= \frac{2}{\sqrt{3}}\cdot\frac{\pi}{6}$

$= \frac{\pi}{3\sqrt{3}}$

(e) $\int_0^1 \frac{5\,dx}{(2x+1)(2-x)} = \int_0^1 \frac{2}{2x+1} + \frac{1}{2-x}\,dx$

$= \left[\ln|2x+1| - \ln|2-x|\right]_0^1$

$= \ln\overset{3}{5} + \ln 2$

$= \ln 1\cancel{0}6$

2. ⓐ $(x+iy)^2 = 7 + 6\sqrt{2}\,i$

$x^2 - y^2 = 7$ and $2xy = 6\sqrt{2}$

$xy = 3\sqrt{2}$

$x = \pm 3$ and $y = \pm\sqrt{2}$

ie. $3 + i\sqrt{2}$ and $-3 - i\sqrt{2}$

ⓑ $\frac{i}{3+i} \times \frac{3-i}{3-i} = \frac{3i - i^2}{9 - i^2}$

$= \frac{1+3i}{10}$

$= \frac{1}{10} + \frac{3i}{10}$

ⓒ $|(x+iy)-(3+6i)| = 2|(x+iy)-(-3-6i)|$

$|(x-3)+(y-6)i| = 2|(x+3)+(y+6)i|$

$\sqrt{(x-3)^2+(y-6)^2} = 2\sqrt{(x+3)^2+(y+6)^2}$

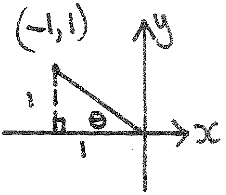
$(x-3)^2 + (y-6)^2 = 4[(x+3)^2+(y+6)^2]$

$\Downarrow$

$(x+5)^2 + (y+10)^2 = 80$

which is a circle with centre $(-5,-10)$ and radius $\sqrt{80}$

ⓓ ⓘ



$\theta = \frac{\pi}{4}$ ∴ $\arg z = \frac{3\pi}{4}$

$r = \sqrt{2}$

∴ $z = \sqrt{2}\,\text{cis}\,\frac{3\pi}{4}$

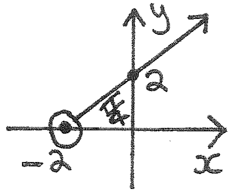
ⓘⓘ $z^6 = (\sqrt{2})^6\,\text{cis}\,\frac{9\pi}{2}$

$= 8\,\text{cis}\,\frac{\pi}{2}$

$= 8(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$

$= 0 + 8i$

ⓔ ⓘ



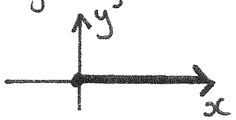
ⓘⓘ let $z = x+iy$

$\text{Re}(z) = |z|$

$x = \sqrt{x^2+y^2}$

$x^2 = x^2+y^2$

$y = 0,\ x \geq 0$



ⓕ $w = \sqrt{3}\,i\,z$

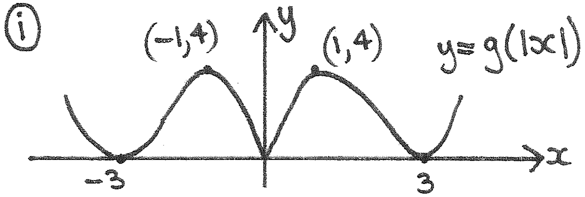
∴ $w^2 = 3i^2z^2$

$w^2 = -3z^2$

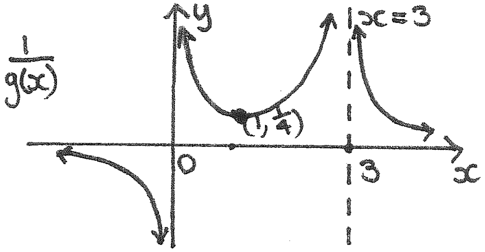
$w^2 + 3z^2 = 0$

$\left[\tan\frac{\pi}{3} = \frac{OB}{OA} \Rightarrow OB = \sqrt{3}OA\right]$

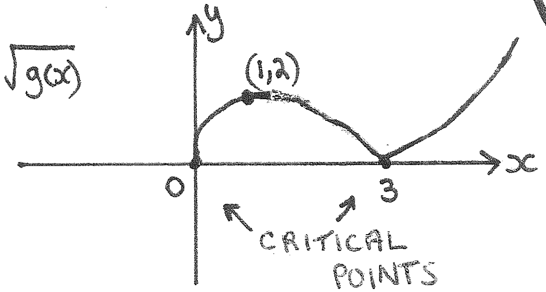
3 ⓐ ⓘ



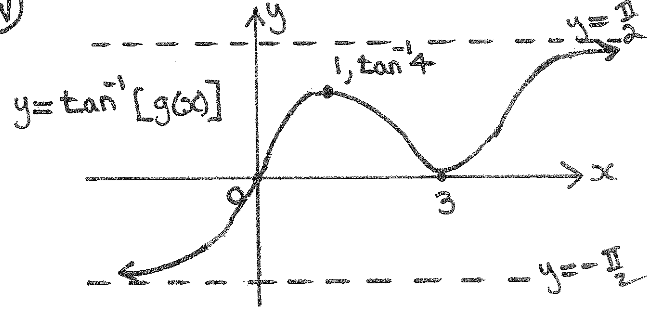
ⓘⓘ $y = \frac{1}{g(x)}$



ⓘⓘⓘ $y = \sqrt{g(x)}$



ⓘⓥ



ⓑ ⓘ x-ints: ±2  y-ints: ±2

ⓘⓘ $2x + 2y\frac{dy}{dx} + y.1 + x\frac{dy}{dx} = 0$

$(x+2y)\frac{dy}{dx} = -(2x+y)$

$\frac{dy}{dx} = \frac{-(2x+y)}{x+2y}$

ⓘⓘⓘ stat. points $\frac{dy}{dx} = 0$ : $2x + y = 0$

$y = -2x$

sub. $y = -2x$ into equn:

$x^2 + 4x^2 - 2x^2 - 4 = 0$

$3x^2 = 4$

$x = \pm\frac{2}{\sqrt{3}}$

∴ stat. points $\left(\frac{2}{\sqrt{3}}, \frac{-4}{\sqrt{3}}\right)$ and $\left(\frac{-2}{\sqrt{3}}, \frac{4}{\sqrt{3}}\right)$

(iv) vert. tangents : $\frac{dy}{dx}$ undefined

i.e. $x+2y=0$

$y=-\frac{1}{2}x$

sub $y=-\frac{1}{2}x$ into equn:

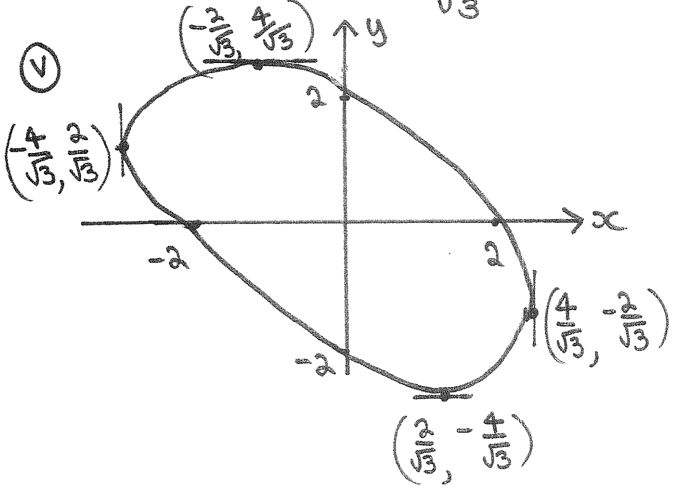
$$x^2+\frac{1}{4}x^2-\frac{1}{2}x^2-4=0$$

$$\frac{3}{4}x^2=4$$

$$x^2=\frac{16}{3}$$

$$x=\pm\frac{4}{\sqrt{3}}$$

(v)



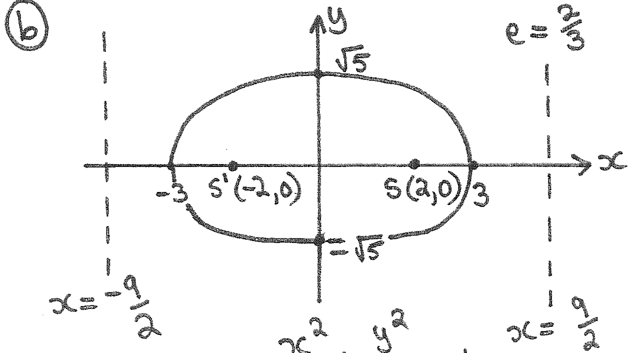
4. (a) $\delta V=2\pi x(x^2-1)\,\delta x$

$$V=2\pi\int_1^4 x^3-x\,dx$$

$$=2\pi\left[\frac{1}{4}x^4-\frac{1}{2}x^2\right]_1^4$$

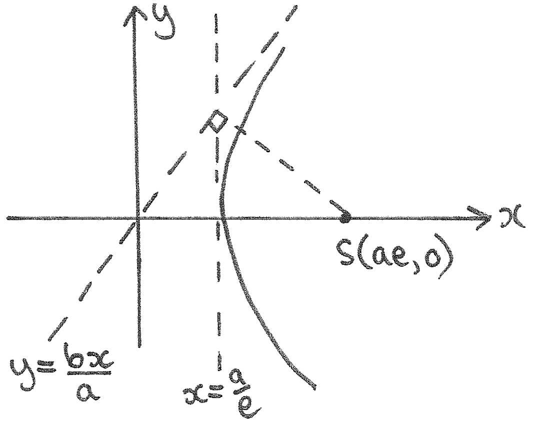
$$=\frac{225\pi}{2}\text{ units}^3$$

(b)



$$\frac{x^2}{9}+\frac{y^2}{5}=1$$

(c)



(i) gradient of line $=-\frac{a}{b}$

equn of line: $y-0=-\frac{a}{b}(x-ae)$

$$ax+by-a^2e=0$$

(ii) asymptote $y=\frac{b}{a}x$ and directrix $x=\frac{a}{e}$ meet at $\left(\frac{a}{e},\frac{b}{e}\right)$

sub $\left(\frac{a}{e},\frac{b}{e}\right)$ into equn of line:

$$LHS=a\times\frac{a}{e}+b\times\frac{b}{e}-a^2e$$

$$=\frac{a^2+b^2}{e}-a^2e$$

$$=\frac{a^2e^2}{e}-a^2e$$

$$=0$$

$$=RHS$$

[SINCE $b^2=a^2(e^2-1)$

$b^2=a^2e^2-a^2$

$a^2+b^2=a^2e^2$

(d) (i) $P(x)=(x+3)(x^2-4x+13)$

∴ ONLY RATIONAL ZERO IS $x=-3$

SINCE $x^2-4x+13=0$ HAS NO RATIONAL ROOTS

(ii) NOW FOR $x^2-4x+13=0$

$$x=\frac{4\pm\sqrt{16-4.1.13}}{2}$$

$$=\frac{4\pm\sqrt{-36}}{2}$$

$$=\frac{4\pm\sqrt{36i^2}}{2}$$

$$=\frac{4\pm6i}{2}$$

$$=2+3i\text{ and }2-3i$$

∴ complex zeros: $-3,\ 2+3i,\ 2-3i$

5. (a) let ∠EWZ = $a^\circ$

∴ ∠ZXY = $a^\circ$ (∠s in same segment circle XYZW)

∴ ∠YEN = $a^\circ$ (alt. segment theorem)

$\therefore \angle MEW = a^\circ$ (vert. opp. $\angle$s)

$\therefore MN \parallel WZ$ (altern. $\angle$s, $\angle MEW = \angle EWZ$)

ⓑ ⓘ $(p-q)^2 \geqslant 0$

$$p^2 - 2pq + q^2 \geqslant 0$$

$$p^2 + q^2 \geqslant 2pq$$

$$pq \leqslant \frac{p^2+q^2}{2}$$

(ii) sub $p = \frac{1}{x}$, $q = \frac{1}{y}$:

$$\frac{1}{x} \cdot \frac{1}{y} \geqslant \frac{\frac{1}{x^2} + \frac{1}{y^2}}{2}$$

$$\geqslant \frac{\frac{y^2}{x^2y^2} + \frac{x^2}{x^2y^2}}{2}$$

$$\geqslant \frac{x^2+y^2}{2x^2y^2}$$

ⓒ area $\Delta = \frac{1}{2} \cdot 2y \cdot 2y \cdot \sin\frac{\pi}{3}$

$$= \sqrt{3}y^2$$

vol. $\Delta = \sqrt{3}\,y^2\,\delta x$

vol. solid $= \int_{-a}^{a} \sqrt{3}\,y^2\,dx$

$$= 2\sqrt{3}\int_0^a y^2\,dx$$

$$= 2\sqrt{3}\int_0^a a^2 - x^2\,dx$$

$$= 2\sqrt{3}\left[a^2x - \tfrac{1}{3}x^3\right]_0^a$$

$$= \frac{4\sqrt{3}\,a^3}{3} \text{ units}^3$$

ⓓ $P(\omega) = \omega^3 + p\omega^2 + q\omega + r = 0$

i.e. $p\omega^2 + q\omega = -r-1$ since $\omega^3 = 1$

$P(\omega^2) = \omega^6 + p\omega^4 + q\omega^2 + r = 0$

i.e. $p\omega^4 + q\omega^2 = -r-1$ since $\omega^6 = (\omega^3)^2 = 1$

$\therefore p\omega^4 + q\omega^2 = p\omega^2 + q\omega$

$p\omega.\omega^3 + q\omega^2 = p\omega^2 + q\omega$

$p\omega + q\omega^2 = p\omega^2 + q\omega$

$\therefore p = q$

If $p = q$ then $p\omega^2 + p\omega = -r-1$

$$p(\omega^2 + \omega) = -r-1$$

$-p = -r-1$ since $\omega^2 + \omega + 1 = 0$

i.e. $p = r+1$

So $p = q = r+1$

ⓔ $$\frac{1}{(x-3)(x^2+1)} \equiv \frac{a}{x-3} + \frac{bx+c}{x^2+1}$$

$1 \equiv a(x^2+1) + (bx+c)(x-3)$

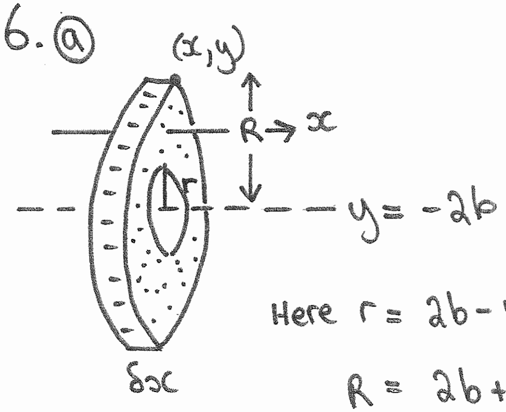
$x = 3$: $1 = 10a \Rightarrow a = \frac{1}{10}$

$x = 0$: $1 = a - 3c \Rightarrow c = -\frac{3}{10}$

equate coeffts of $x^2$: $a + b = 0 \Rightarrow b = -\frac{1}{10}$

i.e. $\dfrac{1}{10(x-3)} - \dfrac{x+3}{10(x^2+1)}$

6. ⓐ



Here $r = 2b - y$

$R = 2b + y$

ⓘ annulus area $= \pi(R^2 - r^2)$

$$= \pi\left[(2b+y)^2 - (2b-y)^2\right]$$

$$= \pi \times 8by$$

$$= 8\pi by$$

∴ volume of slice thickness $\delta x$ is

$$\delta V = 8\pi b y\,\delta x$$

(ii) $$V = 8\pi b \int_{-a}^{a} y\,dx$$

$$= 8\pi b \int_{-a}^{a} \sqrt{\frac{b^2}{a^2}(a^2 - x^2)}\;dx$$

$$= \frac{8\pi b^2}{a} \int_{-a}^{a} \sqrt{a^2 - x^2}\;dx$$

$$= \frac{8\pi b^2}{a} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{a^2 - a^2\sin^2\theta}\,.\;a\cos\theta\,d\theta$$

where $x = a\sin\theta$
so $dx = a\cos\theta\,d\theta$

$$= \frac{8\pi b^2}{a} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} a\cos\theta\,.\,a\cos\theta\,d\theta$$

$$= \frac{8\pi b^2}{a} \cdot a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

$$= 8\pi a b^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tfrac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= 4\pi a b^2 \left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 4\pi^2 a b^2 \text{ units}^3$$

(b) (i) $u = x^n$ $\qquad \frac{dv}{dx} = (1+x)^{\frac{1}{2}}$

$\frac{du}{dx} = nx^{n-1}$ $\qquad v = \frac{2}{3}(1+x)^{3/2}$

INTEGRATION BY PARTS:

$$I_n = \frac{2}{3}\left[x^n . (1+x)^{3/2}\right]_{-1}^{0}$$

$$- \frac{2}{3}\int_{-1}^{0} (1+x)^{3/2} n.x^{n-1}\,dx$$

$$= 0 - \frac{2n}{3}\int_{-1}^{0} x^{n-1}(1+x)^{\frac{1}{2}}(1+x)\,dx$$

$$= -\frac{2n}{3}\int_{-1}^{0} x^{n-1}(1+x)^{\frac{1}{2}} + x^n(1+x)^{\frac{1}{2}}\,dx$$

$$= -\frac{2n}{3}\int_{-1}^{0} x^{n-1}(1+x)^{\frac{1}{2}}dx - \frac{2n}{3}\int_{-1}^{0} x^n(1+x)^{\frac{1}{2}}dx$$

$$= -\frac{2n}{3} I_{n-1} - \frac{2n}{3} I_n$$

ie. $\frac{2n}{3} I_n + I_n = -\frac{2n}{3} I_{n-1}$

$$I_n\left(\frac{2n+3}{3}\right) = -\frac{2n}{3} I_{n-1}$$

$$I_n = \frac{-2n}{2n+3} I_{n-1}$$

(ii) $I_3 = \frac{-6}{9} I_2$

$I_2 = -\frac{4}{7} I_1$

$I_1 = \frac{-2}{5} I_0$

Now $$I_0 = \int_{-1}^{0} x^0 (1+x)^{\frac{1}{2}}\,dx$$

$$= \frac{2}{3}\left[(1+x)^{3/2}\right]_{-1}^{0}$$

$$= \frac{2}{3}$$

$$\therefore I_3 = -\frac{6}{9} \times -\frac{4}{7} \times -\frac{2}{5} \times \frac{2}{3}$$

$$= -\frac{32}{315}$$

(c) $1 + 2x + 3x^2 + \ldots + nx^{n-1}$

$$= \frac{(x-1)(n+1)x^n - (x^{n+1}-1).1}{(x-1)^2}$$

[QUOTIENT RULE]

$$= \frac{(n+1)x^{n+1} - (n+1)x^n - x^{n+1} + 1}{(x-1)^2}$$

$$= \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2}$$

let $x=2$:

$$1+2\times2+3\times4+\_\_\_+n2^{n-1}$$

$$=\frac{n2^{n+1}-(n+1)2^n+1}{(2-1)^2}$$

$$=n2^{n+1}-(n+1)2^n+1$$

$$=2^n(n-1)+1$$

ⓓ other zero $1+2i$

$\therefore$ 2 factors are $[x-(1+2i)][x-(1-2i)]$

$$=x^2-2x+5$$

and $x^2-2x+5\overline{)x^3-5x^2+11x-15}$ with quotient $x-3$

i.e. $P(x)=(x-3)(x-(1+2i))(x-(1-2i))$

7. ⓐ ⓘ $y=c^2x^{-1}$

$$y'=\frac{-c^2}{x^2}$$

At $(cp,\frac{c}{p})$: $m_{TAN}=\frac{-c^2}{c^2p^2}=-\frac{1}{p^2}$

$\therefore m_{NORM}=p^2$

EQUN NORM: $y-\frac{c}{p}=p^2(x-cp)$

$$p^3x-py=c(p^4-1)$$

ⓘⓘ coords Q : sub $y=0$: $Q\left[\frac{c(p^4-1)}{p^3},0\right]$

$x$-coord of M $=\dfrac{\frac{cp^4-p}{p^3}+cp}{2}$

$$=\frac{\frac{cp^4-c}{p^3}+\frac{cp^4}{p^3}}{2}$$

$$=\frac{c(2p^4-1)}{2p^3}$$

$y$-coord of M $=\dfrac{0+\frac{c}{p}}{2}=\dfrac{c}{2p}$

$\therefore M\left[\frac{c(2p^4-1)}{2p^3},\frac{c}{2p}\right]$

ⓘⓘⓘ sub $p=\frac{c}{2y}$ into $x=\frac{c(2p^4-1)}{2p^3}$

$$x=\frac{2\times\frac{c^4}{16y^4}c-c}{2\times\frac{c^3}{8y^3}}$$

$$=\frac{\frac{c^5}{8y^4}-c}{\frac{c^3}{4y^3}}\times\frac{8y^4}{8y^4}$$

$$=\frac{c^5-8cy^4}{2c^3y}$$

$$\therefore 2c^3xy+8cy^4=c^5$$

$$2c^2xy+8y^4=c^4$$

ⓑ ⓘ ie. show $\frac{1}{6},\frac{1}{8},\frac{1}{12}$ is an AP.

$T_3-T_2=\frac{1}{12}-\frac{1}{8}=-\frac{1}{24}$

$T_2-T_1=\frac{1}{8}-\frac{1}{6}=-\frac{1}{24}=T_3-T_2$

$\therefore$ A.P.

$\therefore$ 6, 8, 12 in harmonic progression

ⓘⓘ Find b if $\frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b}$

$\frac{1}{b}+\frac{1}{b}=\frac{1}{a}+\frac{1}{c}$

$\frac{2}{b}=\frac{a+c}{ac}$

$b=\frac{2ac}{a+c}.$

$$\frac{a}{ab}-\frac{b}{ab}=\frac{b}{bc}-\frac{c}{bc}$$

$$\frac{a-b}{ab}=\frac{b-c}{bc}$$

$$\frac{a-b}{a}=\frac{b-c}{c}$$

$$ac-bc=ab-ac$$

$$2ac=b(a+c)$$

i.e. $b=\frac{2ac}{a+c}$

ⓘⓘⓘ Show that $\sqrt{ac}-\frac{2ac}{a+c}\geqslant0$

Now $\sqrt{ac}-\frac{2ac}{a+c}=\frac{\sqrt{ac}(a+c)-2ac}{a+c}$

$$=\frac{\sqrt{ac}\left[a+c-2\sqrt{a+c}\right]}{a+c}$$

$$=\frac{\sqrt{ac}\left[(\sqrt{a})^2-2\sqrt{a}\sqrt{c}+(\sqrt{c})^2\right]}{a+c}$$

$$=\frac{\sqrt{ac}\left[\sqrt{a}-\sqrt{c}\right]^2}{a+c}$$

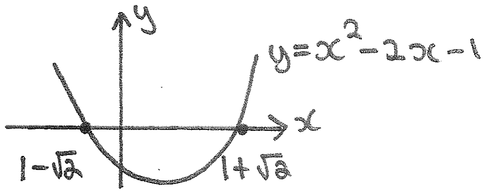
$\geqslant0$ since $[\sqrt{a}-\sqrt{c}]^2\geqslant0$

and $a+c>0$ for $a>0, c>0$

i.e. $\sqrt{ac} > \frac{2ac}{a+c}$

(c) (i) $x$-ints.: $x^2-2x-1=0$

$x = 1 \pm \sqrt{2}$



(ii) STEP 1: Prove $S(5)$ true.

$2^5 = 32$

$5^2 = 25$

$\therefore 2^5 > 5^2$ i.e. $S(5)$ true.

STEP 2: Assume $S(k)$ true

i.e. $2^k > k^2$

Hence prove $S(k+1)$ true

i.e. $2^{k+1} > (k+1)^2$

or $2^{k+1} - (k+1)^2 > 0$.

Now $2^{k+1} - (k+1)^2 = 2.2^k - (k+1)^2$

$> 2.k^2 - (k+1)^2$ by our assumption

$> k^2 - 2k - 1$

$> 0$ for $k > 1+\sqrt{2}$ from part (i)

i.e. If $S(k)$ true then $S(k+1)$ true.

STEP 3: Hence if the result is true for $n=k$, it is true for $n=k+1$. It is true for $n=5$, so by the principle of mathematical induction it is true for all positive integers $n \geqslant 5$.

8. (a) equn is $P\left(\frac{1}{\sqrt{x}}\right) = 0$

i.e. $\left(\frac{1}{\sqrt{x}}\right)^3 + \frac{p}{\sqrt{x}} + m = 0$

$\frac{1}{x\sqrt{x}} + \frac{p}{\sqrt{x}} + m = 0$

$\times x\sqrt{x}$: $1 + px + mx\sqrt{x} = 0$

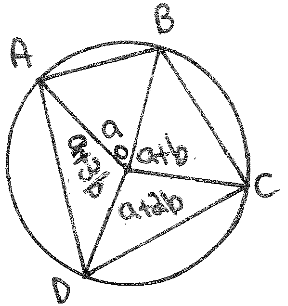
$1 + px = -mx\sqrt{x}$

$(1+px)^2 = m^2x^2.x$

$1 + 2px + p^2x^2 = m^2x^3$

$m^2x^3 - p^2x^2 - 2px - 1 = 0$

(b)



(i) $a + a+d + a+2d + a+3d = 2\pi$ ($\angle$s at a point)

$4a + 6d = 2\pi$

$2a + 3d = \pi$

$\therefore \angle BOD = 2a+3d = \pi$

i.e. BD is a diameter.

(ii) $A = \frac{1}{2}r^2\sin a + \frac{1}{2}r^2\sin(a+d)$
$+ \frac{1}{2}r^2\sin(a+2d) + \frac{1}{2}r^2\sin(a+3d)$

$= \frac{1}{2}r^2\left[\sin a + \sin(a+d) + \sin(\pi-a) + \sin(\pi-(a+d))\right]$

from $2a+3d=\pi$

$a+3d = \pi - a$

$a+2d = \pi-(a+d)$

$= \frac{1}{2}r^2\left[\sin a + \sin(a+d) + \sin a + \sin(a+d)\right]$

$= \frac{1}{2}r^2\left[2\sin a + 2\sin(a+d)\right.$

$= r^2\left[\sin a + \sin(a+d)\right]$

$= r^2\left[2\sin\frac{2a+d}{2}\cos\frac{d}{2}\right]$

$= 2r^2\sin\frac{\pi-2d}{2}\cos\frac{d}{2}$ from $2a+d = \pi-2d$

$= 2r^2\sin\left(\frac{\pi}{2}-d\right)\cos\frac{d}{2}$

$= 2r^2\cos\alpha\cos\frac{\alpha}{2}$

ⓒ ⓘ $p^7 = \left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)^7$

$= \cos 2\pi + i\sin 2\pi$ by de Moivre

$= 1$

∴ p is a root of $x^7 = 1$

i.e. $x^7 - 1 = 0$

$(x-1)(1+x+x^2+\ldots+x^6) = 0$

Since $p \neq 1$ then $1+p+p^2+\ldots+p^6=0$

(ii) Since coeffts of $x^2+ax+b=0$

then other root $\beta = \bar{\alpha}$

$= \overline{p+p^2+p^4}$

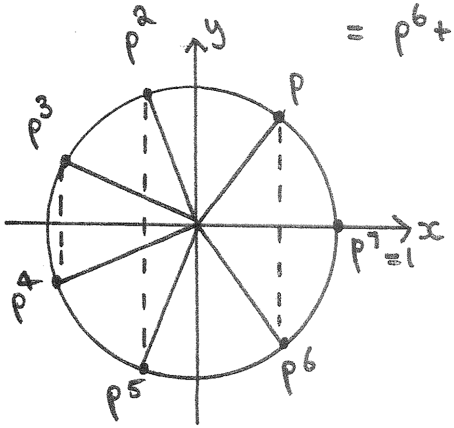
$= \bar{p}+\overline{p^2}+\overline{p^4}$

$= p^6+p^5+p^3$

since $\bar{p}=p^6$

$\overline{p^2}=p^5$

$\overline{p^4}=p^3$



(iii) $\alpha+\beta = -a$

$(p+p^2+p^4)+(p^6+p^5+p^3) = -a$

$p+p^2+\ldots+p^6 = -a$

$-1 = -a$ since from ⓘ $1+p+p^2+\ldots+p^6=0$

i.e. $a=1$

$\alpha\beta = b$

$(p+p^2+p^4)(p^6+p^5+p^3) = b$

$p(1+p+p^3)\,p^3(p^3+p^2+1) = b$

$p^4(1+p+p^3)(1+p^2+p^3) = b$

$b = p^4(1+p^2+p^3+p+p^3+p^4+p^3+p^5+p^6)$

$= p^4(1+p+p^2+p^3+p^4+p^5+p^6+2p^3)$

$= p^4(0+2p^3)$ from ⓘ

$= 2p^7$

$= 2\times1$

$= 2$

(iv) Now $p = \text{cis}\frac{2\pi}{7}$

∴ $\alpha = p+p^2+p^4$

$= \text{cis}\frac{2\pi}{7}+\text{cis}\frac{4\pi}{7}+\text{cis}\frac{8\pi}{7}$

∴ $\text{Im}(\alpha) = \sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}$

$= \sin\frac{2\pi}{7}+\sin\left(\pi-\frac{3\pi}{7}\right)+\sin\left(\pi+\frac{\pi}{7}\right)$

$= \sin\frac{2\pi}{7}+\sin\frac{3\pi}{7}-\sin\frac{\pi}{7}$

Also solving $x^2+ax+b=0$

i.e. $x^2+x+2=0$

$x = \frac{-1\pm\sqrt{1^2-4.1.2}}{2}$

$= \frac{-1\pm\sqrt{-7}}{2}$

$= \frac{-1\pm\sqrt{7i^2}}{2}$

$= \frac{-1\pm\sqrt{7}i}{2}$

Taking $\alpha = \frac{-1+\sqrt{7}i}{2}$ since from diagram $p+p^2+p^4>0$

then $\text{Im}(\alpha) = \frac{\sqrt{7}}{2}$

i.e. $-\sin\frac{\pi}{7}+\sin\frac{2\pi}{7}+\sin\frac{3\pi}{7} = \frac{\sqrt{7}}{2}$