CARINGBAH HIGH SCHOOL

2013

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I Pages 2–5

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 6–12

90 marks

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Which of the following are the foci for the ellipse

$$\frac{x^2}{9}+\frac{y^2}{16}=1$$

(A) $\left(\mathbf{0},\pm\frac{\mathbf{3\sqrt{7}}}{\mathbf{4}}\right)$

(B) $(\pm\sqrt{\mathbf{7}},\mathbf{0})$

(C) $(\mathbf{0},\pm\sqrt{\mathbf{7}})$

(D) $\left(\frac{\mathbf{3\sqrt{7}}}{\mathbf{4}},0\right)$

2 Which is the correct answer to the following integral?

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} sin^7(x)cos^4(x)\,dx$$

(A) $\mathbf{2}\times\int_{\mathbf{0}}^{\frac{\boldsymbol{\pi}}{\mathbf{4}}} \boldsymbol{sin^7(x)cos^4(x)\,dx}$

(B) $\dfrac{\mathbf{1024-533\sqrt{2}}}{\mathbf{36960}}$

(C) $-\dfrac{\mathbf{1024-533\sqrt{2}}}{\mathbf{36960}}$

(D) zero

3 Let α,β and γ be the roots of the cubic equation $x^3-5x^2+13x-7=0$ .Which of the following is the equation with roots α^2,β^2 and γ^2 ?

(A) $\boldsymbol{7x^3-13x^2+5x-1=0}$

(B) $\boldsymbol{x^3+x^2+99x-49=0}$

(C) $\boldsymbol{x^3+5x^2-13x-7=0}$

(D) $\boldsymbol{49x^3+99x^2+x-1=0}$

4 Given that $x^2 + y^2 + xy = 12$, which of the following is true?

(A) $\boldsymbol{\frac{dy}{dx} = \frac{2x + y}{2y + x}}$

(B) $\boldsymbol{\frac{dy}{dx} = -\frac{2x + y}{2y + x}}$

(C) $\boldsymbol{\frac{dy}{dx} = \frac{2x - y}{2y + x}}$

(D) $\boldsymbol{\frac{dy}{dx} = \frac{-2x + y}{2y + x}}$

5 The equation $|z - 1 - 3i| + |z - 9 - 3i| = 10$ corresponds to an ellipse in the Argand diagram. Which of the following is the complex number corresponding to the centre of the ellipse?

(A) $\boldsymbol{5 + 3i}$

(B) $\boldsymbol{-5 + 3i}$

(C) $\boldsymbol{-5 - 3i}$

(D) $\boldsymbol{5 - 3i}$

6 The point $T(acos\theta, asin\theta)$ lies on the circle $x^2 + y^2 = r^2$. Which of the following gives the equation of the tangent at *T*?

(A) $\boldsymbol{xcos\theta + ysin\theta = a}$

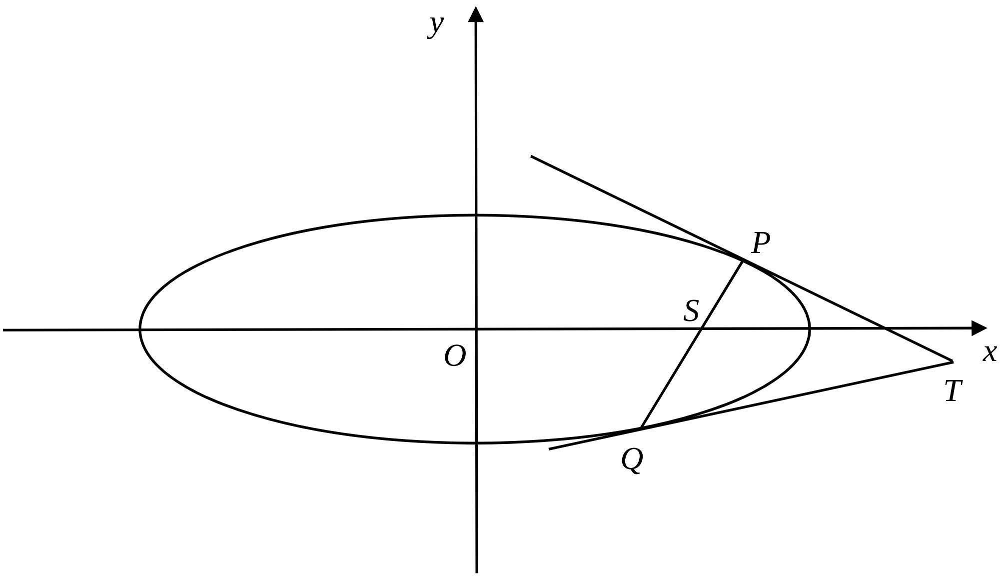
(B) $\boldsymbol{xcos\theta - ysin\theta = a}$

(C) $\boldsymbol{xcos\theta - ysin\theta = a^2}$

(D) $\boldsymbol{xcos\theta + ysin\theta = a^2}$

7

The point P lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The chord through P and the focus $S(ae,0)$ meets the ellipse at Q. The tangents to the ellipse at P and Q meet at the point $T(x_0,y_0)$, so the equation of PQ is $\frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1$. (Note that T lies on the directrix).



What is the value of the ratio $\frac{PS}{ST}$, given that this ratio is constant?

(A) $\boldsymbol{e^2}$

(B) ae

(C) $\boldsymbol{\frac{a}{e}}$

(D) $\boldsymbol{e}$

8 Suppose $\omega^3=1$, $\omega\neq 1$ and k is a positive integer.

What are the two values of $1+\omega^k+\omega^{2k}$?

(A) 3, 0

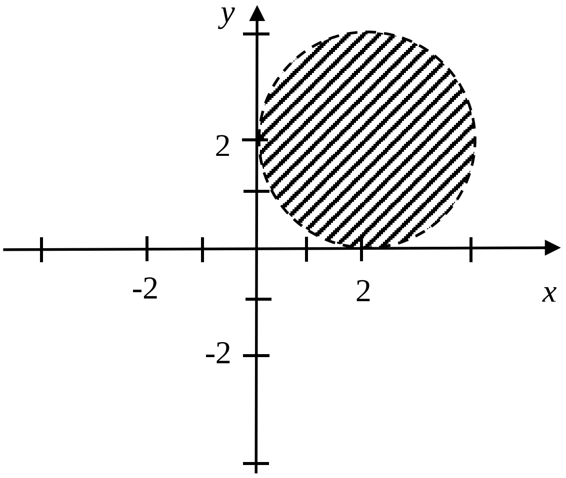
(B) 3, 1

(C) 1, 0

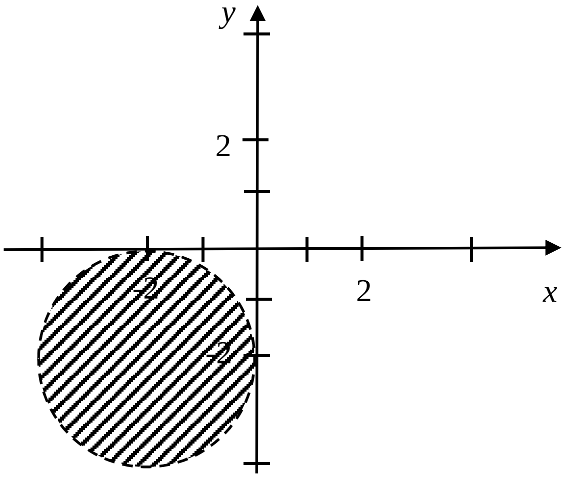
(D) None of the above

9 Which of the following is the graph of $|z - 1 - i| < 2$?

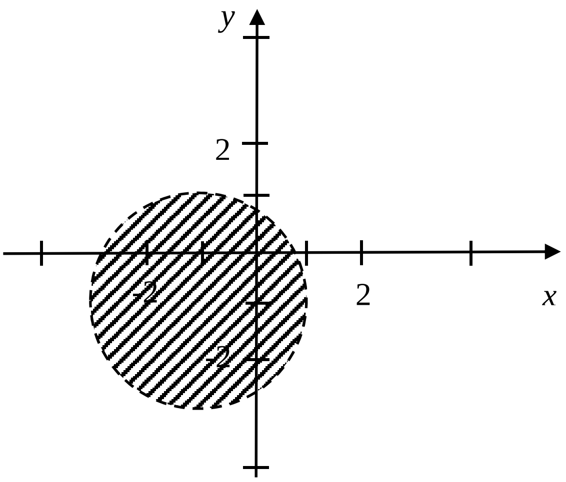
(A)



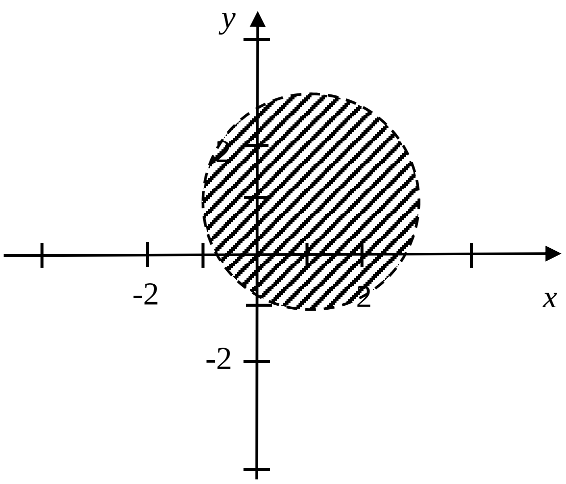
(B)



(C)



(D)



10 Given that $\cos(a+b)\,x + \cos(a-b)\,x = 2\cos(ax)\cos(bx)$, which of the following is the answer for

$$\int \cos(3x)\cos(2x)\,dx \;?$$

(A) $\boldsymbol{\frac{1}{2}(cos5x + cosx) + c}$

(B) $\boldsymbol{\frac{1}{10}cos5x + \frac{1}{2}cosx + c}$

(C) $\boldsymbol{\frac{1}{10}sin5x + \frac{1}{2}sinx + c}$

(D) $\boldsymbol{\frac{1}{2}(sin5x + sinx) + c}$

END OF MULTIPLE CHOICE QUESTIONS

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available. In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a NEW booklet. **Marks**

(a) Find

(i) $\int \frac{t^2-2}{t^3}\,dt$ **2**

(ii) $\int xe^x\,dx$ **2**

(iii) $\int \frac{2x}{(x+1)(x+3)}\,dx$ **3**

(b) By using the substitution $u = x - 4$ evaluate

$$\int_4^{4\cdot 5} \frac{dx}{\sqrt{(x-3)(5-x)}}$$ **3**

(c) (i) If **3**

$$u_n = \int_0^{\frac{\pi}{2}} x^n sinx\,dx\,, \qquad n \geq 2$$

prove that

$$u_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)u_{n-2}$$

(ii) Hence evaluate **2**

$$\int_0^{\frac{\pi}{2}} x^2 sinx\,dx$$

Question 12 (15 marks) Start a NEW booklet. **Marks**

(a) The complex number w is given by $w = -1 + i\sqrt{3}$.

(i) Show that $w^2 = 2\overline{w}$. **2**

(ii) Evaluate $|w|$ and $\arg w$. **2**

(iii) Show that w is a root of $w^3 - 8 = 0$ **1**

(b) Sketch the locus of z satisfying:

(i) $Re(z) = |z|$ **2**

(ii) Both $Re(z) \geq 2$ and $|z - 1| \leq 2$ **3**

(c) Given that a and b are real numbers and **2**

$$\frac{a}{1+i} + \frac{b}{1+2i} = 1$$

find the values of a and b.

(d) The complex numbers z_1, z_2, z_3 and z_4 are represented in the complex plane by the points A, B, C and D respectively. **3**

If $z_1 + z_3 = z_2 + z_4$ prove ABCD is a parallelogram.

Question 13 (15 marks) Start a NEW booklet. **Marks**

(a) The equation $x^3 + bx^2 + x + 2 = 0$, where b is a real number, has roots α, β, γ.

(i) Obtain an expression, in terms of *b*, for **2**

$$\alpha^2 + \beta^2 + \gamma^2$$

(ii) Hence determine the set of possible values of *b* if the roots of the above equation are all real. **1**

(iii) Write down the equation whose roots are **2**

$$2\alpha, 2\beta, 2\gamma$$

(b) Given that the polynomial $P(x) = 8x^4 - 36x^3 - 66x^2 - 35x - 6$ has a zero of multiplicity 3, find all the zeros of $P(x)$. **3**

(c) If *z* represents a complex number such that $z^5 = 1$, where $z \neq 1$.

(i) Deduce that **2**

$$z^2 + z + 1 + \frac{1}{z} + \frac{1}{z^2} = 0$$

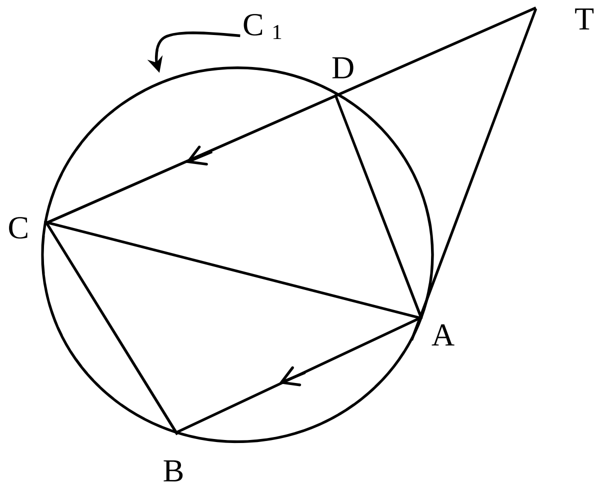
(ii) By substituting $x = z + \frac{1}{z}$ reduce the equation in (i) to a quadratic in *x*. **2**

(iii) Hence deduce that **3**

$$cos\frac{2\pi}{5}\ cos\frac{4\pi}{5} = -\frac{1}{4}$$

Question 14 (15 marks) Start a NEW booklet. **Marks**

(a) The points A, B, C and D lie on the circle $\mathcal{C}_1$. From the exterior point T, a tangent is drawn to point A on $\mathcal{C}_1$. The line CT passes through D and TC is parallel to AB.



(i) Copy or trace the diagram on to your page.

(ii) Prove that ΔADT is similar to ΔABC. **3**

The line BA is produced through A to point M, which lies on a second circle $\mathcal{C}_2$. The points A, D, T also lie on $\mathcal{C}_2$ and the line DM crosses AT at Q.

(iii) Show that ΔQMA is isosceles. **2**

(iv) Show that $TM = BC$. **2**

(b) (i) Prove that the normal to the hyperbola $xy = 4$ at the point $P(2p, \frac{2}{p})$ is given by **2**

$$p^3x - py = 2(p^4 - 1)$$

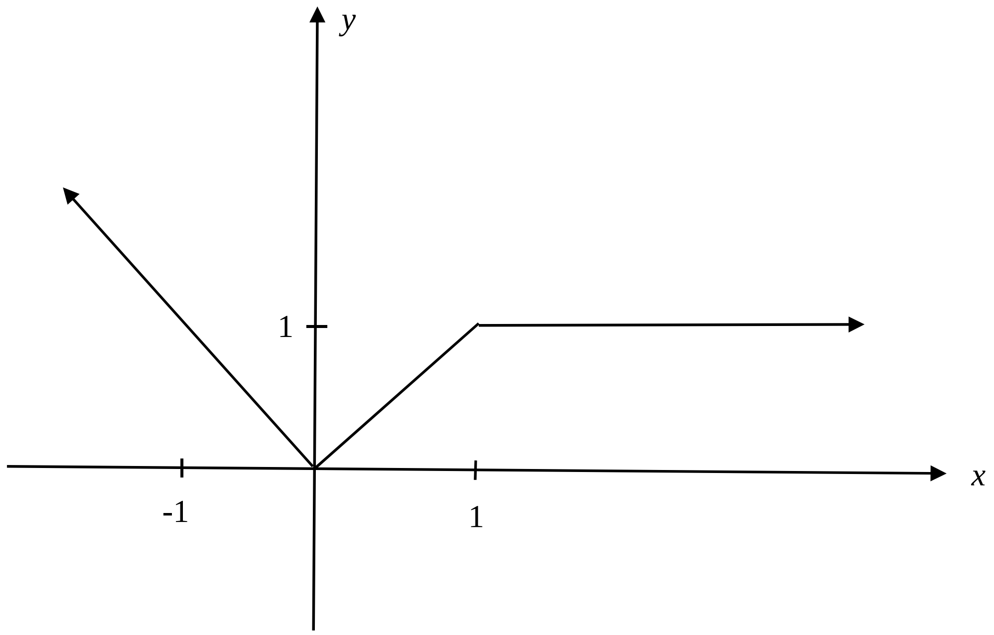
(ii) If this normal meets the hyperbola again at $Q(2q, \frac{2}{q})$ prove that $p^3q = -1$. **2**

(iii) Hence prove that there exists only one chord which is normal to the hyperbola at both ends and find its equation. **4**

Question 15 (15 marks) Start a NEW booklet. **Marks**

(a) Find the equation of the ellipse with centre the origin, which has a focus at $(2,0)$ and the corresponding directrix is $\mathrm{x} = 4$. **3**

(b)



The diagram shows the graph of the function $y = f(x)$

Draw separate sketches of the following:

(i) $y = f(-x)$ **1**

(ii) $y = \dfrac{1}{f(x)}$ **1**

(iii) $y = f(|x|)$ **1**

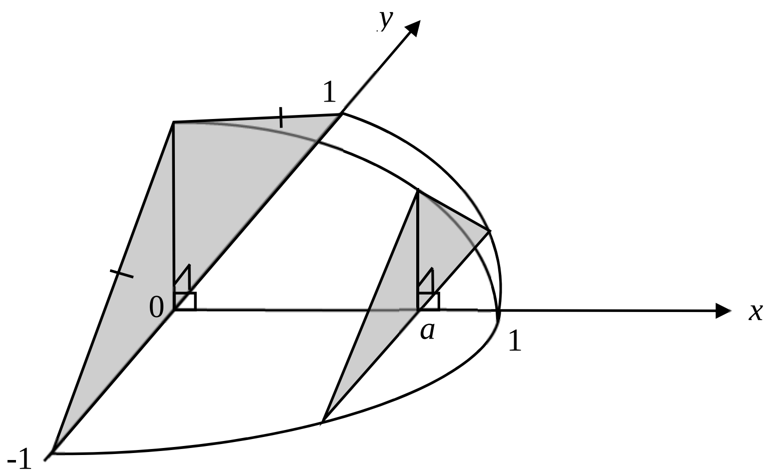
(iv) $y = \ln(f(x))$ **2**

(v) $y = e^{f(x)}$ **1**

(vi) $x = f(y)$ **1**

Question 15 continues on the next page.

(c) The base of a solid is the semi-circular region of radius 1 unit in the *x*-*y* plane as illustrated in the diagram below.



Each cross-section perpendicular to the *x*-axis is an isosceles triangle with each of the two equal sides three quarters the length of the third side..

(i) Show that the area of the triangular cross-section at $x = a$ is **2**

$$\frac{\sqrt{5}}{2}(1 - a^2).$$

(ii) Hence find the volume of the solid. **3**

Question 16 (15 marks) Start a NEW booklet. **Marks**

(a) $P(4,6)$ and $Q(14,24)$ are two points on the hyperbola **3**

$$\frac{x^2}{4} - \frac{y^2}{12} = 1$$

M is the midpoint of PQ and $O(0,0)$ is the origin. The tangents to the hyperbola at P and Q intersect at the point R. Show that the points R, O and M are collinear.

You may assume that the tangent to this hyperbola at $T(x_1, y_1)$ has equation

$$\frac{x_1 x}{4} - \frac{y_1 y}{12} = 1$$

(Do NOT prove this.)

(b) A particle is moving so that $\ddot{x} = 18x^3 + 27x^2 + 9x$.

Initially $x = -2$ and the velocity, v, is -6. It is known that the velocity is always negative.

(i) Show that $v^2 = 9x^2(1+x)^2$. **2**

(ii) Hence, or otherwise, show that **2**

$$\int \frac{1}{x(1+x)} dx = -3t$$

(iii) Find a, b such that **1**

$$\frac{1}{x(1+x)} \equiv \frac{a}{x} + \frac{b}{1+x}$$

(iv) Show that for some constant c, **2**

$$log_e\left(1 + \frac{1}{x}\right) = 3t + c$$

(v) Using this equation and the initial conditions, find x as a function of t. **2**

(c) The angles A, B and C are consecutive terms in an arithmetic series. Show that **3**

$\cos(A)\cos(C) - cos^2(B) = \sin(A)\sin(C) - sin^2(B)$.

END OF EXAM

Candidate Name/Number: ______________________________________

Multiple choice answer page. Fill in either A, B, C or D for questions 1-10.

<u>This page must be handed in with your answer booklets</u>

1.	
2.	
3.	
4.	
5.	

6.	
7.	
8.	
9.	
10.	

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a>0, \quad -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x>0$

CARINGBAH HIGH EXTENSION II 2013

THSC SOLUTIONS

Q1 C	Q6 A
Q2 D	Q7 D
Q3 B	Q8 A
Q4 B	Q9 D
Q5 A	Q10 C

Q11

(a) (i) $\int \frac{t^2-2}{t^3}\,dt = \int\left(\frac{1}{t} - \frac{2}{t^3}\right)dt$

$= \int\left(\frac{1}{t} - 2t^{-3}\right)dt$

$= \ln|t| + \frac{1}{t^2} + c$

(ii) $\int x.e^x\,dx = \overset{u.v}{xe^x} - \int \overset{u'.v}{1.e^x}\,dx$

$= xe^x - e^x + c$

(iii) $I = \int \frac{2x}{(x+1)(x+3)}\,dx$ Using partial fractions

$2x \equiv a(x+3) + b(x+1)$

$x=-1 \Rightarrow a=-1$

$x=-3 \Rightarrow b=3$

$\therefore I = \int\left(\frac{-1}{x+1} + \frac{3}{x+3}\right)dx$

$= 3\ln|x+3| - \ln|x+1| + c$

$= \ln\left|\frac{(x+3)^3}{x+1}\right| + c$

(b) $I = \int_4^{4.5} \frac{dx}{\sqrt{(x-3)(5-x)}}$

If $u = x-4$
$du = dx$

when
$x=4.5,\ u=0.5$
$x=4,\ u=0$

$\int_{u=0}^{u=0.5} \frac{1.du}{\sqrt{(1+u)(1-u)}}$

$= \int_{u=0}^{u=\frac{1}{2}} \frac{1}{\sqrt{1-u^2}}\,du$

$= \left[\sin^{-1} u\right]_0^{1/2}$

$= \frac{\pi}{6} - 0$

$= \frac{\pi}{6}$

(c) (i) $U_n = \left[\overset{u.v}{-x^n.\cos x}\right]_0^{\pi/2} + n\int_0^{\pi/2} \overset{u'.v}{x^{n-1}.\cos x.dx}$

$= 0 + n\int_0^{\pi/2} x^{n-1}.\cos x.\,dx$

Now use parts a second time.

$= n\left[\overset{u\quad v}{x^{n-1}.\sin x}\right]_0^{\pi/2} - n(n-1)\int_0^{\pi/2} \overset{u'.v}{x^{n-2}.\sin x.dx}$

$= n\left(\left(\frac{\pi}{2}\right)^{n-1}.\sin\frac{\pi}{2} - 0\right) - n(n-1)U_{n-2}$

$= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)U_{n-2}$

(ii) $n=2 \Rightarrow U_2 = 2\left(\frac{\pi}{2}\right)^1 - 2.1.U_0$

$= \pi - 2\int_0^{\pi/2} \sin x\,.dx$

$= \pi + 2\left[\cos x\right]_0^{\pi/2}$

$= \pi + 2(0-1)$

$= \pi - 2$

Q12 (a) $w = -1 + i\sqrt{3}$.

(i) $w^2 = (-1+i\sqrt{3})(-1+i\sqrt{3})$

$= 1 - 3 - 2i\sqrt{3}$

$= -2 - 2i\sqrt{3}$

$\bar{w} = -1 - i\sqrt{3}$

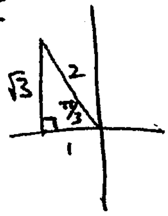
$2\bar{w} = 2(-1-i\sqrt{3})$

$= -2 - 2i\sqrt{3}$

$= w^2$

(ii) $|w| = \sqrt{1+3}$ $\quad \arg w = \frac{2\pi}{3}$

$= 2$



(iii) $\therefore w = 2 \operatorname{cis} \frac{2\pi}{3}$

$w^3 = 2^3 \operatorname{cis}\left(3 \cdot \frac{2\pi}{3}\right)$

$= 8 \operatorname{cis} 2\pi$

$= 8 \times 1$

$w^3 = 8$

$\therefore \underline{w^3 - 8 = 0}$

(b) (i) Let $z = x+iy$.

$Re(z) = x$

$|z| = \sqrt{x^2+y^2}$

as $Re(z) = |z|$

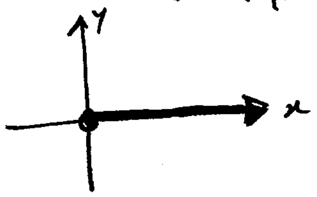
then $x = \sqrt{x^2+y^2}$

$x^2 = x^2+y^2$

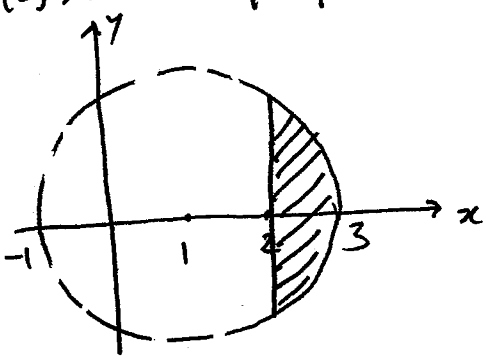
$y^2 = 0$

$\therefore y = 0$ ie x-axis

But $|z| \geq 0$ $\therefore$ only $x \not< 0$



(ii) $Re(z) \geq 2$ and $|z-1| \leq 2$



(c) $\therefore a(1+2i) + b(1+i) = (1+i)(1+2i)$

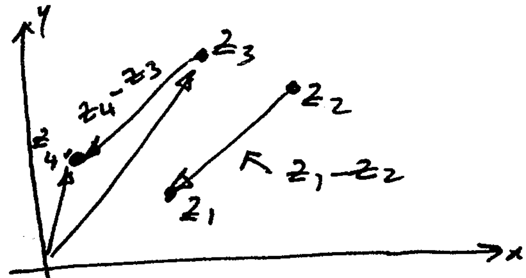
equating real parts $\Rightarrow$ $a+b = -1$

" imaginary parts $\Rightarrow$ $2a+b = 3$

$\therefore \underline{a = 4}$

$\underline{b = -5}$

(d) $\therefore z_1 - z_2 = z_4 - z_3$



If $z_1 - z_2 = z_4 - z_3$

then $|z_1 - z_2| = |z_4 - z_3|$

and $\arg(z_1 - z_2) = \arg(z_4 - z_3)$

$\therefore AB = CD$

and $AB \parallel CD$

$\therefore$ ABCD is a parallelogram.

Q13

(a) (i) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= (-b)^2 - 2\times 1$

$= \underline{b^2 - 2}$

(ii) real roots $\Rightarrow \alpha^2+\beta^2+\gamma^2 \geq 0$

but here $\alpha, \beta, \gamma \neq 0$

$\therefore \alpha^2+\beta^2+\gamma^2 > 0$

$\therefore b^2 - 2 > 0$

$\therefore \underline{b < -\sqrt{2} \text{ or } b > \sqrt{2}}$

(iii) $2\alpha = x \quad \therefore \frac{x}{2} = \alpha$

in $\alpha^3 + b\alpha^2 + \alpha + 2 = 0$

$\therefore \left(\frac{x}{2}\right)^3 + b\left(\frac{x}{2}\right)^2 + \frac{x}{2} + 2 = 0$

$\frac{x^3}{8} + \frac{bx^2}{4} + \frac{x}{2} + 2 = 0$

$\therefore \underline{x^3 + 2bx^2 + 4x + 16 = 0}$

(b) $P'(x) = 32x^3 - 108x^2 - 132x - 35$

$P''(x) = 96x^2 - 216x - 132$

$= 12(8x^2 - 18x - 11)$

$= 12(2x+1)(4x-11)$

$\therefore x = -\frac{1}{2}$ or $\frac{11}{4}$

$P(\frac{11}{4}) \neq 0$, $P(-\frac{1}{2}) = 0$

$\therefore P(x) = (2x+1)^3 Q(x)$

$= (2x+1)^3 \cdot (x-6)$

$\therefore \underline{x = -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, 6}$

(c) (i) $\therefore z^5 - 1 = 0$

ie $(z-1)(z^4+z^3+z^2+z+1) = 0$

$z \neq 1$ so $z^4+z^3+z^2+z+1 = 0$

$z \neq 0$ so $\div$ by z^2

ie $\underline{z^2+z+1+\frac{1}{z}+\frac{1}{z^2} = 0}$

(ii) $\therefore \left(z^2+\frac{1}{z^2}\right) + \left(z+\frac{1}{z}\right) + 1 = 0$

$\left(z+\frac{1}{z}\right)^2 - 2 + \left(z+\frac{1}{z}\right) + 1 = 0$

$\therefore \underline{x^2 + x - 1 = 0}$

(iii) From $z^5 = 1$

$$\left.\begin{aligned} z_1 &= cis\,\tfrac{2\pi}{5} \\ z_2 &= cis\,\tfrac{4\pi}{5} \\ \tfrac{1}{z_2} = z_3 &= cis\,\tfrac{6\pi}{5} \\ \tfrac{1}{z_1} = z_4 &= cis\,\tfrac{8\pi}{5} \end{aligned}\right\} \text{These are the 4 roots of (i)}$$

$z_5 = cis\,\frac{10\pi}{5} = 1$

from (ii) $x = x_1$ or x_2

$x_1 = z_1 + \frac{1}{z_1} = cis\,\frac{2\pi}{5} + cis\,-\frac{2\pi}{5}$

$= \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} + \cos -\frac{2\pi}{5} + i\sin -\frac{2\pi}{5}$

as cos EVEN function and sin ODD function

$\therefore x_1 = \underline{2\cos\left(\frac{2\pi}{5}\right)}$

and $x_2 = z_2 + \frac{1}{z_2}$

$= \underline{2\cos\left(\frac{4\pi}{5}\right)}$

from (ii) $x_1 \cdot x_2 = -1$

$\therefore 2\cos\left(\frac{2\pi}{5}\right) \cdot 2\cos\left(\frac{4\pi}{5}\right) = -1$

$\therefore \underline{\cos\left(\frac{2\pi}{5}\right) \cdot \cos\left(\frac{4\pi}{5}\right) = -\frac{1}{4}}$

Q14 (a) Let $\angle DAT = \beta$, Let $\angle ABC = \alpha$

(ii) $\angle DAT = \angle ACD$ ($\angle$s in alt. seg.)

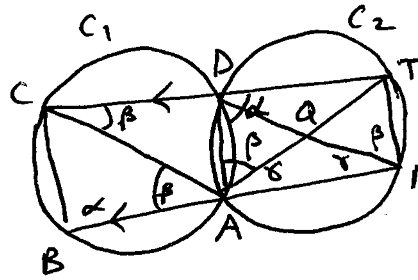
$\angle ACD = \angle BAC$ (alt $\angle$s, $CD \parallel AB$)

$\angle ABC = \angle ADT$ (ext $\angle$ cyclic quad ABCD)

(iii) $\angle MAT = \angle MDT$ ($\angle$s in same seg)

$\angle MDT = \angle AMD$ (alt $\angle$s, $DT \parallel AM$)

Let $\angle MAT = \gamma$



$\therefore \triangle QMA$ is isos.

(iv) $\angle DMT = \angle DAT$ ($\angle$s in same seg)

$\therefore \angle AMT = \beta + \gamma$

$\angle ATD = \angle AMD$ ($\angle$s in same seg)

$= \gamma$

$\therefore$ From $\triangle ADT$ $\alpha + \beta + \gamma = 180°$

$\therefore \angle BCA = \gamma$

$\therefore \angle BCD = \beta + \gamma = \angle AMT$

$\therefore$ BCTM is a parallelogram (1 pair of sides parallel plus one pair of opp. $\angle$s equal).

$\therefore CB = TM$.

(b) (i)

$$y = \frac{4}{x}$$

$$y' = -\frac{4}{x^2}$$

$$\therefore m_{T_P} = -\frac{4}{4p^2} = -\frac{1}{p^2}$$

$$m_{N_P} = p^2$$

$$\therefore y - \frac{2}{p} = p^2(x - 2p)$$

$$py - 2 = p^3x - 2p^4$$

$$\underline{p^3x - py = 2(p^4 - 1)} \quad — (1)$$

(ii) $\therefore$ Q satisfies $xy = 4$ then $y = \frac{4}{x}$

so $p^3 \cdot x - p \cdot \frac{4}{x} = 2(p^4 - 1)$

or $p^3x^2 - 2(p^4 - 1)x - 4p = 0$

$$\therefore x = \frac{2(p^4-1) \pm \sqrt{4(p^4-1)^2 + 16p^4}}{2p^3}$$

$$= \frac{p^4 - 1 \pm \sqrt{(p^4-1)^2 + 4p^4}}{p^3}$$

$$= \frac{p^4 - 1 \pm (p^4 + 1)}{p^3}$$

$$= \frac{p^4 - 1 + p^4 + 1}{p^3} \text{ or } \frac{p^4 - 1 - p^4 - 1}{p^3}$$

$$= \frac{2p^4}{p^3}, \quad \frac{-2}{p^3}$$

$$= 2p$$

Now if $x = 2p$ then we have $P\left(2p, \frac{2}{p}\right)$

so $Q\left(2q, \frac{2}{q}\right) \Rightarrow 2q = \frac{-2}{p^3}$

or $\underline{\underline{p^3q = -1}}$

(iii) By symmetry $q^3p = -1$ also

$$\therefore p^3q = pq^3$$

$$\text{or } p^3q - q^3p = 0$$

$$pq(p^2 - q^2) = 0$$

$$pq(p-q)(p+q) = 0$$

$\therefore pq = 0$ — not possible

or $p = q$ — not possible

$\therefore \underline{p = -q}$.

$$\therefore q^3 \cdot -q = -1$$

$$q^4 = 1$$

$$\therefore q = \pm 1$$

$$\therefore P \equiv (\pm 2, \pm 2)$$

$\therefore$ chord is $y = x$: $-2 \le x \le 2$

Q15

(a) $ae = 2$ — (1)

$a/e = 4$ — (2)

$(1) \times (2) \Rightarrow a^2 = 8$

$(1) \div (2) \Rightarrow e^2 = \frac{1}{2}$

ellipse $\Rightarrow e^2 = 1 - \frac{b^2}{a^2}$

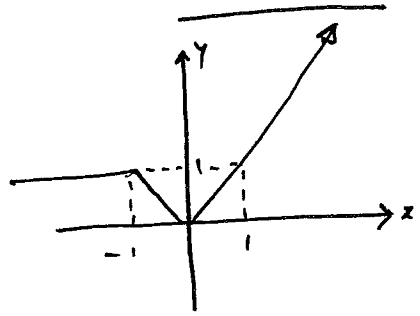
$$\frac{1}{2} = 1 - \frac{b^2}{8}$$

$$4 = 8 - b^2$$

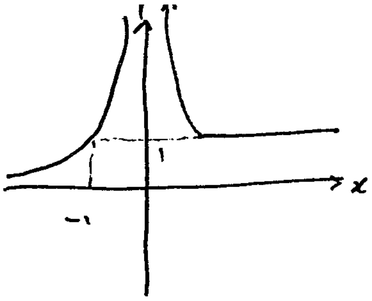
$$\underline{b^2 = 4}$$

$$\therefore \underline{\frac{x^2}{8} + \frac{y^2}{4} = 1}$$

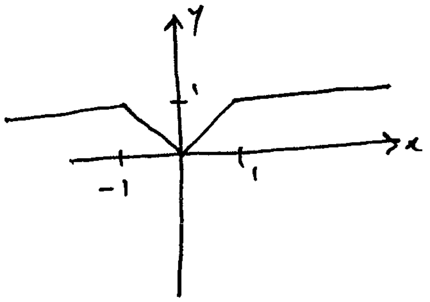
(b) (i)



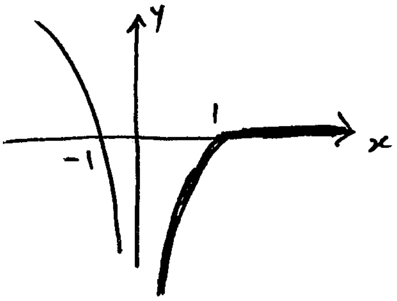
(ii)



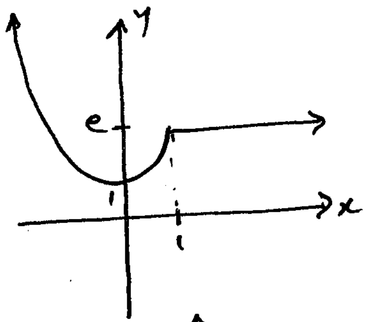
(ii)



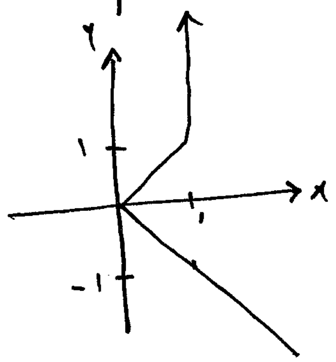
(iv)



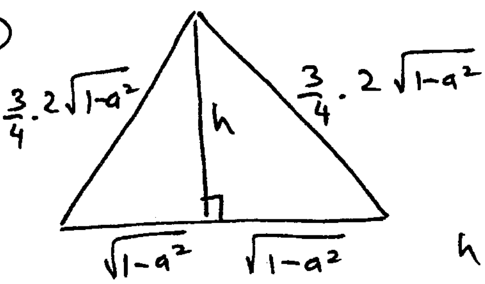
(v)

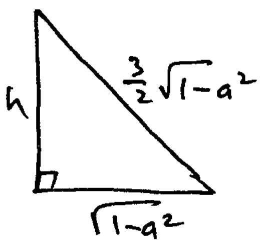


(vi)



(c)





$$h^2 = \frac{9}{4}(1-a^2) - (1-a^2)$$
$$= \frac{5}{4}(1-a^2)$$

$$\therefore h = \frac{\sqrt{5}}{2}\sqrt{1-a^2}$$

$$\therefore \text{Area} = \frac{1}{2}\left(2\sqrt{1-a^2}\right)\cdot\frac{\sqrt{5}}{2}\sqrt{1-a^2}$$
$$= \frac{\sqrt{5}}{2}(1-a^2)$$

(ii)
$$V = \frac{\sqrt{5}}{2}\int_{x=0}^{x=1}(1-x^2)\,dx$$
$$= \frac{\sqrt{5}}{2}\left[x - \frac{x^3}{3}\right]_0^1$$
$$= \frac{\sqrt{5}}{2}\left(1 - \frac{1}{3} - (0-0)\right)$$
$$= \frac{\sqrt{5}}{3}\,u^2$$

Q16 (a) $O \equiv (0,0)$ $M \equiv \left(\frac{18}{2}, \frac{30}{2}\right) \equiv (9,15)$

$$\text{Tangent}_P \equiv T_P \equiv \frac{4x}{4} - \frac{6y}{12} = 1$$
$$\therefore x - \frac{y}{2} = 1 \quad \text{—— ①}$$

$$T_Q \equiv \frac{14x}{4} - \frac{24y}{12} = 1$$
$$\therefore \frac{7x}{2} - 2y = 1 \quad \text{—— ②}$$

② − 4×①
$$\frac{7x}{2} - 4x = 1 - 4$$
$$7x - 8x = -6$$
$$-x = -6$$
$$x = 6$$

sub into ②
$$6 - \frac{y}{2} = 1$$
$$y/2 = 5$$
$$y = 10$$

$$\therefore R \equiv (6,10)$$

$$m_{RO} = \frac{10}{6} = \frac{5}{3}$$
$$m_{MO} = \frac{15}{9} = \frac{5}{3}$$
$\Big\}$ $m_{RO} = m_{MO}$ $\Rightarrow$ RMO collinear

(i) $\displaystyle\int \frac{d}{dx}\frac{1}{2}v^2\,dx = \int (18x^3+27x^2+9x)\,dx$

$$\frac{1}{2}v^2 = \frac{9x^4}{2} + 9x^3 + \frac{9x^2}{2} + c$$

$$v^2 = 9x^4 + 18x^3 + 9x^2 + c'$$

$x=-2$, $v=-6$ $\therefore c'=0$.

$$\therefore v^2 = 9x^2(x^2+2x+1)$$
$$= 9x^2(x+1)^2$$

(ii) $\therefore v = 3x(1+x)$ or $-3x(1+x)$

here $x=-2 \Rightarrow V = -6\times-1$ $=6$ ~~not -6~~	here $x=-2$ $V = 6\times-1$ $=-6$

$$\therefore \underline{V = -3x(1+x)}$$

$$\therefore \frac{dx}{dt} = -3x(1+x)$$

$$\frac{dt}{dx} = -\frac{1}{3}\frac{1}{x(1+x)}$$

integrate both sides w.r.t x gives

$$t = -\frac{1}{3}\int \frac{1}{x(1+x)}\,dx$$

or $$-3t = \int \frac{1}{x(1+x)}\,dx$$

(iii) $1 \equiv a(1+x) + bx$

$x=0 \Longrightarrow a=1$

$x=-1 \Longrightarrow b=-1$

(iv) $$\therefore -3t = \int\left(\frac{1}{x} - \frac{1}{1+x}\right)dx$$

$$= \ln|x| - \ln|1+x| + c$$

$$3t = \ln\left|\frac{1+x}{x}\right| + c$$

$$3t + c' = \ln\left|1+\frac{1}{x}\right|$$

(v) $t=0$ $x=-2$

$\therefore \ln\left(\frac{1}{2}\right) = c'$

$$\therefore 1+\frac{1}{x} = e^{3t+\ln 1/2}$$
$$= \frac{1}{2}e^{3t}$$
$$\frac{1}{x} = \frac{1}{2}e^{3t} - 1$$
$$x = \frac{1}{1/2\, e^{3t} - 1}$$
$$x = \frac{2}{e^{3t}-2}$$

(c) AP $\Rightarrow A, B, C \equiv B-d, B, B+d$

so we need only show.

$$\cos A\cos C - \sin A\sin C = \cos^2 B - \sin^2 B$$
$$= \cos(2B)$$

$$LHS = \cos(A+C)$$
$$= \cos(B-d+B+d)$$
$$= \cos(2B)$$
$$= RHS$$