Caringbah High School

2015

Trial HSC Examination

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen (Black pen is preferred)

- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper

- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I Pages 2 – 5

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 6 – 14

90 marks

- Attempt Questions 11–16
- Allow about 2 hour and 45 minutes for this section

Question 1 - 10 (1 mark each) Answer on page provided.

1 The argument of $z = -3i$ is:

A) $\frac{\pi}{2}$ B) $\frac{3\pi}{2}$ C) $-\frac{\pi}{2}$ D) $-\frac{3\pi}{2}$

2 Consider a polynomial $P(x)$ that has real coefficients. Which of the following could not be a pair of possible solutions to $P(x) = 0$?

A) $x_1 = -1 + i;\ x_2 = -1 - i$ B) $x_1 = 1 + i;\ x_2 = 1 - i$

C) $x_1 = -2 + i;\ x_2 = 2 - i$ D) $x_1 = -2 + i;\ x_2 = -2 - i$

3 Given that $z^3 = 1$ and ω is a complex solution, the value of

$\omega^3 + \omega^4 + \omega^5$ is:

A) 0 B) 1

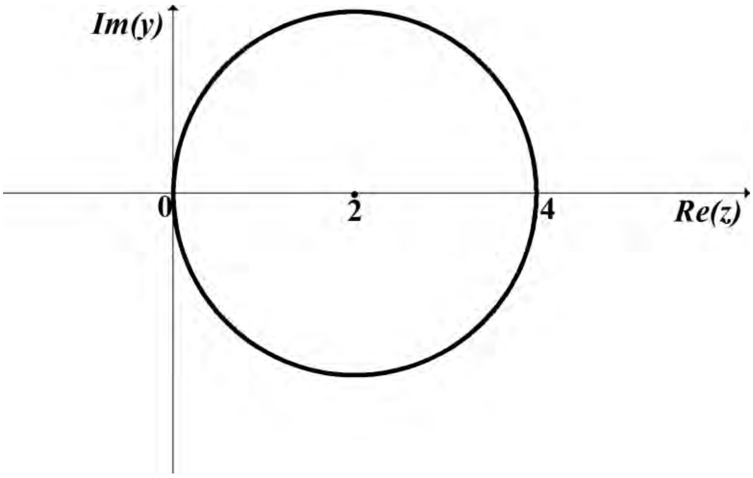
C) ω D) $-\omega$

4 The coordinates of the foci of the hyperbola $\frac{x^2}{4} - y^2 = 1$ are given by:

A) $\left(0, \pm\sqrt{5}\right)$ B) $\left(\pm\sqrt{5}, 0\right)$

C) $\left(0, \pm 2\sqrt{5}\right)$ D) $\left(\pm 2\sqrt{5}, 0\right)$

5 Which of the following is the equation of the circle shown below?



A) $(z+2)(\overline{z}+2)=4$

B) $(z-2)(\overline{z}-2)=4$

C) $(z+2i)(\overline{z}-2i)=4$

D) $(z-2i)(\overline{z}+2i)=4$

6 The roots of the polynomial $4x^3+4x-5=0$ are α, β and γ.

What is the value of $(\beta+\gamma-3\alpha)(\alpha+\beta-3\gamma)(\gamma+\alpha-3\beta)$?

A) 16

B) 80

C) −16

D) −80

7 The equation $x^2+2y^2-2xy+x=8$ defines *y* implicitly as a function of *x*.

What is the value of $\frac{dy}{dx}$ at the point $(3,2)$?

A) $\frac{1}{4}$

B) $-\frac{1}{4}$

C) $\frac{3}{2}$

D) $-\frac{3}{2}$

8 All of the integrals below are of the form $\int_{-1}^{1} f(x)\,dx$.

Without evaluating, which of the integrals can be rewritten as $2\int_{0}^{1} f(x)\,dx$?

A) $\int_{-1}^{1} e^x \tan^{-1}(x^2)dx$

B) $\int_{-1}^{1} \frac{x^2 \sin x}{x^2+5}\,dx$

C) $\int_{-1}^{1} \sqrt{x^2+e^x}\,dx$

D) $\int_{-1}^{1} x^3 \sin^{-1} x\,dx$

9 Using the recurrence relation $U_n = \int \tan^n x\,dx = \frac{\tan^{n-1} x}{n-1} - U_{n-2}$

what is the value of $\int \tan^6 x\,dx$?

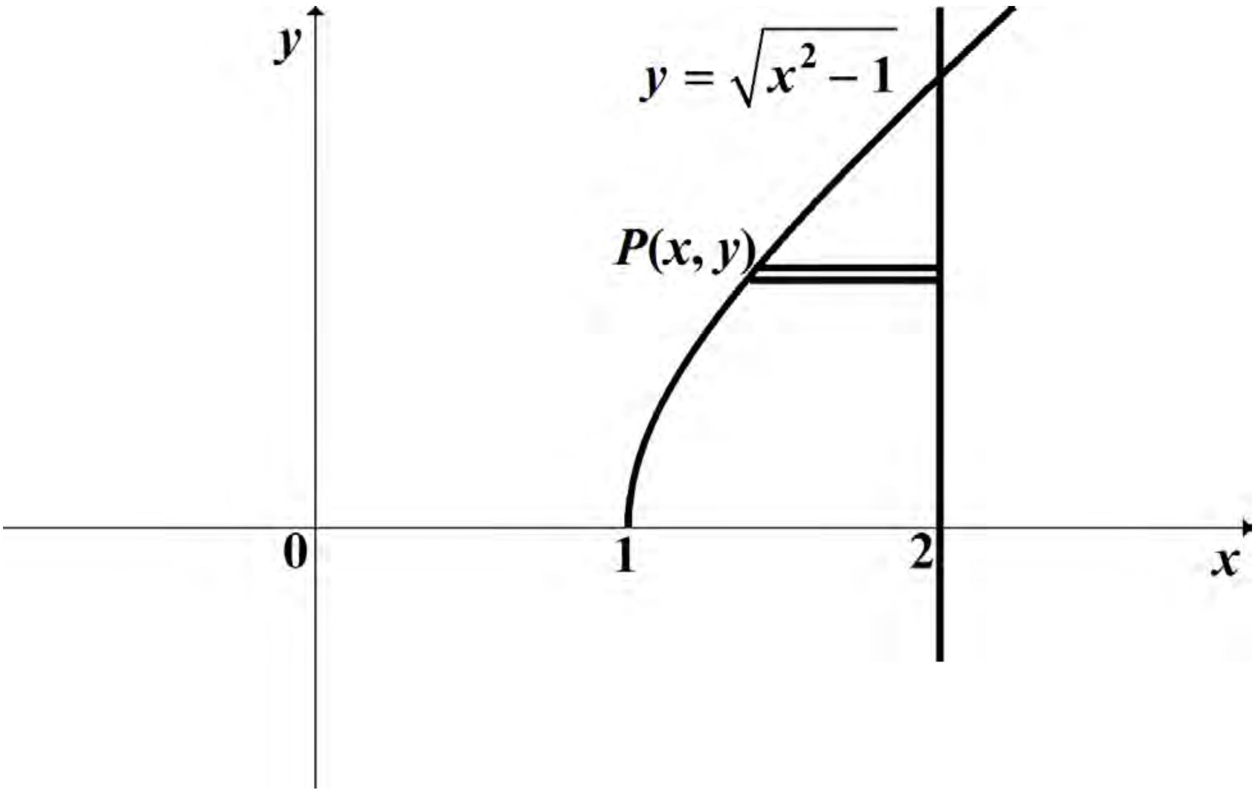
A) $\frac{\tan^5 x}{5} - \frac{\tan^4 x}{4} + \frac{\tan^3 x}{3} - \frac{\tan^2 x}{2} + \tan x + c$

B) $\frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x + c$

C) $\frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x + c$

D) $\frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} - \tan x - x + c$

10



The region bounded by the x-axis, the curve $y = \sqrt{x^2 - 1}$ and the line $x = 2$

is rotated around the y-axis.

The slice at $P(x, y)$ on the curve is perpendicular to the axis of rotation.

What is the volume δV of the annular slice formed?

A) $\pi\left(3 - y^2\right)\delta y$

B) $\pi\left(4 - \left(y^2 + 1\right)^2\right)\delta y$

C) $\pi\left(4 - x^2\right)\delta x$

D) $\pi\left(2 - x^2\right)\delta x$

END OF MULTIPLE CHOICE QUESTIONS

Section II

60 marks
Attempt all questions 11−14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11−14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a NEW booklet. **Marks**

a) Let $z = 1 + i$ and $u = 2 - i$. Find:

(i) $\text{Im}(u\,z)$. 1

(ii) $|u - z|$. 1

(iii) $-i\bar{u}$. 1

b) Evaluate $\int_0^1 x^2\sqrt{1-x}\;dx$. 3

c) It is given that $Z = \frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2}i$.

(i) Show that Z can be expressed in the form $\sqrt{2}\,cis\frac{\pi}{6}$. 2

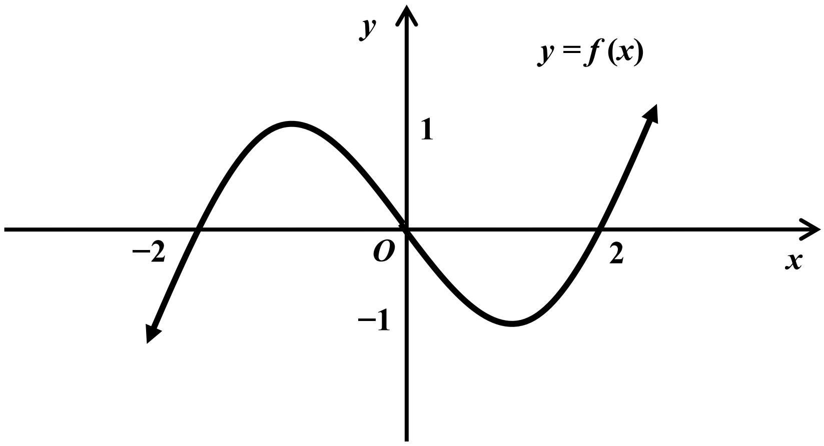
(ii) Hence, express Z^8 in $x + iy$ form. 2

d) (i) Prove that if the polynomial $P(x)$ has a root α of multiplicity m then $P'(x)$ has a root α of multiplicity $m-1$. 2

(ii) Hence, given that the polynomial $P(x) = x^4 + x^3 - 3x^2 - 5x - 2$ has a root of multiplicity 3, find all the roots of $P(x)$. 3

Question 12 (15 marks) Start a NEW booklet. **Marks**

a) Consider the function $y = f(x)$ drawn below.



On separate sketches, showing any important features, neatly draw the graphs of

(i) $y = 1 - f(x)$ 1

(ii) $y = \sqrt{f(x)}$ 1

(iii) $y = f\left(\frac{1}{x}\right)$ 2

(iv) $y = e^{f(x)}$ 2

b) Evaluate $\int_8^{12} \frac{1}{x^2 - 16x + 80}\, dx$ 3

Question 12 continues on page 8

Question 12 (continued) **Marks**

c) The equation $x^3 + 2x - 1 = 0$ has roots α, β and γ. Find

(i) the value of $\alpha^2 + \beta^2 + \gamma^2$. 2

(ii) the value of $\alpha^3 + \beta^3 + \gamma^3$. 2

(iii) the equation with roots $-\alpha, -\beta, -\gamma$. 2

End of Question 12

Question 13 (15 marks) Start a NEW booklet. **Marks**

a) Consider the locus of the point $P(x, y)$ whose co-ordinates satisfy the parametric equations $x = \cos t,\ y = 1 - \cos 2t.$

(i) State the domain and range of the locus of P. 2

(ii) Determine the Cartesian equation of the locus of P. 2

(iii) Neatly sketch the curve traced by the point $P(x, y)$. 1

b) Find $\displaystyle\int \frac{\cos x}{\sin x + \sin^2 x}\, dx$. 4

c) An ellipse has the equation $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$

(i) Prove that the tangent to the ellipse at $P(4\cos\theta, 3\sin\theta)$ has equation $\dfrac{x\cos\theta}{4} + \dfrac{y\sin\theta}{3} = 1$. 3

(ii) The ellipse meets the y-axis at B and B'. The tangents at B and B' meet the tangent at P at the points Q and Q'. 3

Find $BQ \times B'Q'$.

Question 14 (15 marks) Start a NEW booklet. **Marks**

a) Consider the functions $f(x) = |x| + 1$ and $g(x) = \dfrac{6}{|x|}$.

(i) Solve the equation $f(x) = g(x)$. 2

(ii) Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the same axes. 2

(iii) Hence or otherwise solve the inequality $g(x) > f(x)$. 1

b) i) Prove that $a^2 + b^2 \geq 2ab$. 1

ii) Hence or otherwise prove that $(p+2)(q+2)(p+q) \geq 16pq$ 2

where p and q are positive real numbers.

c) The base of a solid is the region between the lines $y = 3x$ and $y = -x$ from $x = 0$ to $x = 2$.

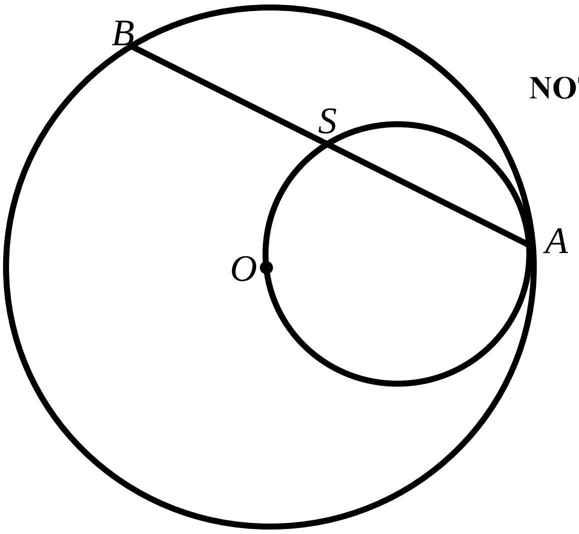
Each cross section by planes perpendicular to the x-axis is a square.

Calculate the volume of the solid. 3

Question 14 continues on page 11

Question 14 (continued) **Marks**

d) In the diagram below the two circles touch internally at *A*. *O* is the centre of the larger circle. *B* is a point on the larger circle and chord *AB* cuts the smaller circle at *S*.



Answer this question on the page provided

Prove that chord *AB* is bisected at *S*. 4

[Ensure that any constructions are clearly shown and labelled]

End of Question 14

Question 15 (15 marks) Start a NEW booklet. **Marks**

a) Let z be a complex number such that $|z| = r$ and $\arg z = \theta$ for $0 < \theta < \dfrac{\pi}{2}$.

Prove that $\arg\left(r^2 - z^2\right) = \theta - \dfrac{\pi}{2}$. 3

b) (i) Show that $\dfrac{d}{dx}\left(\dfrac{x^2}{2}\ln x - \dfrac{x^2}{4}\right) = x\ln x$ 1

(ii) Using $\dfrac{1}{3}$ of a page neatly sketch the region in the number plane that contains all points satisfying simultaneously the inequalities 1

$$x \le 1,\ y \ge 1,\ y \le e^x.$$

(iii) This region is rotated through one complete revolution about the *x*-axis.

Use the method of cylindrical shells to find the volume of the resulting solid. 4

c) If n is a positive integer and $f(x) = e^{-x}\left(1 + x + \dfrac{x^2}{2!} + \ldots\ldots\ldots\ldots\ldots + \dfrac{x^n}{n!}\right),\ x \ge 0$

(i) Show that $f(x)$ is a decreasing function. 3

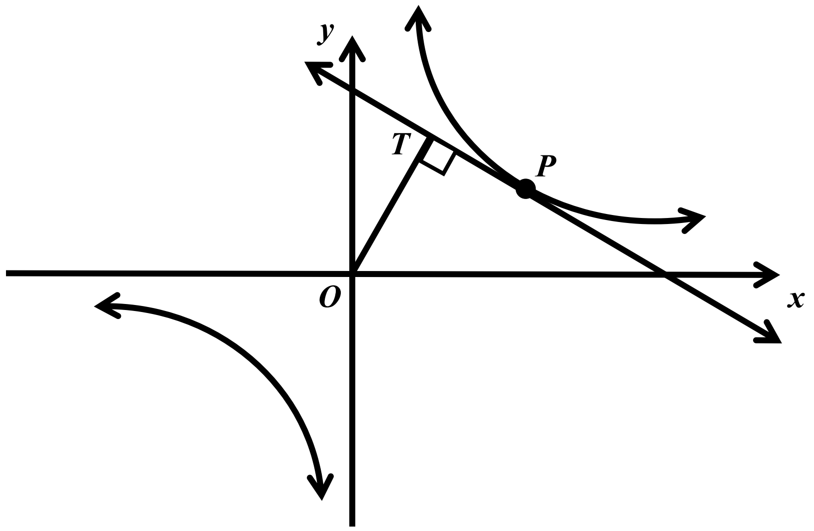
(ii) Deduce that for $x > 0$ and n any positive integer, 3

$$e^x \ge 1 + x + \frac{x^2}{2!} + \ldots\ldots\ldots\ldots\ldots + \frac{x^n}{n!}$$

Question 16 (15 marks) Start a NEW booklet. **Marks**

a) Find $\displaystyle\int \frac{e^{2x}}{e^x - 1}\,dx$. 3

b)



The point $P\left(cp, \dfrac{c}{p}\right)$ lies on the hyperbola $xy = c^2$. The point T lies at the foot of the perpendicular drawn from the origin O to the tangent at P.

(i) Show that the tangent at P has equation $x + p^2 y = 2cp$. 2

(ii) If the coordinates of T are (x_1, y_1) show that $y_1 = p^2 x_1$. 1

(iii) Show that the locus of T is given by $\left(x^2 + y^2\right)^2 = 4c^2 xy$. 2

Question 16 continues on page 14

Question 16 (continued) **Marks**

c) Consider the integral $I_n = \int_0^1 x^{2n+1} e^{-x^2}\, dx$.

(i) Use integration by parts to show that $I_n = -\frac{1}{2e} + nI_{n-1}$, for $n \geq 1$. 2

(ii) Show that $I_0 = \frac{1}{2} - \frac{1}{2e}$ and $I_1 = \frac{1}{2} - \frac{1}{e}$ 1

(iii) Prove by mathematical induction that for all $n \geq 1$: 3

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots\dots\dots + \frac{1}{n!} = e - \frac{2eI_n}{n!}$$

(iv) It is given that $0 \leq I_n \leq 1$ because $0 \leq x^{2n+1} e^{-x^2} \leq 1$, for $0 \leq x \leq 1$.

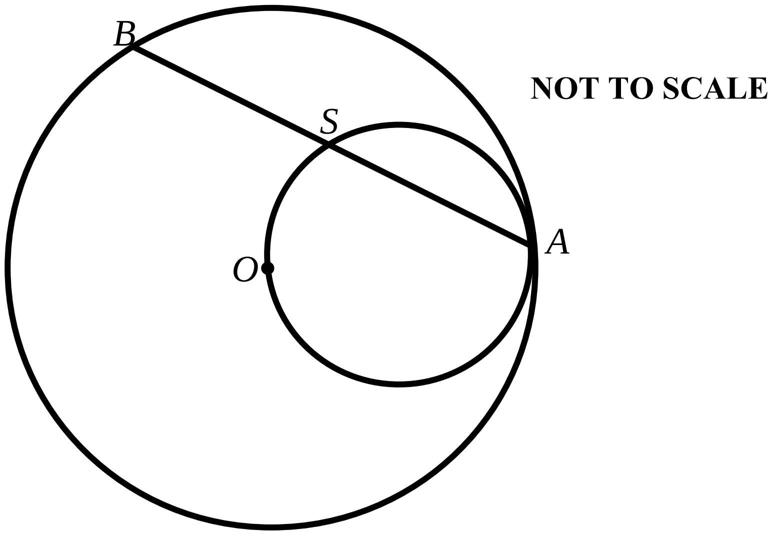
[Do not prove this]

Use this fact to help to evaluate $\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots\dots\dots$ 1

giving your answer in exact form.

END OF EXAM

Question 14d (additional diagram)   Candidate Name/Number: ________________________________



Hand in this page inside the Question 14 booklet

CHS YEAR 12 EXTENSION 2 2015	TRIAL HSC SOLUTIONS

Multiple Choice Section:

1.C	2.C	3.A	4.B	5.B
6.D	7.D	8.D	9.C	10.A

Question 1.

$-\frac{\pi}{2}$ ---------- $\boxed{C}$

Question 2.

Solutions need to be complex conjugates as coefficients are real

$x_1 = -2 + i;\ x_2 = 2 - i$ are not ---------- $\boxed{C}$

Question 3.

$\omega^3 - 1 = 0 \;\rightarrow\; (\omega - 1)(1 + \omega + \omega^2) = 0$

and since $\omega \neq 1$ (must be non-real), then

$\omega^3 + \omega^4 + \omega^5 = 1 + \omega + \omega^2$

$= 0$ ---------- $\boxed{A}$

Question 4.

$a = 2; b = 1;\ b^2 = a^2(e^2 - 1);$ Foci $(\pm ae, 0)$

$\therefore\ e^2 = \frac{5}{4} \;\rightarrow\; e = \frac{\sqrt{5}}{2}$

$\therefore$ Foci $(\pm\sqrt{5}, 0)$ ------------ $\boxed{B}$

Question 5.

Equation in Cartesian form is $(x-2)^2 + y^2 = 4$

$(z-2)(\bar{z}-2) = 4 \;\rightarrow\; z\bar{z} - 2(z + \bar{z}) + 4 = 4$

$\therefore\ x^2 + y^2 - 2(2x) + 4 = 4$

$\therefore\ x^2 - 4x + 4 + y^2 = 4$

$\therefore\ (x-2)^2 + y^2 = 4$ ------------ $\boxed{B}$

Question 6.

$(\beta + \gamma - 3\alpha)(\alpha + \beta - 3\gamma)(\gamma + \alpha - 3\beta)$

$= (\alpha + \beta + \gamma - 4\alpha)(\alpha + \beta + \gamma - 4\beta)(\alpha + \beta + \gamma - 4\gamma)$

Now $\alpha + \beta + \gamma = -\frac{b}{a} = 0$ and $\alpha\beta\gamma = \frac{c}{a} = \frac{5}{4}$

$\therefore\ = (-4\alpha)(-4\beta)(-4\gamma)$

$= -64\alpha\beta\gamma$

$= -64 \times \frac{5}{4} = -80$ ------------ $\boxed{D}$

Question 7.

$x^2 + 2y^2 - 2xy + x = 8$

$\therefore\ 2x + 4y\frac{dy}{dx} - 2\left(x \times \frac{dy}{dx} + y \times 1\right) + 1 = 0$

$\therefore\ 2(2y - x)\frac{dy}{dx} = 2y - 2x - 1$

$\therefore\ \frac{dy}{dx} = \frac{2y - 2x - 1}{2(2y - x)}$ and at $P(3,2)$

$\frac{dy}{dx} = \frac{4 - 6 - 1}{2(4 - 3)} = -\frac{3}{2}$ ------------ $\boxed{D}$

Question 8.

Let $f(x) = x^3 \sin^{-1} x$

$\therefore\ f(-x) = (-x)^3 \sin^{-1}(-x)$

$= -x^3 \times -\sin^{-1} x$

$= x^3 \sin^{-1} x = f(x)$

hence $f(x)$ is an even function

hence ------------ $\boxed{D}$

Question 9.

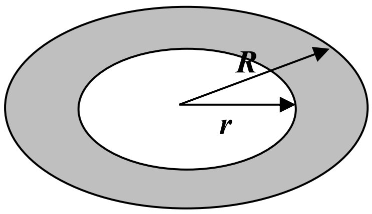
$$U_6 = \frac{\tan^5 x}{5} - U_4$$
$$U_4 = \frac{\tan^3 x}{3} - U_2$$

$$U_2 = \frac{\tan x}{1} - U_0$$

$$U_0 = \int 1\, dx = x + c$$

$$\therefore U_6 = \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x + c \quad --- \boxed{C}$$

Question 10.



where $\boldsymbol{R} = 2$
and $\boldsymbol{r} = \boldsymbol{x}$

$$\therefore \text{ Area of slice} = \pi\left(2^2 - x^2\right)$$
$$= \pi\left(4 - \left(y^2 + 1\right)\right)$$
$$\therefore \text{ Volume of slice} = \pi\left(3 - y^2\right)\delta y \quad ------- \boxed{A}$$

Question 11

a) i) $\text{Im}\left[(1+i)(2-i)\right] = \text{Im}(3 + 1i) = 1$

ii) $|u - z| = |1 - 2i| = \sqrt{5}$

iii) $-i\bar{u} = -i \times (2 + i) = 1 - 2i$

b) Let $u = 1 - x \;\to\; x = 1 - u$ and $du = -dx$
When $x = 0, u = 1;\; x = 1, u = 0.$

$$\therefore \quad I = -\int_1^0 (1-u)^2 \sqrt{u}\; du$$

$$= \int_0^1 u^{\frac{1}{2}} - 2u^{\frac{3}{2}} + u^{\frac{5}{2}}\, du$$

$$= \left[\frac{2}{3}u^{3/2} - \frac{4}{5}u^{5/2} + \frac{2}{7}u^{7/2}\right]_0^1$$

$$= \frac{2}{3} - \frac{4}{5} + \frac{2}{7} = \frac{16}{105}$$

c) i) $|Z| = r = \sqrt{\frac{6}{4} + \frac{2}{4}} = \sqrt{2}$

$$\arg Z = \theta = \tan^{-1}\left(\frac{\sqrt{2}/2}{\sqrt{6}/2}\right)$$

$$\therefore \; \theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

ii) $$Z^8 = \left(\sqrt{2}\,cis\left(\frac{\pi}{6}\right)\right)^8$$
$$= 2^4 cis\left(\frac{8\pi}{6}\right) \quad \textbf{\textit{DMT}}$$

$$= 16\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$$

$$= 16\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -8\left(1 + \sqrt{3}i\right)$$

d) i) Let $P(x) = (x-\alpha)^m Q(x)$

$$\therefore P'(x) = m(x-\alpha)^{m-1}Q(x) + (x-\alpha)^m Q'(x)$$

$$= (x-\alpha)^{m-1}\left(mQ(x) + (x-\alpha)Q'(x)\right)$$

hence $P'(x)$ has a root α of multiplicity $m - 1$.

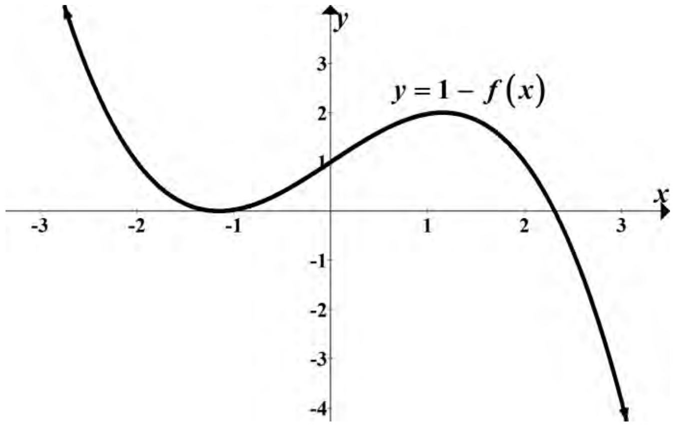
ii) $P(x) = x^4 + x^3 - 3x^2 - 5x - 2$
$P'(x) = 4x^3 + 3x^2 - 6x - 5$
$P''(x) = 12x^2 + 6x - 6$
$P''(x) = 0 \;\to\; 2x^2 + x - 1 = 0$
$\therefore (2x-1)(x+1) = 0 \;\to\; x = \frac{1}{2}$ or $x = -1$

when $x = -1,\; P'(-1) = -4 + 3 + 6 - 5 = 0$
when $x = -1,\; P(-1) = 1 - 1 - 3 + 5 - 2 = 0$
hence $x = -1$ is a root of multiplicity 3

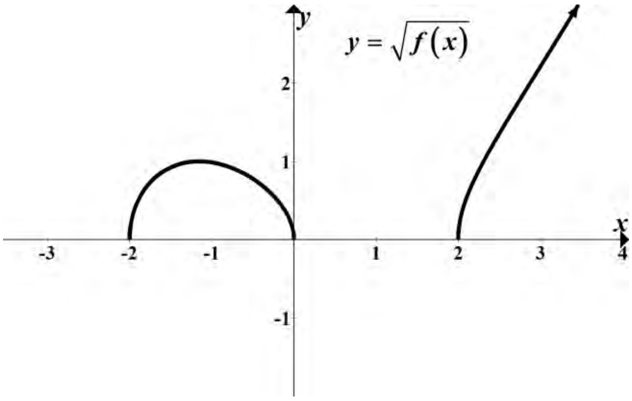
$\therefore P(x) = (x+1)^3(x-2)$ by inspection
hence $P(x)$ has roots $x = -1, -1, -1, 2$.

Question 12.

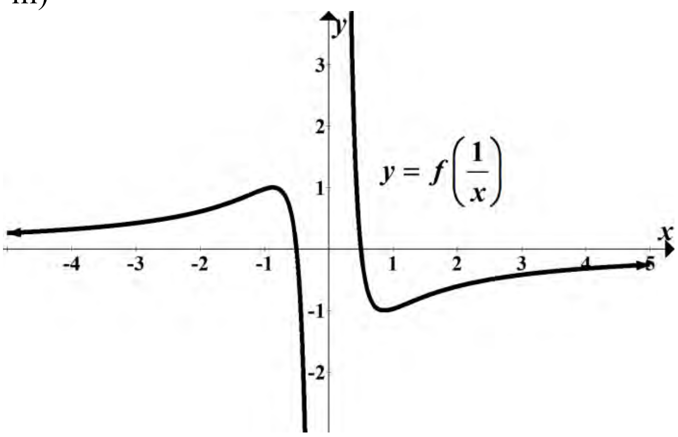
a) i)



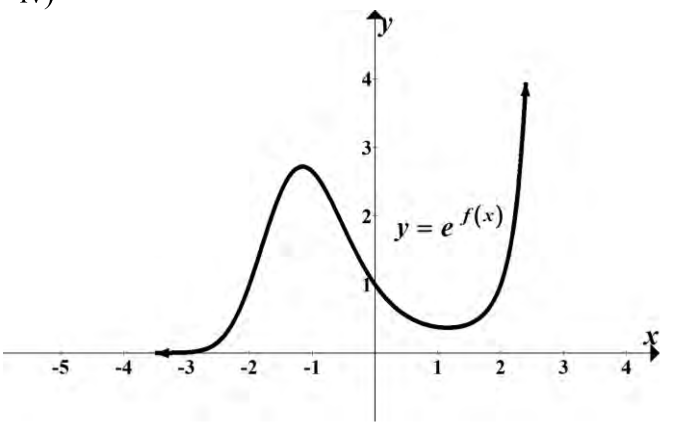
ii)



iii)



iv)



b) $I = \int_8^{12} \frac{1}{\left(x^2 - 16x + 64\right) + 16}\, dx$

$= \int_8^{12} \frac{1}{(x-8)^2 + 4^2}\, dx$

$= \frac{1}{4}\left[\tan^{-1}\left(\frac{x-8}{4}\right)\right]_8^{12}$

$= \frac{1}{4}\left(\tan^{-1}1 - \tan^{-1}0\right) = \frac{\pi}{16}$

c) i)

$\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$= (0)^2 - 2(2) = -4$

ii) $x^3 + 2x - 1 = 0 \rightarrow x^3 = 1 - 2x$

$\therefore\ \alpha^3 = 1 - 2\alpha$

$\beta^3 = 1 - 2\beta$

$\gamma^3 = 1 - 2\gamma$

$\therefore\ \alpha^3 + \beta^3 + \gamma^3 = 3 - 2(\alpha+\beta+\gamma)$

$= 3 - 2(0) = 3$

iii) Let $x = -\alpha \rightarrow \alpha = -x$

$\therefore$ the required equation is given by

$\therefore\ (-x)^3 + 2(-x) - 1 = 0$

$\therefore\ -x^3 - 2x - 1 = 0$

$\therefore\ x^3 + 2x + 1 = 0$

Question 13.

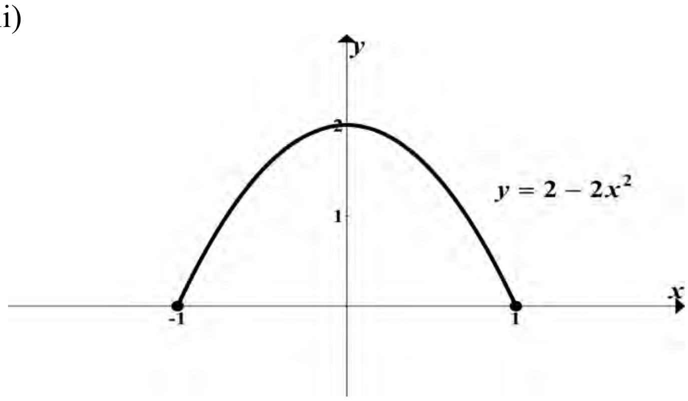
a) i) D: $-1 \le x \le 1$; R: $0 \le y \le 2$

ii) $y = 1 - \left(2\cos^2 t - 1\right)$

$= 2 - 2(\cos t)^2$

$\therefore\ y = 2 - 2x^2$

iii)



13b) $I=\int\frac{\cos x}{\sin x+\sin^2 x}\,dx$

Let $u=\sin x \;\rightarrow\; du=\cos x\,dx$

$\therefore\; I=\int\frac{1}{u+u^2}\,du=\int\frac{1}{u(1+u)}\,du$

Now $\frac{1}{u(1+u)}=\frac{1}{u}-\frac{1}{u+1}$ using partial fractions

$\therefore\; I=\int\frac{1}{u}-\frac{1}{u+1}\,du$

$=\ln u-\ln(1+u)$

$=\ln\left(\frac{u}{1+u}\right)$

$\therefore\; I=\ln\left(\frac{\sin x}{1+\sin x}\right)+c$

c) i) $\frac{x^2}{16}+\frac{y^2}{9}=1 \;\rightarrow\; a=4,\, b=3$

$\therefore\; \frac{2x}{16}+\frac{2y}{9}\cdot\frac{dy}{dx}=0 \;\rightarrow\; \frac{dy}{dx}=-\frac{9x}{16y}$

$\therefore$ at $P(4\cos\theta, 3\sin\theta)$, $\frac{dy}{dx}=-\frac{36\cos\theta}{48\sin\theta}=-\frac{3\cos\theta}{4\sin\theta}$

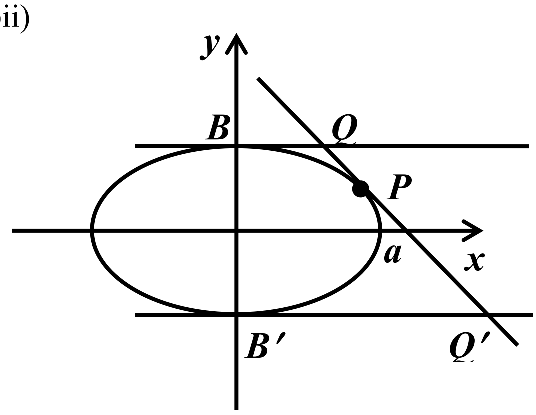
$\therefore$ eq^n of T at P: $y-3\sin\theta=-\frac{3\cos\theta}{4\sin\theta}(x-4\cos\theta)$

$4y\sin\theta-12\sin^2\theta=-3x\cos\theta+12\cos^2\theta$

$3x\cos\theta+4y\sin\theta=12(\sin^2\theta+\cos^2\theta)$

$\therefore\; \frac{x\cos\theta}{4}+\frac{y\sin\theta}{3}=1\;\left[\sin^2\theta+\cos^2\theta=1\right]$

13c)ii)



At Q: $y=3 \;\rightarrow\; x=\frac{4(1-\sin\theta)}{\cos\theta}$

At Q′: $y=-3 \;\rightarrow\; x=\frac{4(1+\sin\theta)}{\cos\theta}$

$\therefore\; BQ\times B'Q'=\frac{4(1-\sin\theta)}{\cos\theta}\times\frac{4(1+\sin\theta)}{\cos\theta}$

$=\frac{16(1-\sin^2\theta)}{\cos^2\theta}=16$

Question 14.

a) i) $|x|+1=\frac{6}{|x|}$

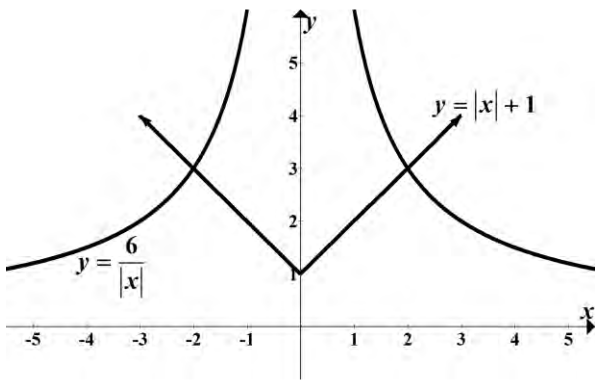
$\therefore\; |x|^2+|x|-6=0$

$\therefore\; (|x|+3)(|x|-2)=0$

$\therefore\; |x|=-3 \;\rightarrow$ no solution

or $|x|=2 \;\rightarrow\; x=\pm 2$

ii)



iii) $-2<x<0$ and $0<x<2$

b) i) $(a-b)^2 \geq 0$

$\therefore\ a^2 - 2ab + b^2 \geq 0$

$\therefore\ a^2 + b^2 \geq 2ab$

ii) using (i) let $a = \sqrt{p}$ and $b = \sqrt{q}$

$\therefore\ \sqrt{p}^2 + \sqrt{q}^2 \geq 2\sqrt{p}\sqrt{q}$

$\therefore\ p + q \geq 2\sqrt{p}\sqrt{q}$ -------- $\boxed{1}$

similarly let $a = \sqrt{p}$ and $b = \sqrt{2}$

$\therefore\ p + 2 \geq 2\sqrt{p}\sqrt{2}$ --------- $\boxed{2}$

also let $a = \sqrt{q}$ and $b = \sqrt{2}$

$\therefore\ q + 2 \geq 2\sqrt{q}\sqrt{2}$ --------- $\boxed{3}$

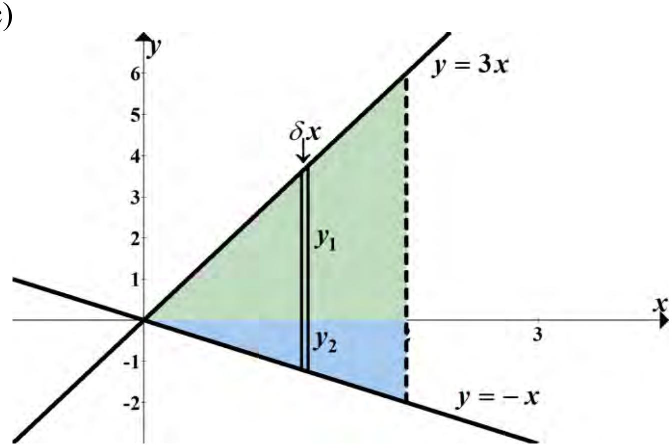
hence $\boxed{1} \times \boxed{2} \times \boxed{3}$ gives:

$(p+2)(q+2)(p+q) \geq 2\sqrt{p}\sqrt{q} \times 2\sqrt{p}\sqrt{2} \times 2\sqrt{q}\sqrt{2}$

$\therefore\ (p+2)(q+2)(p+q) \geq 16\left(\sqrt{p}\right)^2\left(\sqrt{q}\right)^2$

$\therefore\ (p+2)(q+2)(p+q) \geq 16pq$

c)



The area of a typical cross section is $\delta A = (y_1 + y_2)^2$

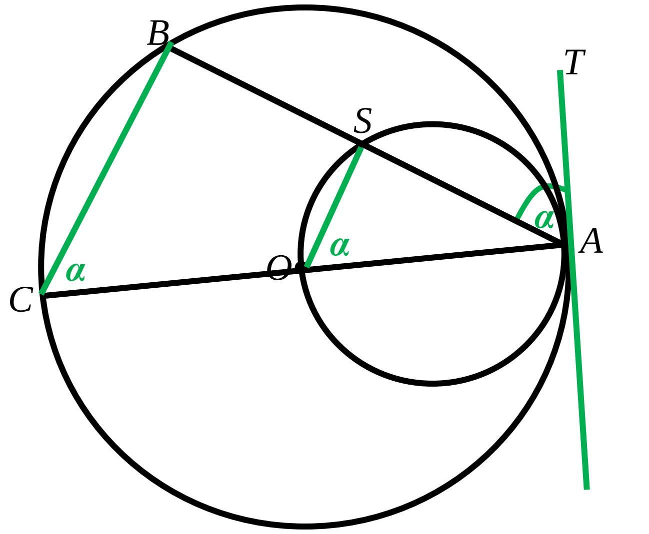
$\therefore\ \delta A = (3x + x)^2 = 16x^2$

$\therefore\ \delta V = 16x^2\,\delta x$

$\therefore\ V = 16\int_0^2 x^2\,dx$

$= 16\left[\frac{x^3}{3}\right]_0^2 = \frac{128}{3}u^3$

d)



Construct tangent *TA* at *A*. Construct diameter *AOC*.
Join *OS* and *BC*.

Let $\angle TAS = \alpha$.

$\therefore \angle SOA = \alpha$ [$\angle$ in alternate segment (ΔAOS)]

$\therefore \angle BCA = \alpha$ [$\angle$ in alternate segment (ΔBCA)]

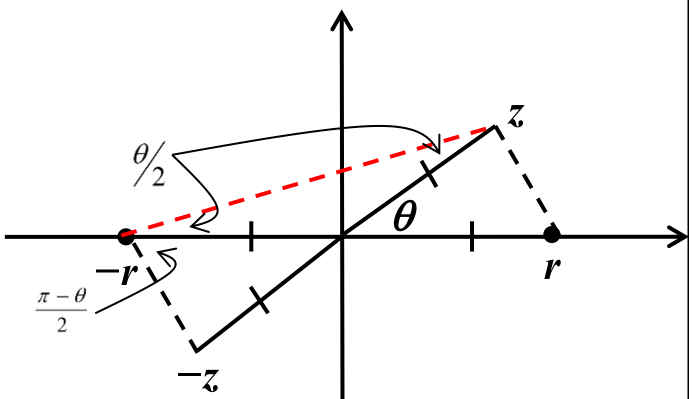
$\therefore \angle SOA = \angle BCA$ and since they are equal corresponding angles then *BC* is parallel to *OS*.

Also $OA = OC$ (equal radii)
and since a family of parallel lines divides transversals in the same ratio, then $BS = SA$.

Hence chord *AB* is bisected at *S*.
[Note: Conguent trianges can also be used.]

Question 15.

a)



$\arg(r^2 - z^2) = \arg(r - z) + \arg(r + z)$

Now $\arg(r + z) = \frac{\theta}{2}$ (by isoscelesΔ)

also $\arg(r - z) = \frac{\theta}{2} - \frac{\pi}{2}$ (by isoscelesΔ)

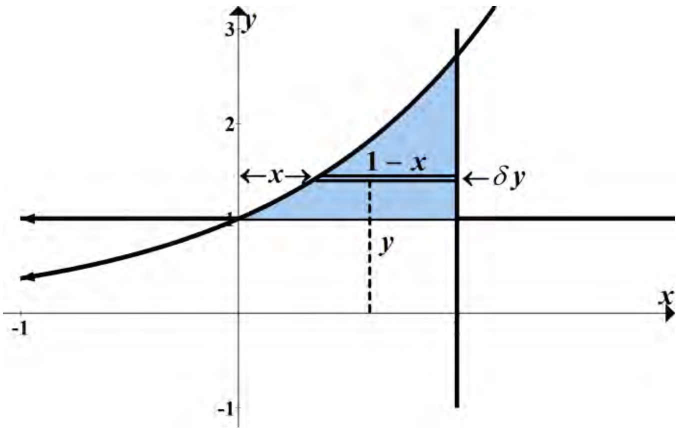
$\therefore \arg(r^2 - z^2) = \frac{\theta}{2} - \frac{\pi}{2} + \frac{\theta}{2} = \theta - \frac{\pi}{2}$

15b) i)

$$\frac{d}{dx}\left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right) = \frac{x^2}{2}\times\frac{1}{x} + \frac{2x}{2}\times\ln x - \frac{2x}{4}$$

$$= \frac{x}{2} + x\times\ln x - \frac{x}{2} = x\ln x$$

ii)



Taking the middle of the shell at a distance ‘ y ‘ units from the x-axis gives the following two radii:

$R = y + \frac{\delta y}{2}$ and $r = y - \frac{\delta y}{2}$

$\therefore$ Volume δV of typical cylinder $= \pi(R+r)(R-r)h$

$$= \pi\left(y + \frac{\delta y}{2} + y - \frac{\delta y}{2}\right)\left(y + \frac{\delta y}{2} - y + \frac{\delta y}{2}\right)h$$

$= 2\pi y h \delta y$ where $h = 1 - x$

Now $y = e^x \rightarrow x = \ln y$ [when $x = 1, y = e$]

$$\therefore \quad V = 2\pi\int_1^e y(1 - \ln y)\,dy$$

$$\therefore \quad V = 2\pi\int_1^e y\,dy - 2\pi\int_1^e y\ln y\,dy$$

$$= 2\pi\left[\frac{y^2}{2}\right]_1^e - 2\pi\left[\frac{y^2}{2}\ln y - \frac{y^2}{4}\right]_1^e$$

$$= 2\pi\left(\frac{e^2}{2} - \frac{1}{2}\right) - 2\pi\left[\left(\frac{e^2}{2}\ln e - \frac{e^2}{4}\right) - \left(\frac{1}{2}\ln 1 - \frac{1}{4}\right)\right]$$

$$= \pi\left(e^2 - 1\right) - \pi\left(e^2 - \frac{e^2}{2} + \frac{1}{2}\right)$$

$$= \frac{\pi}{2}\left(e^2 - 3\right)u^3$$

c) i) $f(x) = e^{-x}\left(1 + x + \frac{x^2}{2!} + \ldots\ldots\ldots\ldots + \frac{x^n}{n!}\right)$

$$\therefore f'(x) = e^{-x}\left(1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \ldots\ldots\ldots\ldots + \frac{nx^{n-1}}{n!}\right)$$

$$- e^{-x}\left(1 + x + \frac{x^2}{2!} + \ldots\ldots\ldots\ldots + \frac{x^n}{n!}\right)$$

$$\therefore f'(x) = e^{-x}\left(1 + x + \frac{x^2}{2!} + \ldots\ldots\ldots\ldots + \frac{x^{n-1}}{(n-1)!}\right)$$

$$- e^{-x}\left(1 + x + \frac{x^2}{2!} + \ldots\ldots\ldots\ldots + \frac{x^n}{n!}\right)$$

$$= -e^{-x}\left(\frac{x^n}{n!}\right) < 0 \text{ for all } x \geq 0.$$

since $e^{-x} > 0$ and $\frac{x^n}{n!} > 0$.

hence the curve is decreasing for all $x \geq 0$ as $f'(x) < 0$.

ii) When $x = 0$ $f(0) = 1$

and since the curve is always decreasing $f(x) \leq 1$.

$$\therefore e^{-x}\left(1 + x + \frac{x^2}{2!} + \ldots\ldots\ldots\ldots + \frac{x^n}{n!}\right) \leq 1$$

$$\therefore 1 + x + \frac{x^2}{2!} + \ldots\ldots\ldots\ldots + \frac{x^n}{n!} \leq \frac{1}{e^{-x}}$$

hence $e^x \geq 1 + x + \frac{x^2}{2!} + \ldots\ldots\ldots\ldots + \frac{x^n}{n!}$.

Question 16.

a) $\int \frac{e^{2x}}{e^x - 1}\,dx = \int \frac{e^x.e^x}{e^x - 1}\,dx$

Let $u = e^x - 1 \rightarrow du = e^x\,dx$

$$\therefore I = \int \frac{u+1}{u}\,du = \int 1 + \frac{1}{u}\,du$$

$$= u + \ln u$$

$$= e^x - 1 + \ln\left(e^x - 1\right) + C_1$$

$$= e^x + \ln\left(e^x - 1\right) + C_2$$

Question 16 continued:

b) i) $xy = c^2 \quad \rightarrow \quad y = c^2 x^{-1}$

$\therefore \quad y' = \frac{c^2}{x^2} \quad \rightarrow \quad m = -\frac{1}{p^2}$ when $x = cp$

Equation of tangent at P :

$y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$

$p^2 y - cp = -x + cp$

$\therefore \quad x + p^2 y = 2cp$

ii) since OT is perpendicular tangent at P it has p^2 as its gradient.

$\therefore$ equation of OT is $y = p^2 x$, and since $T(x_1, y_1)$ lies on OT then $y_1 = p^2 x_1$.

iii) Since T satisfies the equation of the tangent:

$x_1 + p^2 y_1 = 2cp$

and from (ii) $p^2 = \frac{y_1}{x_1}$

$\therefore \quad x_1 + \frac{y_1}{x_1} \bullet y_1 = 2c\sqrt{\frac{y_1}{x_1}}$

$\therefore \quad x_1^2 + y_1^2 = 2c\, x_1 \sqrt{\frac{y_1}{x_1}}$

$\therefore \quad x_1^2 + y_1^2 = 2c\sqrt{x_1 y_1}$

$\therefore \quad \left(x_1^2 + y_1^2\right)^2 = 4c^2 x_1 y_1$

hence the locus of T is given by

$\left(x^2 + y^2\right)^2 = 4c^2 xy.$

c) i) $I_n = \int_0^1 x^{2n+1} e^{-x^2}\, dx = \int_0^1 x^{2n} \cdot x e^{-x^2}\, dx$

$u = x^{2n}, \qquad v' = x e^{-x^2}$

$u' = 2n x^{2n-1}, \qquad v = -\frac{1}{2} e^{-x^2}$

$$\therefore \quad I_n = \left[-\frac{1}{2} x^{2n} e^{-x^2}\right]_0^1 + n\int_0^1 x^{2n-1} e^{-x^2}\, dx$$

$$= -\frac{1}{2} e^{-1} - 0 + n I_{n-1}$$

$$\therefore \quad I_n = -\frac{1}{2e} + n I_{n-1} \quad \text{-------} \boxed{*}$$

ii) $I_0 = \int_0^1 x e^{-x^2}\, dx$

$$= \left[-\frac{1}{2} e^{-x^2}\right]_0^1 = -\frac{1}{2}\left(e^{-1} - e^0\right)$$

$$= -\frac{1}{2}\left(\frac{1}{e} - 1\right) = \frac{1}{2} - \frac{1}{2e}$$

also $I_1 = -\frac{1}{2e} + I_0$

$$= -\frac{1}{2e} + \frac{1}{2} - \frac{1}{2e} = \frac{1}{2} - \frac{1}{e}$$

iii) When $n = 1$: $LHS = 1 + \frac{1}{1!} = 2$

$$RHS = e - \frac{2e I_1}{1!}$$

$$= e - 2e\left(\frac{1}{2} - \frac{1}{e}\right)$$

$$= e - e + 2 = 2 = LHS$$

hence true for $n = 1$

Assume true for $n = k$:

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{k!} = e - \frac{2e I_k}{k!}$$

Prove true for $n = k + 1$:

i.e. $S_k + T_{k+1} = S_{k+1}$ $\left(\text{where } S_{k+1} = e - \frac{2e I_{k+1}}{(k+1)!}\right)$

$$LHS = e - \frac{2e I_k}{k!} + \frac{1}{(k+1)!}$$

$$= e - \frac{2e(k+1) I_k}{(k+1)!} + \frac{1}{(k+1)!}$$

$$= e + \frac{1 - 2e(k+1) I_k}{(k+1)!}$$

$$= e + \frac{1 - 2e\left(\frac{1}{2e} + I_{k+1}\right)}{(k+1)!}$$

$$= e + \frac{1 - 1 - 2eI_{k+1}}{(k+1)!}$$

$$= e - \frac{2eI_{k+1}}{(k+1)!} = S_{k+1}$$

Hence by induction is true for all $n \geq 1$.

NOTE: $I_{k+1} = -\frac{1}{2e} + (k+1)I_k$ using $\boxed{*}$

$\therefore \; (k+1)I_k = \frac{1}{2e} + I_{k+1}$

iv) $0 \leq I_n \leq 1$, hence as $n \to \infty$, $\frac{I_n}{n!} \to 0$

$$\therefore \quad 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots\dots\dots + \frac{1}{n!} = e - 2e \times 0 = e$$

$$\therefore \quad \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots\dots\dots = e - 1$$