Caringbah High School

2018

Trial HSC Examination

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen only
- NESA approved calculators may be used
- A reference sheet is provided
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I **10 marks**
Pages 2 – 6
- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II **90 marks**
Pages 7 – 15
- Attempt Questions 11–16
- Allow about 2 hour and 45 minutes for this section

Question 1 - 10 (1 mark each) Answer on page provided.

1 Let $z = 4 - 3i$. What is the value of $\overline{iz}$?

A) $3 + 4i$

B) $3 - 4i$

C) $-3 + 4i$

D) $-3 - 4i$

2 What is the remainder when $x^3 + x^2 + 5x + 6$ is divided by $(x + i)$?

A) $7 - 4i$

B) $7 - 6i$

C) $5 - 4i$

D) $5 + 6i$

3 The roots of the equation $2x^3 - 3x^2 + 2x + 2 = 0$ are α, β and γ.
What is the value of $\alpha + \beta - \dfrac{1}{\alpha\beta}$?

A) -1

B) 1

C) $-\dfrac{3}{2}$

D) $\dfrac{3}{2}$

4 What are the coordinates of the foci of the ellipse $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$?

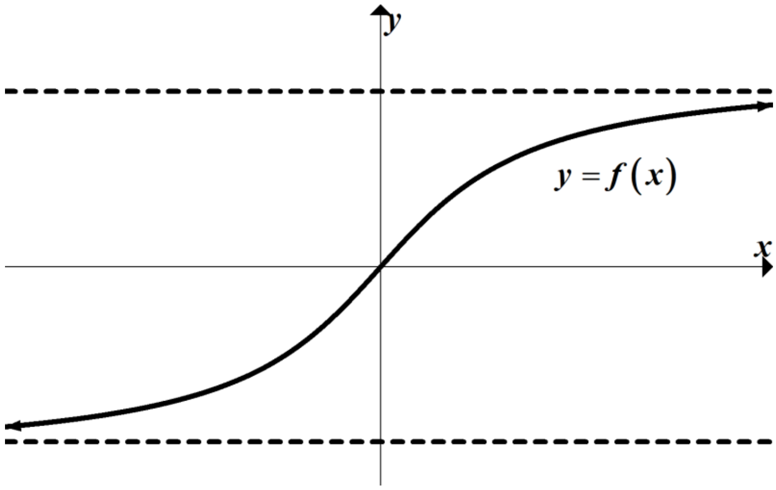
A) $\left(\pm\dfrac{2\sqrt{5}}{3}, 0\right)$

B) $\left(0, \pm\dfrac{2\sqrt{5}}{3}\right)$

C) $\left(\pm\sqrt{5}, 0\right)$

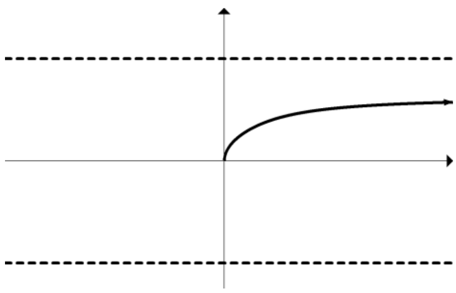
D) $\left(0, \pm\sqrt{5}\right)$

5 The graph of $y = f(x)$ is shown below?

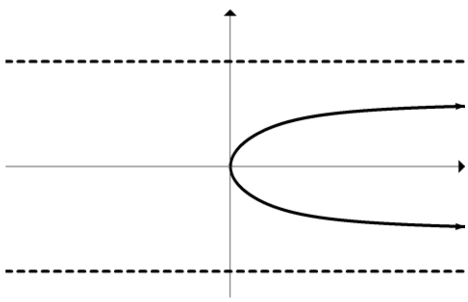


Which of the following graphs best represents $y^2 = f(x)$?

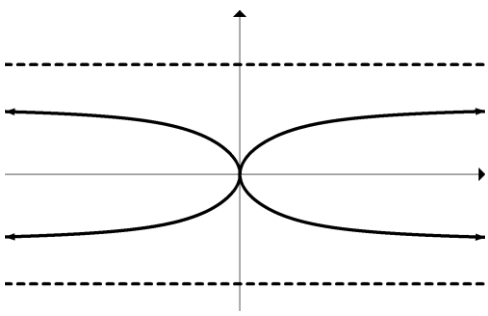
A)



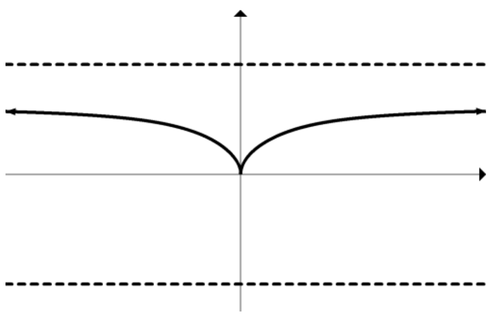
B)



C)



D)



6 Consider $I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2 x}{(1 + \tan x)^2} \, dx$.

After using an appropriate substitution, which of the following is equivalent to I ?

A) $\int_0^2 \frac{1}{u^2} \, du$

B) $\int_0^2 \frac{u^2}{(1+u)^3} \, du$

C) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{u} \, du$

D) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{u^3} \, du$

7 If $e^x + e^y = 1$, which of the following is an expression for $\frac{dy}{dx}$?

A) $-e^{x-y}$

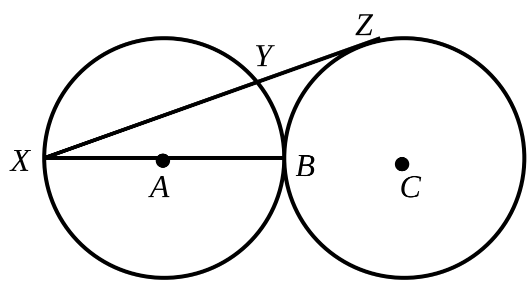
B) $-e^{y-x}$

C) e^{x-y}

D) e^{y-x}

8 Two equal circles touch externally at B. XB is a diameter of the circle centred at A.

XZ is the tangent from X to the circle centred at C and cuts the first circle at Y.



Which is the correct expression that relates XZ to XY ?

A) $3XZ = 4XY$

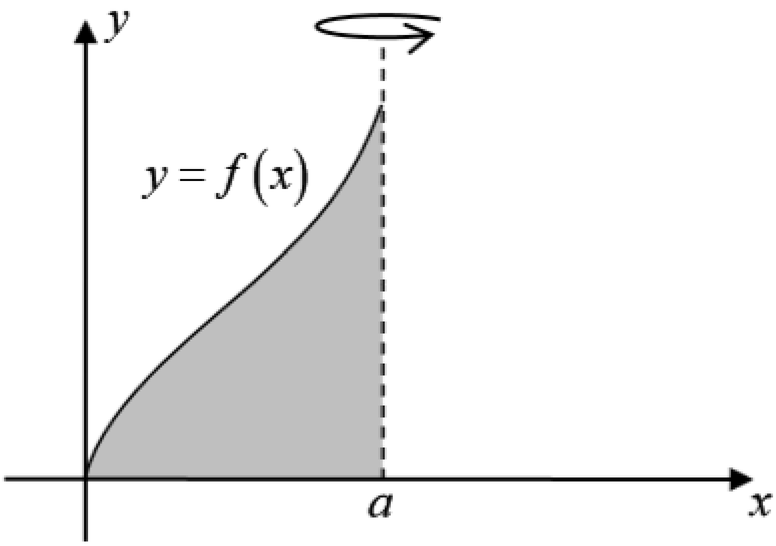
B) $XZ = 2XY$

C) $2XZ = 3XY$

D) $2XZ = 5XY$

9 The function $y = f(x)$ is monotonic increasing over the interval $0 \le x \le a$.

The region bounded by this function, the x–axis and the line $x = a$ is to be rotated about the line $x = a$ to form a solid of revolution.



Which of the following integrals represents the volume of this solid?

A) $\pi \int_0^a \left[a - f(x)\right]^2 \, dx$

B) $\pi \int_0^a [a - x]^2 \, dx$

C) $\pi \int_0^{f(a)} \left[a - f^{-1}(y)\right]^2 \, dy$

D) $\pi \int_0^{f(a)} [a - y]^2 \, dy$

10 Consider $\omega = \frac{1}{2}\left(-1 + i\sqrt{3}\right)$.

Which of the following is a correct expression for $|z|$, where $z = a + b\omega$ and a and b are constants.

A) $\sqrt{(a-b)^2 + 2ab}$

B) $\sqrt{(a-b)^2 + ab}$

C) $\sqrt{(a-b)^2 - ab}$

D) $\sqrt{(a-b)^2 - 2ab}$

END OF MULTIPLE CHOICE QUESTIONS

Section II

90 marks
Attempt all questions 11−16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11−16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a NEW booklet. **Marks**

a) Let $z = 1 - i$ and $u = 3 + 2i$. Find:

(i) $\text{Im}(u - z)$. **1**

(ii) $z\bar{u}$. **1**

b) Let the point P represent the complex number z in the Argand diagram. **1**

Describe the features of the new point Q representing the complex number $2iz$.

c) Find $\displaystyle\int \frac{dx}{x^2 + 6x + 13}$. **2**

d) It is given that $Z = 1 - \sqrt{3}\,i$.

(i) Express Z in modulus – argument form. **2**

(ii) Hence, express Z^{10} in $x + iy$ form. **2**

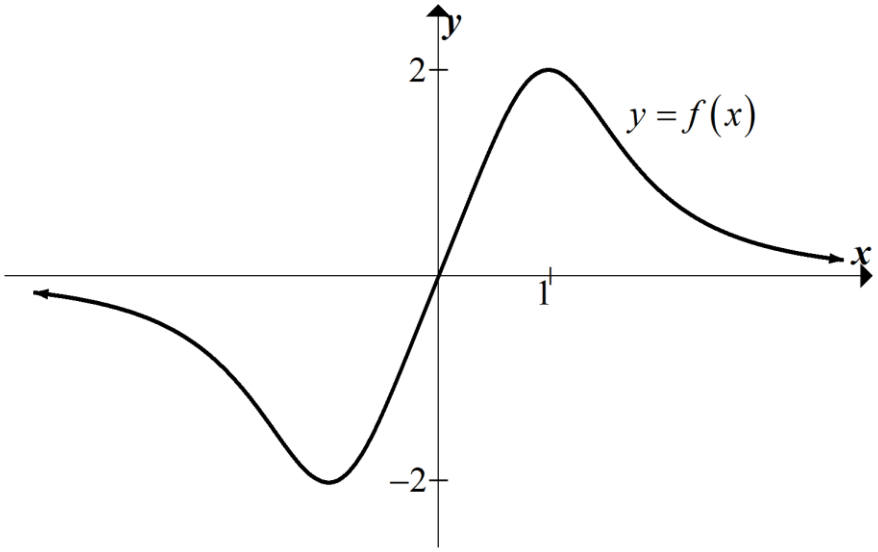
e) Find the x-coordinates of the points on the curve $2x^2 + 2xy + 3y^2 = 15$ where the tangents to the curve are vertical. **3**

f) (i) Find the domain and range of $f(x) = \tan^{-1}(e^x)$. **2**

(ii) Sketch $y = f(x)$ showing any intercepts and asymptotes. **1**

Question 12 (15 marks) Start a NEW booklet. **Marks**

a) The diagram below shows the graph of $y = f(x)$ which is an odd function.



On separate sketches, showing any important features, neatly draw the graphs of

(i) $y = f(-x)$ **1**

(ii) $y = \sqrt{f(x)}$ **2**

(iii) $y = \dfrac{1}{f(x)}$ **2**

(iv) $y = x + f(x)$ **2**

(v) $y = f'(x)$ **2**

Question 12 continues on page 9

Question 12 (continued) **Marks**

b) Use the substitution $t = \tan\frac{\theta}{2}$ to evaluate $\int_0^{\frac{\pi}{3}} \frac{d\theta}{1+\sin\theta}$. **3**

c) C is a unit circle with its centre at the origin on the Argand diagram. If the point representing z_1 moves on the circle C and $z_2 = \frac{1-i}{z_1}$, find the locus of z_2 and describe it geometrically. **3**

End of Question 12

Question 13 (15 marks) Start a NEW booklet. **Marks**

a) Consider the polynomial $P(x) = x^4 - 8x^3 + 18x^2 - 27$. **3**

Factorise $P(x)$, given that it has a root of multiplicity 3.

b) Find $\int \frac{1}{x^2}\sqrt{1+\frac{4}{x}}\,dx$. **3**

c) Express $\dfrac{1+\cos\theta - i\sin\theta}{1+\cos\theta + i\sin\theta}$ in the form $a + ib$. **3**

d) The roots of the cubic equation $x^3 - 4x^2 - 12 = 0$ are α, β and γ.

Find the equation with roots:

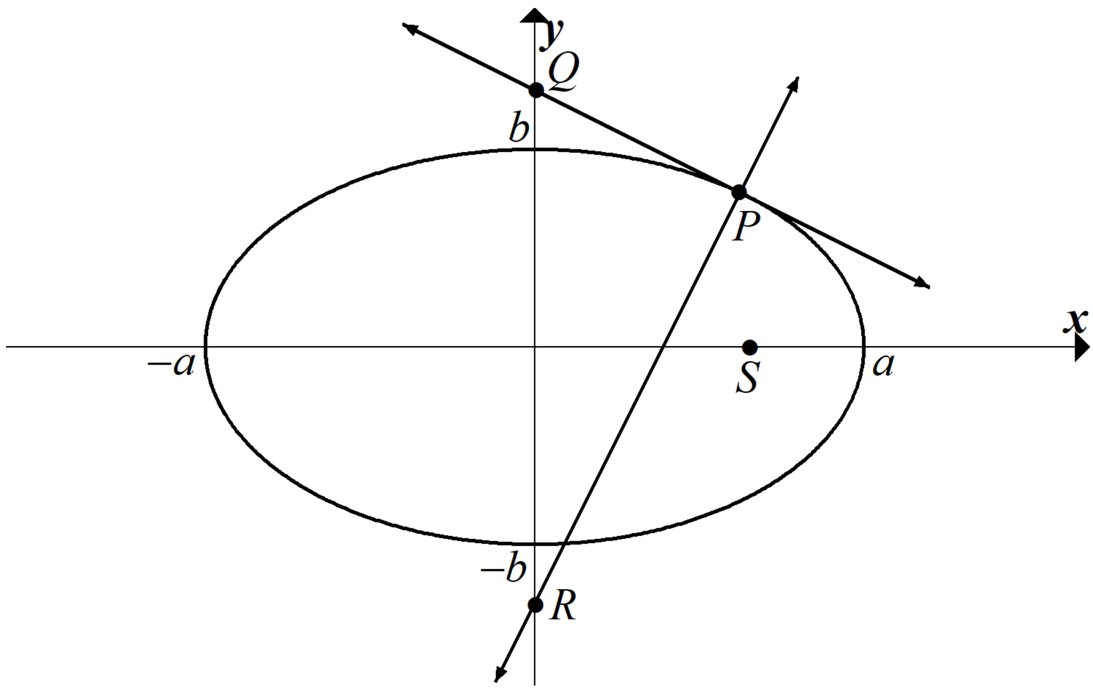
(i) $-\alpha, -\beta, -\gamma$. **2**

(ii) $\alpha + \beta, \beta + \gamma, \gamma + \alpha$. **2**

e) Solve $a^x = e^{2x-1}$ for x, expressing the answer in terms of a. $(a > 0)$. **2**

Question 14 (15 marks) Start a NEW booklet. **Marks**

a) The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse E: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ as shown below.



The tangent at P cuts the y-axis at Q. Also, the normal at P cuts the y-axis at R.

The normal at P is: $\boldsymbol{\dfrac{ax}{\cos\theta} - \dfrac{by}{\sin\theta} = a^2 - b^2}$. [Do not prove this]

(i) Show that the equation of the tangent at P is given by **2**

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1.$$

(ii) Show that the y-coordinate of Q is $\dfrac{b}{\sin\theta}$. **1**

(iii) Write down the coordinates of the point R. **1**

(iv) If S is the focus $(ae, 0)$, prove that P, Q, R and S are concyclic points. **3**

Question 14 continues on page 12

Question 14 (continued) **Marks**

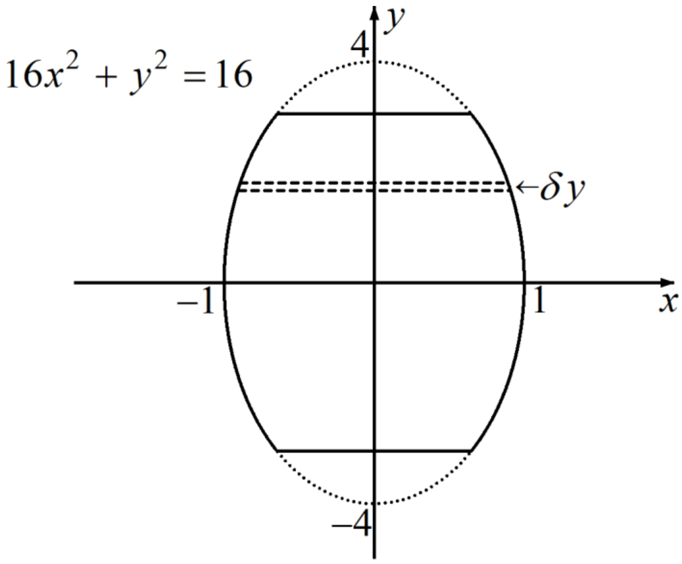
b) Consider the function $f(x) = e^{-x} - 1 + x$.

By finding expressions for $f'(x)$ and $f''(x)$ show that **3**

$e^{-x} \geq 1 - x$, for all x.

c) At a children's playground there is a narrow 6 metre long tunnel with vertical walls. The base (or floor) of the tunnel is in the shape of an ellipse with equal 1 metre amounts cut from each end.

Vertical cross-sections perpendicular to the major axis of the ellipse are in the shape of a rectangle topped by a semi-circle. The base of the rectangle is twice its height and the ellipse has equation $16x^2 + y^2 = 16$.



(i) With the aid of a diagram show that the area of a typical cross-section is given by $A_s = \left(\dfrac{4 + \pi}{2}\right)x^2$. **2**

(ii) Hence, find the volume of the tunnel in exact form. **3**

End of Question 14

Question 15 (15 marks) Start a NEW booklet. **Marks**

a) Find the general solution in radians to the equation $\dfrac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta} = \dfrac{3}{4}$. **3**

b) (i) Let x be a fixed, non-zero number satisfying $x > -1$. **3**

Use the method of mathematical induction to prove that for $n = 2, 3, 4, \ldots$.

$$(1+x)^n > 1 + nx$$

(ii) Deduce that $\left(1 - \dfrac{1}{2n}\right)^n > \dfrac{1}{2}$ for $n = 2, 3, 4, \ldots$. **1**

c) If $a > 0$ and $b > 0$ show that $a^3 + b^3 \geq a^2b + ab^2$. [Do not use Induction] **2**

d) $P\left(5p, \dfrac{5}{p}\right), p > 0$ and $Q\left(5q, \dfrac{5}{q}\right), q > 0$ are two points on the hyperbola $xy = 25$.

The equation of the chord PQ is given by $\boldsymbol{x + pqy = 5(p+q)}$. [Do not prove this]

(i) Hence or otherwise, state the equations of the tangents at P and Q. **1**

(ii) The tangents at P and Q intersect at R. **2**

Show that the coordinates of R are given by $R\left(\dfrac{10pq}{p+q}, \dfrac{10}{p+q}\right)$.

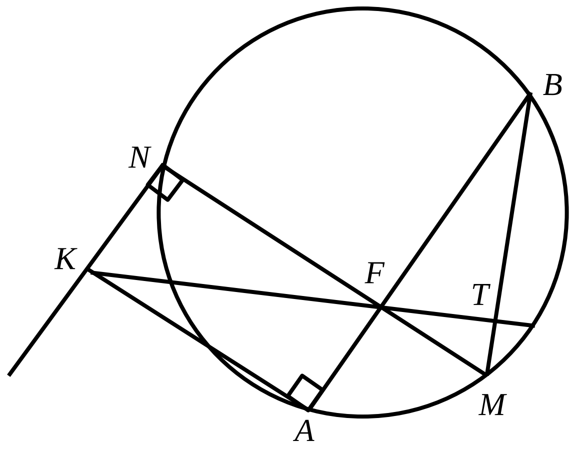
(iii) If the secant PQ passes through the point $S(15, 0)$, find the locus of R. **3**

Question 16 (15 marks) Start a NEW booklet. **Marks**

a) The area enclosed between the curves $y = \sqrt{x}$ and $y = x^2$ is rotated about the y – axis through one complete revolution. Use the method of cylindrical shells to find the volume of the solid that is generated. **3**

b) If $I_n = \int_0^1 x^n e^{-x}\, dx$, show that $I_n = -\frac{1}{e} + n I_{n-1}$. **3**

c) As shown below, a circle has two chords *AB* and *MN* intersecting at *F*. Perpendiculars are drawn to these chords at *A* and *N* intersecting at *K*. *KF* produced meets *MB* at *T*.



Answer this question on the page provided

(i) Explain why *AKNF* is a cyclic quadrilateral. **1**

(ii) By letting $\angle AKF = \theta$, prove that *KT* is perpendicular to *MB*. **3**

[Use and hand in the attached diagram - see page 18]

Question 16 continues on page 15

Question 16 (continued) **Marks**

d) Water flows into an aquarium at a rate proportional to the amount of water Q in the aquarium. At the same time, evaporation occurs at a rate proportional to the square of the quantity of water in the aquarium. Thus at any time $'t'$, it is known that

$$\frac{dQ}{dt} = aQ - bQ^2$$

where $'a'$ and $'b'$ are constants. Initially $Q = Q_0$.

i) Using partial fractions, show that an expression for Q in terms of t is given by $Q = \dfrac{aQ_0 e^{at}}{\left(a - bQ_0\right) + bQ_0 e^{at}}$. **4**

ii) Show that the quantity of water tends to a limit as time increases. **1**

END OF EXAM

Candidate Name/Number: ____________________________________

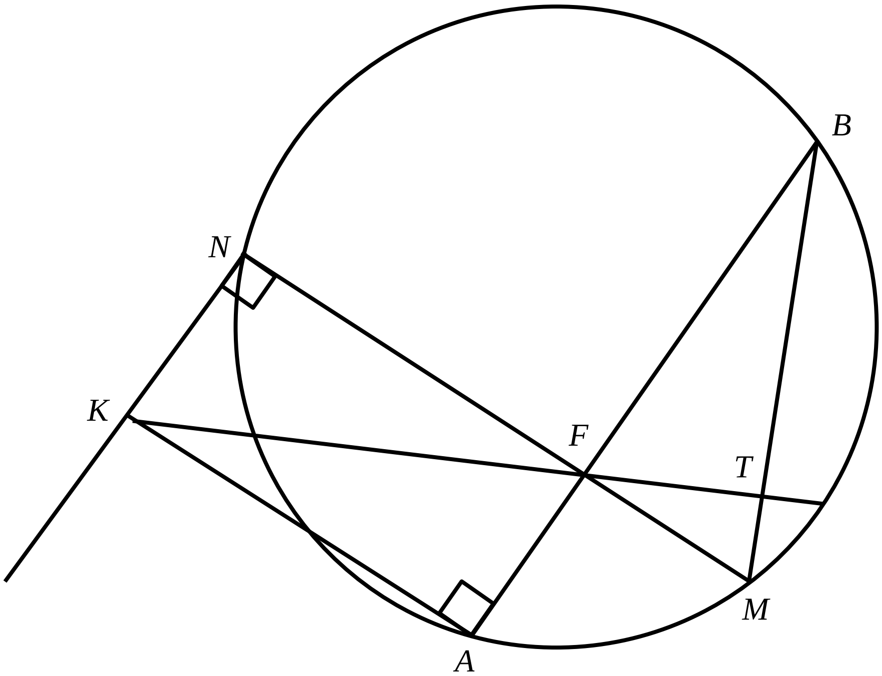
Multiple choice answer page. Fill in either A, B, C or D for questions 1-10.

This page must be handed in with your answer booklets

1.	
2.	
3.	
4.	
5.	

6.	
7.	
8.	
9.	
10.	

Question 16c (additional diagram)   Candidate Name/Number: ___________________________________



Hand in this page inside the Question 16 booklet

CHS YEAR 12 EXTENSION 2 2018	TRIAL HSC SOLUTIONS

Multiple Choice Section:

1.B 2.C 3.D 4.D 5.B

6.A 7.A 8.C 9.C 10.B

Question 1.

$iz = i(4-3i)$
$iz = 3 + 4i$
$\therefore\ \overline{iz} = 3 - 4i$ ---------- $\boxed{B}$

Question 2.

The remainder is given by
$P(i) = (-i)^3 + (-i)^2 + 5(-i) + 6$
$= i - 1 - 5i + 6$
$= 5 - 4i$ ---------- $\boxed{C}$

Question 3.

$\alpha + \beta + \gamma = \frac{3}{2}$

$\alpha\beta\gamma = -1 \rightarrow \gamma = -\frac{1}{\alpha\beta}$

$\therefore\ \alpha + \beta - \frac{1}{\alpha\beta} = \frac{3}{2}$ ---------- $\boxed{D}$

Question 4.

$a = 2; b = 3;\ a^2 = b^2(1-e^2)$; Foci $(0, \pm be)$

$\therefore\ e^2 = \frac{5}{9} \rightarrow e = \frac{\sqrt{5}}{3}$

$\therefore$ Foci $(0, \pm\sqrt{5})$ ------------ $\boxed{D}$

Question 5.

------------ $\boxed{B}$

Question 6.

let $u = 1 + \tan x \rightarrow du = \sec^2 x\, dx$

when $x = \frac{\pi}{4}, u = 2;\ x = -\frac{\pi}{4}, u = 0.$

$\therefore\ I = \int_0^2 \frac{1}{u^2}\, du$ ------------ $\boxed{A}$

Question 7.

$e^x + e^y = 1$ and using implicit differentiation

$e^x + e^y \frac{dy}{dx} = 0$

$\therefore\ \frac{dy}{dx} = -\frac{e^x}{e^y} = -e^{x-y}$ ------------ $\boxed{A}$

Question 8.

$\angle XYB = 90^0$ and $\angle XZXC = 90^0$.

$\therefore\ \frac{XZ}{XY} = \frac{XC}{XB} = \frac{3ZC}{2ZC} = \frac{3}{2}$

$\therefore\ 2XZ = 3XY$ ------------ $\boxed{C}$

Question 9.

The radius of a typical slice is $r = a - x$

$\therefore\ V_s = \pi r^2 \delta y$

Volume $= \pi\int_0^{f(a)} (a-x)^2\, dy$

$= \pi\int_0^{f(a)} \left[a - f^{-1}(y)\right]^2 dy$ ------ $\boxed{C}$

Question 10.

$$|z| = |a + b\omega|$$

$$= \left| a + b\left(\frac{1}{2}\left(-1 + i\sqrt{3}\right)\right)\right|$$

$$= \left|\left(a - \frac{b}{2}\right) + \frac{b\sqrt{3}}{2}i\right|$$

$$= \sqrt{\left(a - \frac{b}{2}\right)^2 + \left(\frac{b\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{a^2 - ab + \frac{b^2}{4} + \frac{3b^2}{4}}$$

$$= \sqrt{a^2 - 2ab + b^2 + ab}$$

$$= \sqrt{(a-b)^2 + ab}$$ ———————————— $\boxed{B}$

Question 11

a) i) $\text{Im}(u - z) = \text{Im}(2 + 3i) = 3$

ii) $z\bar{u} = (1 - i)(3 - 2i) = 1 - 5i$

b) It is rotated 90° in an anti-clockwise direction and $|Q| = 2|P|$.

c) $$\int \frac{dx}{x^2 + 6x + 13} = \int \frac{dx}{\left(x^2 + 6x + 9\right) + 4}$$

$$= \int \frac{dx}{(x+3)^2 + 2^2}$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right) + c$$

d) i) $Z = 1 - \sqrt{3}\,i$

$\therefore \arg(Z) = -\dfrac{\pi}{3}$ and $|Z| = 2$

$\therefore\ Z = 2cis\left(-\dfrac{\pi}{3}\right)$

d) ii) $$Z^{10} = \left[2cis\left(-\frac{\pi}{3}\right)\right]^{10}$$

$$= 2^{10} cis\left(-\frac{10\pi}{3}\right)$$

$$= 1024\left[\cos\left(\frac{10\pi}{3}\right) - i\sin\left(\frac{10\pi}{3}\right)\right]$$

$$= 1024\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$$= 512\left(-1 + \sqrt{3}i\right)$$

e) $2x^2 + 2xy + 3y^2 = 15$

Using implicit differentiation

$4x + 2x \cdot y' + 2y + 6y \cdot y' = 0$

$2x + x \cdot y' + y + 3y \cdot y' = 0$

$y'(x + 3y) = -2x - y$

$\therefore\ \dfrac{dy}{dx} = \dfrac{-2x - y}{x + 3y}$

vertical tangents will occur when $x + 3y = 0$

i.e. when $y = -\dfrac{x}{3}$

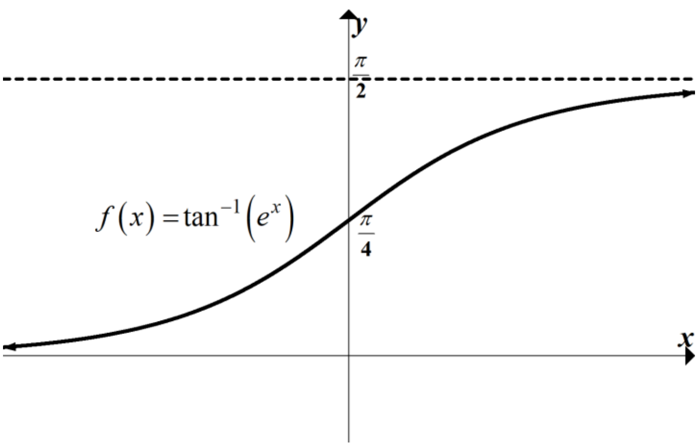
$\therefore\ 2x^2 + 2x\left(-\dfrac{x}{3}\right) + 3\left(-\dfrac{x}{3}\right)^2 = 15$

$\therefore\ x^2 = 9 \quad \rightarrow \quad x = \pm 3$

f) i) D: all real x.

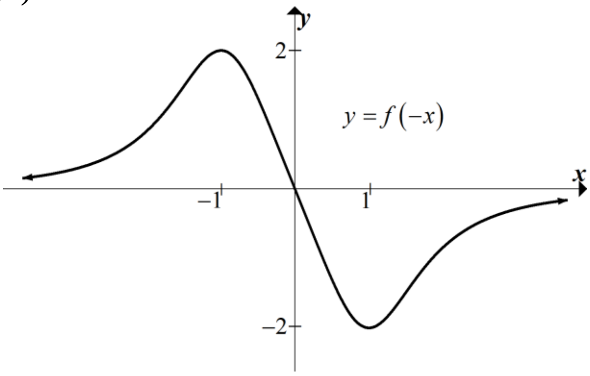
R: $0 < y < \dfrac{\pi}{2}$.

ii)

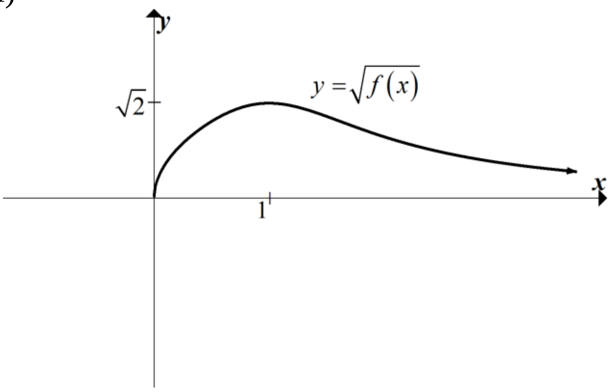


Question 12.

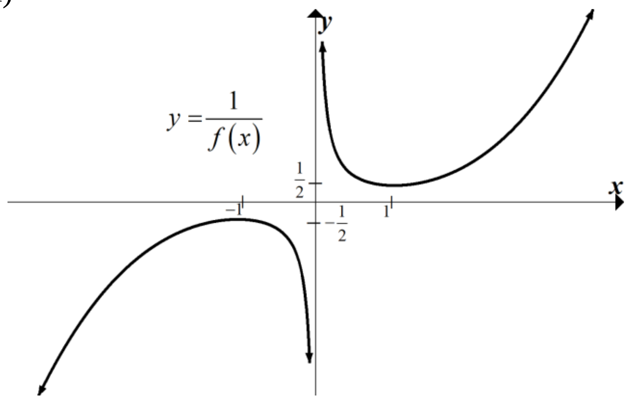
a) i)



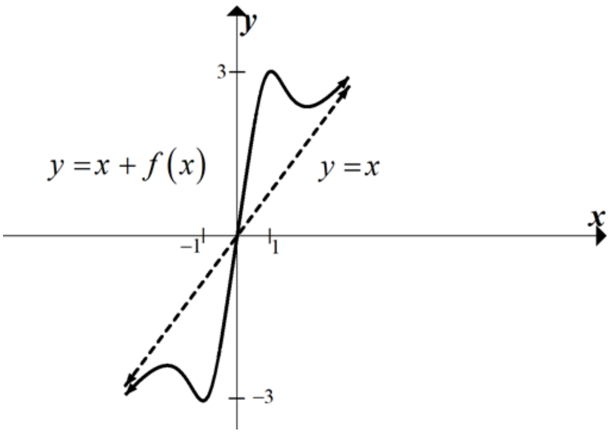
ii)



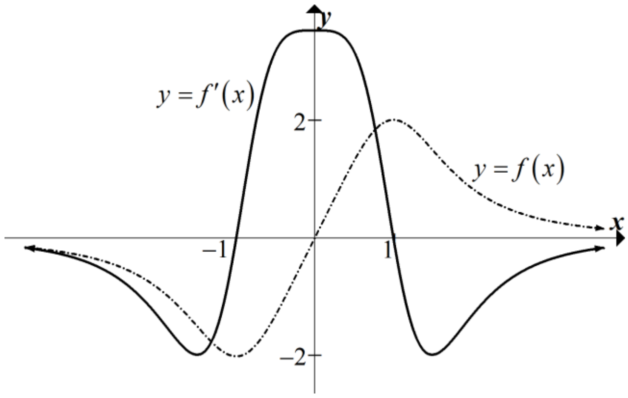
iii)



iv)



v)



b) $t=\tan\frac{\theta}{2} \;\rightarrow\; d\theta=\frac{2}{1+t^2}\,dt$

When $\theta=0,\ t=0;\ \theta=\frac{\pi}{3},\ t=\frac{1}{\sqrt{3}}$. Also $\sin\theta=\frac{2t}{1+t^2}$

$$I=\int_0^{\frac{1}{\sqrt{3}}}\frac{1}{1+\frac{2t}{1+t^2}}\bullet\frac{2t}{1+t^2}\;dt$$

$$=2\int_0^{\frac{1}{\sqrt{3}}}\frac{1}{1+2t+t^2}\;dt\;=\;2\int_0^{\frac{1}{\sqrt{3}}}\frac{1}{(1+t)^2}\;dt$$

$$=-2\left[\frac{1}{1+t}\right]_0^{\frac{1}{\sqrt{3}}}\;=\;\frac{2}{1+\sqrt{3}}$$

c) $z_2=\frac{1-i}{z_1} \quad\rightarrow\quad |z_2|=\left|\frac{1-i}{z_1}\right|$

$\therefore\ |z_2|=\frac{|1-i|}{|z_1|}=\frac{\sqrt{2}}{1}$ as z_1 is on the unit circle.

$\therefore\ |z_2|=\sqrt{2}$

and hence the locus of z_2 is a circle centre (0,0) and radius $\sqrt{2}$.

Question 13.

a) $P(x) = x^4 - 8x^3 + 18x^2 - 27.$

$P'(x) = 4x^3 - 24x^2 + 36x$

$P''(x) = 12x^2 - 48x + 36$

For a triple root $P''(x) = 0$

$\therefore\ 12x^2 - 48x + 36 = 0$

$\therefore\ 12(x-3)(x-1) = 0$

$\therefore\ x = 3$ or $x = 1$

also since $P(3) = 3^4 - 8\times 3^3 + 18\times 3^2 - 27 = 0$

then $x = 3$ is the root of multiplicity three.

[Note $P(1) = 1 - 8 + 18 - 27 \neq 0$]

$\therefore\ P(x) = (x-3)^3(x+1)$

b) Let $I = \int \frac{1}{x^2}\sqrt{1+\frac{4}{x}}\,dx$

Let $u = \frac{1}{x} \rightarrow du = -\frac{1}{x^2}dx$

$\therefore\ I = -\int \sqrt{1+4u}\ du$

$= -\int (1+4u)^{1/2}\ du$

$= -\frac{(1+4u)^{3/2}}{4}\times\frac{2}{3}$

$= -\frac{(1+4u)^{3/2}}{6}$

$= -\frac{1}{6}\left(1+\frac{4}{x}\right)^{3/2} + c$

c) $= \frac{(1+\cos\theta - i\sin\theta)}{(1+\cos\theta + i\sin\theta)}\times\frac{(1+\cos\theta - i\sin\theta)}{(1+\cos\theta - i\sin\theta)}$

$= \frac{(1+\cos\theta - i\sin\theta)^2}{(1+\cos\theta)^2 - (i\sin\theta)^2}$

$= \frac{(1+\cos\theta)^2 - 2(1+\cos\theta)i\sin\theta + (i\sin\theta)^2}{(1+\cos\theta)^2 - (i\sin\theta)^2}$

$= \frac{1+2\cos\theta+\cos^2\theta - 2(1+\cos\theta)i\sin\theta - \sin^2\theta}{1+2\cos\theta+\cos^2\theta+\sin^2\theta}$

$= \frac{(\cos^2\theta+\sin^2\theta)+2\cos\theta+\cos^2\theta - 2(1+\cos\theta)i\sin\theta - \sin^2\theta}{2+2\cos\theta}$

$= \frac{2\cos^2\theta + 2\cos\theta - 2(1+\cos\theta)i\sin\theta}{2(1+\cos\theta)}$

$= \frac{\cos\theta(1+\cos\theta) - (1+\cos\theta)i\sin\theta}{(1+\cos\theta)}$

$= \cos\theta - i\sin\theta$ (see end for alternate solution)

d) i) Let $x = -\alpha \rightarrow \alpha = -x$

$\therefore$ the required equation is given by

$\therefore\ (-x)^3 - 4(-x)^2 - 12 = 0$

$\therefore\ -x^3 - 4x^2 - 12 = 0$

hence $x^3 + 4x^2 + 12 = 0$

ii) The roots are of the form $\alpha+\beta+\gamma-\gamma$

$= (\alpha+\beta+\gamma) - \gamma$

$= 4 - \gamma$ using sum of the roots.

Hence let $x = 4-\gamma \rightarrow \gamma = 4 - x$

$\therefore$ the required equation is given by

$\therefore\ (4-x)^3 - 4(4-x)^2 - 12 = 0$

$\therefore\ 64 - 48x + 12x^2 - x^3 - 64 + 32x - 4x^2 - 12 = 0$

$\therefore\ x^3 - 8x^2 + 16x + 12 = 0$

e) $a^x = e^{2x-1}$

$\therefore \ \ln\left(a^x\right) = \ln\left(e^{2x-1}\right)$

$\therefore \ x\ln a = 2x - 1$

$\therefore \ 1 = x(2 - \ln a)$

$\therefore \ x = \dfrac{1}{2 - \ln a}$

Question 14.

a) i) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ by implicit differentiation

$\therefore \ \dfrac{2x}{a^2} + \dfrac{2y}{b^2}\cdot\dfrac{dy}{dx} = 0 \quad \rightarrow \quad \dfrac{dy}{dx} = -\dfrac{b^2x}{a^2y}$

$\therefore$ at $P(a\cos\theta, b\sin\theta)$, $\dfrac{dy}{dx} = -\dfrac{b\cos\theta}{a\sin\theta}$

$\therefore$ eq^n of T at P: $y - b\sin\theta = -\dfrac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

$ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$

$bx\cos\theta + ay\sin\theta = ab\left(\sin^2\theta + \cos^2\theta\right)$

Hence $\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1 - - - - - \boxed{1}$

ii) Using $\boxed{1}$ and putting $x = 0$

$\dfrac{y\sin\theta}{b} = 1 \quad \rightarrow \quad y = \dfrac{b}{\sin\theta}$

iii) Using the equation of the normal [given]
and putting $x = 0$

$-\dfrac{by}{\sin\theta} = a^2 - b^2 \quad \rightarrow \quad y = \dfrac{b^2 - a^2}{b}\sin\theta$

$\therefore \ R\left(0, \dfrac{b^2 - a^2}{b}\sin\theta\right)$

iii) $\angle QPR = 90^\circ$ ($\angle$ btwn tangent and normal)

$m_{QS} = \dfrac{0 - \dfrac{b}{\sin\theta}}{ae - 0} = \dfrac{-b}{ae\sin\theta}$

$m_{RS} = \dfrac{\dfrac{b^2 - a^2}{b}\sin\theta - 0}{0 - ae} = \dfrac{\dfrac{b^2 - a^2}{b}\sin\theta}{-ae}$

Now $b^2 = a^2\left(1 - e^2\right) \quad \rightarrow \quad b^2 - a^2 = -a^2e^2$

$\therefore \ m_{RS} = \dfrac{-a^2e^2\sin\theta}{-abe} = \dfrac{ae\sin\theta}{b}$

$\therefore \ m_{QS} \times m_{RS} = \dfrac{-b}{ae\sin\theta} \times \dfrac{ae\sin\theta}{b}$

$= -1$

$\therefore \ \angle QSR = 90^\circ = \angle QPR$

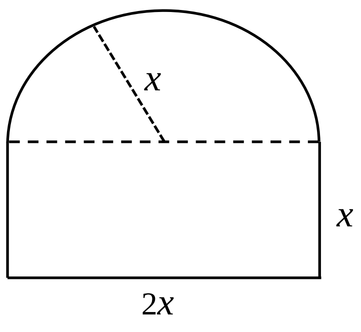
Hence $QPSR$ are concyclic points
[$\angle$'s in the same segment]

b) $f(x) = e^{-x} - 1 + x$

$\therefore \ f'(x) = -e^{-x} + 1$ and $f''(x) = e^{-x}$

As $f''(x) = e^{-x} > 0$ for all x
then $f(x)$ is concave up
Also $f'(x) = 0 \quad \rightarrow e^{-x} = 1$
$\therefore x = 0$ and hence there is a mininmum turning pt at $(0,0)$
$\therefore \ e^{-x} - 1 + x \geq 0 \quad \rightarrow \quad e^{-x} \geq 1 - x.$

c) i)



$A = 2x \times x + \dfrac{\pi x^2}{2}$

$= 2x^2 + \dfrac{\pi x^2}{2}$

$= \left(2 + \dfrac{\pi}{2}\right)x^2$

$= \left(\dfrac{4 + \pi}{2}\right)x^2$

cii) $V = \left(\frac{4+\pi}{2}\right)\int_0^3 2x^2\,dy$

$= (4+\pi)\int_0^3 \frac{16-y^2}{16}\,dy$

$= \left(\frac{4+\pi}{16}\right)\int_0^3 16-y^2\,dy$

$= \left(\frac{4+\pi}{16}\right)\left[16y - \frac{y^3}{3}\right]_0^3$

$= \left(\frac{4+\pi}{16}\right)(48-9)$

$= \frac{39}{16}(4+\pi)\ \text{m}^3$

Question 15.

a) $\frac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta} = \frac{3}{4}.$

$\therefore\ \frac{(s+c)\left(s^2 - sc + c^2\right)}{s+c} = \frac{3}{4}$

$\therefore\ 1 - \sin\theta\cos\theta = \frac{3}{4}$

$\therefore\ \sin\theta\cos\theta = \frac{1}{4}$

$\therefore\ \sin 2\theta = \frac{1}{2}$

$\therefore\ 2\theta = n\pi + (-1)^n\left(\frac{\pi}{6}\right)$

$\therefore\ \theta = \frac{n\pi}{2} + (-1)^n\left(\frac{\pi}{12}\right)$

b) i) When $n=2$ $LHS = (1+x)^2 = 1 + 2x + x^2$

$RHS = 1 + 2x$

Since $x^2 > 0,\ 1 + 2x + x^2 > 1 + 2x$

Hence true for $n=2$.

Assume true for $n=k,\ k \ge 2$

i.e. $(1+x)^k > 1 + kx$ ------ $\boxed{1}$

Prove true for $n = k+1$

i.e. $(1+x)^{k+1} > 1 + (k+1)x$

Now $(1+x)^{k+1} = (1+x)(1+x)^k$

$> (1+x)(1+kx)$ using $\boxed{1}$

and since $x > -1 \rightarrow 1 + x > 0$

$= 1 + kx + x + kx^2$

$> 1 + (k+1)x$, since $kx^2 > 0$.

Hence by induction is true for all $n \ge 2$.

ii) Let $x = -\frac{1}{2n}$. If $n \ge 2$, then this satisfies the condition of part (i), including $x > -1$.

$\therefore\ (1+x)^n > 1 + nx$

$\therefore\ \left(1 - \frac{1}{2n}\right)^n > 1 - \frac{n}{2n}$

$= \frac{1}{2}$

$\therefore\ \left(1 - \frac{1}{2n}\right)^n > \frac{1}{2}$ for $n = 2, 3, 4, \ldots$

c) i.e. prove that $a^3 + b^3 - a^2b - ab^2 \geq 0$

$$\text{LHS} = (a+b)(a^2 - ab + b^2) - ab(a+b)$$

$$= (a+b)(a^2 - 2ab + b^2)$$

$$= (a+b)(a-b)^2 \geq 0, \text{ since } a > 0 \text{ and } b > 0.$$

$$\therefore \quad a^3 + b^3 \geq a^2b + ab^2$$

d) i) By letting $p = q$ in $x + pqy = 5(p+q)$ the equations of the tangents at P and Q are

$x + p^2y = 10p$ and $x + q^2y = 10q$ respectively.

ii) By solving simultaneously

$$x + p^2y = 10p \text{ -----} \boxed{1}$$
$$x + q^2y = 10q \text{ -----} \boxed{2}$$

$$\boxed{1} - \boxed{2}: \ (p^2 - q^2)y = 10(p-q)$$
$$(p-q)(p+q)y = 10(p-q)$$

$$\therefore \quad y = \frac{10}{p+q} \qquad (p \neq q)$$

Substituting $y = \dfrac{10}{p+q}$ in $\boxed{1}$ gives

$$x + p^2\left(\frac{10}{p+q}\right) = 10p$$

$$\therefore \quad x = 10p - \frac{10p^2}{p+q}$$

$$= \frac{10p^2 + 10pq - 10p^2}{p+q}$$

$$= \frac{10pq}{p+q}$$

$\therefore$ R has coordinates $\left(\dfrac{10pq}{p+q}, \dfrac{10}{p+q}\right)$.

iii) Since P passes through $(15, 0)$

$$15 + 0 = 5(p+q) \quad \to \quad p + q = 3$$

$\therefore$ R has coordinates $\left(\dfrac{10pq}{3}, \dfrac{10}{3}\right)$.

Hence the locus of R is $y = \dfrac{10}{3}$. [indep of p and q]

Since $p > 0$ and $q > 0$ then $pq > 0$ and so $x > 0$.

Also since the tangents intersect below the curve:

When $y = \dfrac{10}{3}$, $x = \dfrac{15}{2}$, then the locus of R is

$y = \dfrac{10}{3}$, for $0 < x < \dfrac{15}{2}$.

Question 16.

a) $V_S = 2\pi xy\delta x$

$$\therefore \quad V = \lim_{\delta x \to 0} \sum_{x=0}^{1} 2\pi xy\delta x$$

$$= 2\pi \int_0^1 x\left(\sqrt{x} - x^2\right) dx$$

$$= 2\pi \int_0^1 x^{3/2} - x^3 \ dx$$

$$= 2\pi \left[\frac{2}{5}x^{5/2} - \frac{1}{4}x^4\right]_0^1$$

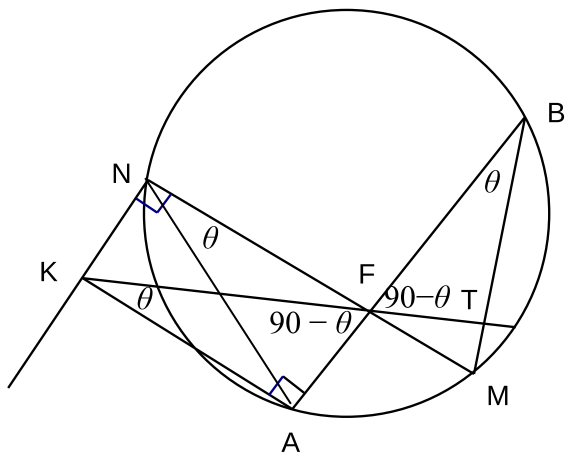
$$= 2\pi\left(\left(\frac{2}{5} - \frac{1}{4}\right) - 0\right) = \frac{3\pi}{10} \ u^3$$

b) Let $u = x^n$, $\quad v' = e^{-x}$
$\quad u' = nx^{n-1}$, $v = -e^{-x}$

$$\therefore \quad I_n = -e^{-x}x^n]_0^1 + \int_0^1 nx^{n-1}e^{-x} \ dx$$

$$= -e^{-1} - 0 + nI_{n-1}$$
$$= -\frac{1}{e} + nI_{n-1}$$

c) i)



Since $\angle KNF + \angle KAF = 180°$
Opposite angles in cyclic quad supplementary
hence *AKNF* is a cyclic quadrilateral.

ii) Join AN
Let $\angle AKF = \theta$
$\therefore \angle AFK = 90 - \theta$ ($\angle$ sum of ΔAKF)

Also

$\angle ANF = \theta$ ($\angle$ in same segment in cyclic quad $AKNF$)
$\therefore \angle BFT = 90 - \theta$ (vertically opp $\angle AFK$)
and $\angle FBM = \theta$ ($\angle$ in same segment)
$\therefore \angle FTB = 90°$ ($\angle$ sum of ΔFTB)
$\therefore KT$ is perpendicular to MB.

d) i) $\dfrac{dQ}{dt} = aQ - bQ^2$

$$\therefore \frac{dt}{dQ} = \frac{1}{Q(a - bQ)}$$

Using partial fractions let

$$\frac{1}{Q(a - bQ)} = \frac{r}{Q} + \frac{s}{a - bQ}$$

$$\therefore 1 = r(a - bQ) + sQ$$

$$= (s - br)Q + ar$$

$s - br = 0$ and $ar = 1 \rightarrow r = \dfrac{1}{a}$ and $s = \dfrac{b}{a}$

$$\therefore t = \int \frac{1}{a}\cdot\frac{1}{Q} + \frac{b}{a}\cdot\frac{1}{a - bQ}\, dQ$$

$$t = \frac{1}{a}\ln Q - \frac{1}{a}\ln(a - bQ) + c$$

When $t = 0,\ Q = Q_0 \Rightarrow c = -\dfrac{1}{a}\ln\left(\dfrac{Q_0}{a - bQ_0}\right)$

$$\therefore t = \frac{1}{a}\ln\left(\frac{Q}{a - bQ}\right) - \frac{1}{a}\ln\left(\frac{Q_0}{a - bQ_0}\right)$$

$$at = \ln\left(\frac{Q}{a - bQ} \times \frac{a - bQ_0}{Q_0}\right)$$

$$\therefore e^{at} = \frac{Q}{a - bQ} \times \frac{a - bQ_0}{Q_0}$$

$$\therefore \frac{Q_0 e^{at}}{a - bQ_0} = \frac{Q}{a - bQ}$$

$Q = \dfrac{aQ_0 e^{at}}{(a - bQ_0) + bQ_0 e^{at}}$ on rearranging

ii) Hence $Q = \dfrac{aQ_0}{bQ_0 + (a - bQ_0)e^{-at}}$

As $t \rightarrow \infty,\ e^{-at} \rightarrow 0$

Hence $Q \rightarrow \dfrac{a}{b}$ which is constant.

Q13c alternate solution:

Let $z = \cos\theta + i\sin\theta$ and hence $\dfrac{1}{z} = \cos\theta - i\sin\theta$

$$\therefore \frac{1 + \frac{1}{z}}{1 + z} = \frac{\frac{z + 1}{z}}{1 + z}$$

$$= \frac{1}{z} = \cos\theta - i\sin\theta$$