Caringbah High School

2019

Trial HSC Examination

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen only
(Black pen is preferred)

- NESA approved calculators may be used

- A reference sheet is provided

- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I **10 marks**
Pages 2 – 5
- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II **90 marks**
Pages 6 – 14
- Attempt Questions 11–16
- Allow about 2 hour and 45 minutes for this section

Question 1 - 10 (1 mark each) Answer on page provided.

1 Let $z = 1 + 2i$ and $\omega = 3i - 4$. What is the value of $z\overline{\omega}$?

A) $4 + i$

B) $2 - 3i$

C) $2 - 11i$

D) $-2 + 11i$

2 Let A, B and C be three consecutive terms of an arithmetic sequence.

Which of the following is a simplification of $\dfrac{\sin(A+C)}{\sin B}$?

A) $2\cos B$

B) $\sin 2B$

C) $\cot B$

D) 2

3 The eccentricity of the ellipse $\dfrac{x^2}{100} + \dfrac{y^2}{36} = 1$ is $e = \dfrac{4}{5}$.

Which of the following is the distance between the two foci?

A) 8

B) 16

C) 20

D) 25

4 The polynomial equation $x^3 - 3x^2 + 2 = 0$ has roots α, β and γ.

What is the value of $\alpha^3 + \beta^3 + \gamma^3$?

A) 9

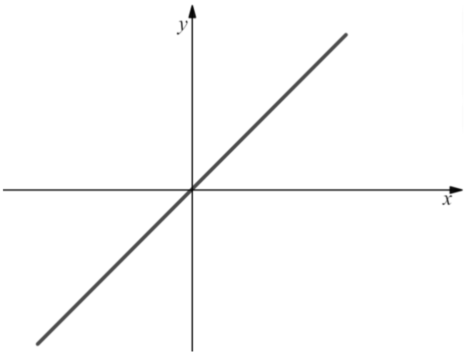
B) 13

C) 21

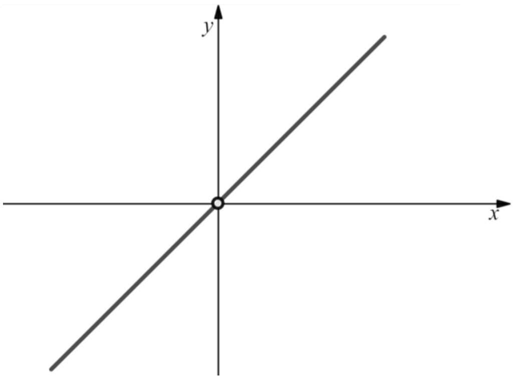
D) 25

5 The equation $\dfrac{x}{y}+\dfrac{y}{x}=2$ defines y implicitly as a function of x.
Which of the following graphs best represents this implicit function?

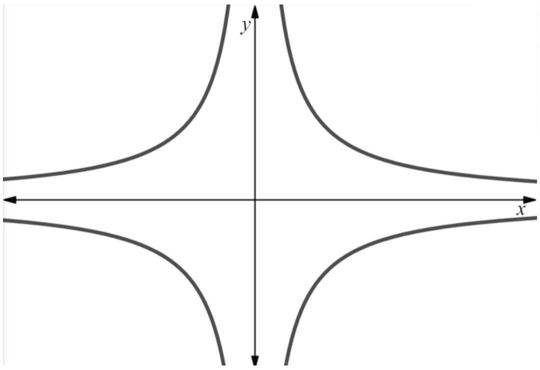
A)



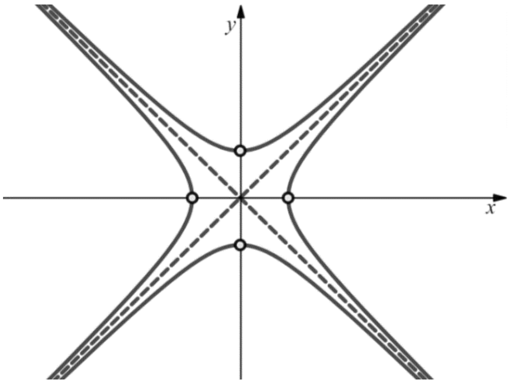
B)



C)



D)



6 If $x=1-\cos\theta$ and $y=\theta-\sin\theta,$ which is the result for $\dfrac{dy}{dx}$?

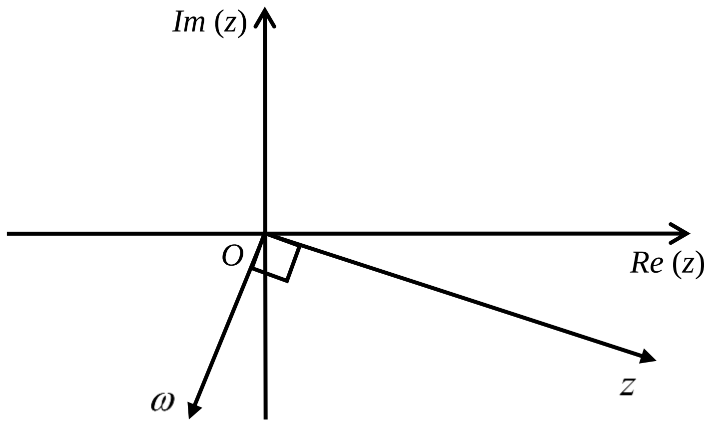
A) $\dfrac{dy}{dx}=-\cot\dfrac{\theta}{2}$

B) $\dfrac{dy}{dx}=-\tan\dfrac{\theta}{2}$

C) $\dfrac{dy}{dx}=\cot\dfrac{\theta}{2}$

D) $\dfrac{dy}{dx}=\tan\dfrac{\theta}{2}$

7 Consider the following diagram, drawn to scale, showing the complex numbers z and ω.



Which of statements below is false?

A) $|z+\omega| = |z-\omega|$

B) $\text{Re}\left(\dfrac{\omega}{z}\right) = 0$

C) $z^2 = k\omega^2$, where k is real

D) $-\pi < \arg(z-\omega) < 0$

8 Which of the following is the domain of the function $y = \cos^{-1}\left(\dfrac{x-a}{b}\right)$, $b > 0$?

A) $a-b \le x \le a+b$

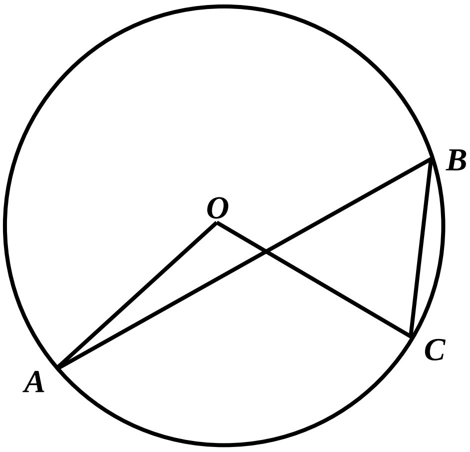
B) $a-1 \le x \le a+1$

C) $a \le x \le a+b\pi$

D) $-1 \le x \le 1$

9 In the diagram below, O is the centre of the circle.

It is given that $\angle OAB = 20^\circ$ and $\angle OCB = 52^\circ$.



What is the size of $\angle ABC$?

A) 49°

B) 32°

C) 56°

D) 64°

10 A particle is moving in a straight line with velocity at any particular time given by $v = \sin^{-1} x$.

Which of the following represents the acceleration of the particle?

A) $\dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$

B) $\dfrac{-\sin^{-1} x}{\sqrt{1-x^2}}$

C) $\dfrac{1}{\sqrt{1-x^2}}$

D) $\dfrac{-\cos x}{\sin^2 x}$

END OF MULTIPLE CHOICE QUESTIONS

Section II (90 marks)

Attempt all questions 11−16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11−16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a NEW booklet. **Marks**

(a) Let $z=\sqrt{3}+i$.

(i) Express z in modulus – argument form. **2**

(ii) Hence find z^4 in $x+iy$ form. **2**

(b) A point $P(z)$ moving in the complex plane has its locus in terms of z

defined by $|z-1|=|z+2-3i|$.

Find the cartesian equation of the locus. **2**

(c) Find $\int \sin\theta\cos^5\theta\, d\theta$. **2**

(d) (i) Find the values of A, B and C so that **2**

$$\frac{5}{(x+1)(x^2+4)} \equiv \frac{A}{x+1}+\frac{Bx+C}{x^2+4}.$$

(ii) Hence find $\int \frac{5}{(x+1)(x^2+4)}\, dx$. **3**

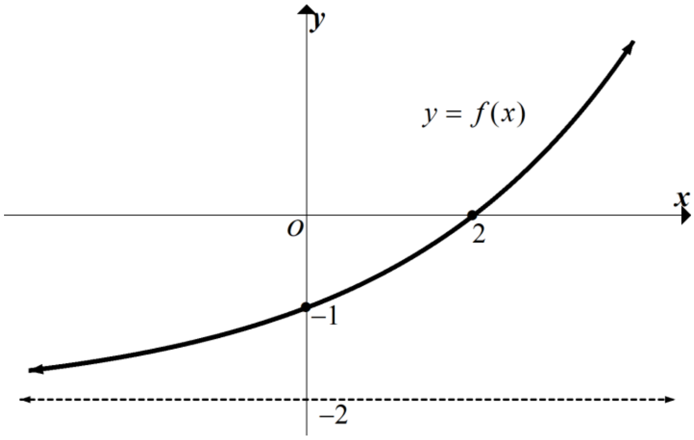
(e) Consider the polynomial $P(x)=x^3+3x^2-24x+k$.

If $P(x)=0$ has a zero of multiplicity 2, find all possible values of k. **2**

Question 12 (15 marks) Start a NEW booklet. **Marks**

(a) Find $\int \frac{x^2}{x+1}\,dx$. **2**

(b) The diagram below shows the graph of $y = f(x)$.



On separate sketches, showing any important features, neatly draw the graphs of

(i) $y = f(-x)$ **1**

(ii) $y = \sqrt{f(x)}$ **1**

(iii) $y = \left[f(x)\right]^2$ **2**

(iv) $y = \frac{1}{f(x)}$ **2**

(v) $y = \log_e\left[f(x)\right]$ **2**

(vi) $y = 1 + \frac{d}{dx}\left[f(x)\right]$ **2**

Question 12 continues on page 8

Question 12 (continued) **Marks**

(c) On the Argand diagram shade the region defined by **3**

$$|z - i| \leq 2 \text{ and } 0 \leq \arg(z - 1) \leq \frac{3\pi}{4}.$$

End of Question 12

Question 13 (15 marks) Start a NEW booklet. **Marks**

(a) Consider the hyperbola $x^2 - 9y^2 = 1$. Find:

(i) the eccentricity. **1**

(ii) the coordinates of the foci. **1**

(iii) the equation of the directrices. **1**

(b) Consider the ellipse $E: \dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$.

Normals to E at the points $P(3\cos\alpha, 2\sin\alpha)$ and $Q(3\cos\beta, 2\sin\beta)$ are at right angles to each other.

(i) Show that the gradient of the normal at P is $\dfrac{3\sin\alpha}{2\cos\alpha}$. **2**

(ii) Hence or otherwise, find the value of $\cot\alpha\cot\beta$. **2**

c) If α, β and γ are the roots of the equation $x^3 + 6x + 1 = 0$, find in simplest form the polynomial equation with roots $\alpha\beta, \beta\gamma$ and $\alpha\gamma$. **3**

d) Consider the function $f(x) = x - \ln(1 + x^2)$.

(i) Explain why $f(x)$ is an increasing function for all x except one value. Find this value and state what happens on the curve at this value. **3**

(ii) Hence show that $e^x \geq 1 + x^2$ for $x \geq 0$. **2**

Question 14 (15 marks) Start a NEW booklet. **Marks**

(a) (i) If ω is a non-real cube root of unity show that $1 + \omega + \omega^2 = 0$. **1**

(ii) Hence evaluate $\left(1 - 3\omega + \omega^2\right)\left(1 + \omega - 8\omega^2\right)$. **2**

(b) It is given that *a*, *b* and *c* are any real numbers.

(i) Prove that $a^2 + b^2 + c^2 \geq ab + bc + ca$. **2**

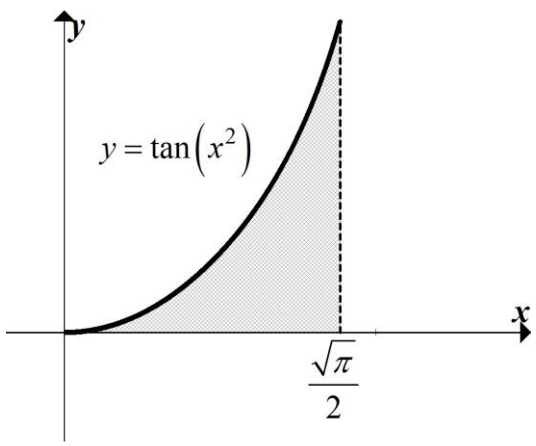
(ii) Hence or otherwise, prove that $3\left(a^2 + b^2 + c^2\right) \geq (a + b + c)^2$. **2**

(c) (i) Find $\frac{d}{dx}\left[\ln\left(\cos x^2\right)\right]$. **1**

(ii) The region in the plane which is bounded by the curve $y = \tan\left(x^2\right)$,

the x - axis and the line $x = \frac{\sqrt{\pi}}{2}$, shown in the diagram below,

is rotated about the y - axis to produce a solid S.



By using the method of cylindrical shells, find the volume of S in exact form. **3**

Question 14 continues on page 11

Question 14 (continued) **Marks**

(d) Let $I_n = \int_0^1 \left(1 - x^2\right)^n dx$ for $n \geq 1$.

(i) Show that $I_n = \dfrac{2n}{2n+1} I_{n-1}$ **3**

(ii) Hence evaluate I_2. **1**

End of Question 14

Question 15 (15 marks) Start a NEW booklet. **Marks**

(a) (i) Show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. **2**

(ii) Hence deduce that $\int_0^{\frac{\pi}{4}} \frac{1-\sin 2x}{1+\sin 2x}\,dx = \int_0^{\frac{\pi}{4}} \tan^2 x\,dx$ **2**

(iii) Hence evaluate $\int_0^{\frac{\pi}{4}} \frac{1-\sin 2x}{1+\sin 2x}\,dx$. **2**

(b) $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ where $p, q > 0$ are two distinct points on the Hyperbola, H, $xy = c^2$.

(i) Show that the equation of the tangent to H at P is given by **2**

$x + p^2 y = 2cp$.

(ii) The tangents to H at P and Q meet at T. **2**

Find the coordinates of T.

(iii) Show that the equation of the chord PQ is given by **2**

$x + pqy = c(p+q)$.

(iv) The chord PQ passes through the point $(2c, 0)$. **1**

Find the relationship between p and q.

(v) Hence, find the equation of the locus of T. **2**

For full marks a complete description must be given.

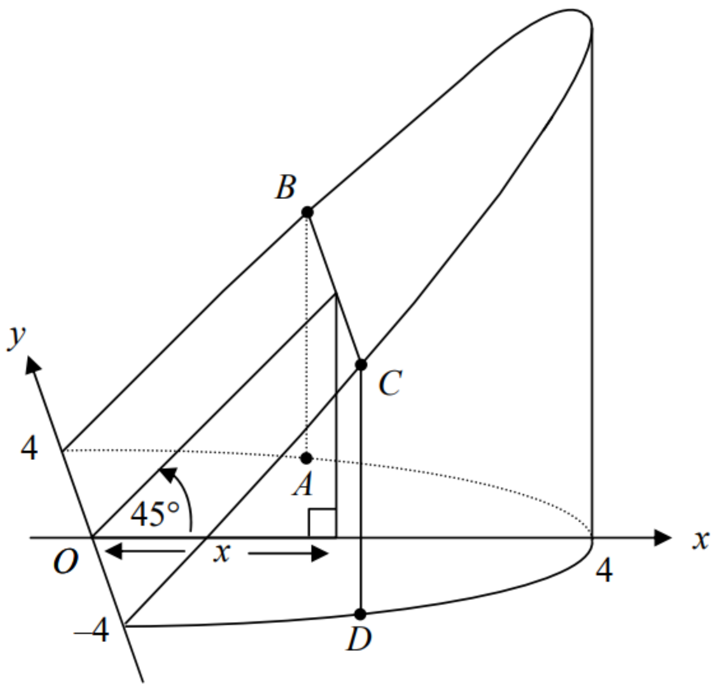
Question 16 (15 marks) Start a NEW booklet. **Marks**

a) Let $P(z) = 2z^3 - 5z^2 + qz - 5$, where q is a real number.

(i) If $P(1-2i) = 0$, solve $P(z) = 0$. **2**

(ii) Hence determine the value of q if $P(1-2i) = 0$. **1**

b) Lachlan's sculpture, "The Wedge", was obtained by cutting a right cylinder of radius 4 units at 45° through a diameter of its base as shown below.



A rectangular slice *ABCD*, of thickness δx, is taken perpendicular to the base of the wedge at a distance *x* from the *y* – axis.

(i) Show that the area of $ABCD$ is given by $2x\sqrt{16 - x^2}$. **2**

(ii) Find the exact volume of the wedge. **3**

Question 16 continues on page 14

Question 16 (continued) **Marks**

(c) The equation $x^2 - x + 1 = 0$ has roots α and β, and $A_n = \alpha^n + \beta^n$ for $n \geq 1$.

(i) Without solving the equation, show that $A_1 = 1$ and $A_2 = -1$. **2**

(ii) Show that $A_n = A_{n-1} - A_{n-2}$ for $n \geq 3$. **2**

(iii) Use mathematical induction to prove that $A_n = 2\cos\frac{n\pi}{3}$ for $n \geq 1$. **3**

END OF EXAM

Candidate Name/Number: ______________________________________

Multiple choice answer page. Fill in either A, B, C or D for questions 1-10.

This page must be handed in with your answer booklets

1.	
2.	
3.	
4.	
5.	

6.	
7.	
8.	
9.	
10.	

CHS YEAR 12 EXTENSION 2 2019	TRIAL HSC SUGGESTED SOLUTIONS

Multiple Choice Section:

1.C	2.A	3.B	4.C	5.B
6.D	7.D	8.A	9.B	10.A

Question 1.

$(1+2i)(-4-3i) = 2-11i$ ---------- $\boxed{C}$

Question 2.

$B - A = C - B \rightarrow 2B = A + C$

$$\frac{\sin(A+C)}{\sin B} = \frac{\sin 2B}{\sin B} = \frac{2\sin B\cos B}{\sin B} = 2\cos B$$ ------ $\boxed{A}$

Question 3.

$e = \frac{4}{5},\ a = 10 \rightarrow ae = 8$

$\therefore -ae$ to $ae = 16$ ---------- $\boxed{B}$

Question 4.

Since α, β, γ satisfy $x^3 - 3x^2 + 2 = 0$

Then $\alpha^3 = 3\alpha^2 - 2$

$\beta^3 = 3\beta^2 - 2$

$\gamma^3 = 3\gamma^2 - 2$

Hence $\alpha^3 + \beta^3 + \gamma^3 = 3(\alpha^2 + \beta^2 + \gamma^2) - 6$

$= 3(\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) - 6$

$= 3(3)^2 - 2(0) - 6$

$= 21$ ------------ $\boxed{C}$

Question 5.

$\frac{x}{y} + \frac{y}{x} = 2 \rightarrow (x-y)^2 = 0\ [x \neq 0, y \neq 0]$

$\therefore y = x$ ------------ $\boxed{B}$

Question 6.

$x = 1 - \cos\theta,\quad y = \theta - \sin\theta$

$\frac{dx}{d\theta} = \sin\theta,\quad \frac{dy}{d\theta} = 1 - \cos\theta$

$\therefore \frac{dy}{dx} = \frac{1-\cos\theta}{\sin\theta}$

$$= \frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$$ ------ $\boxed{D}$

Question 7.

$\arg(z - \omega)$ is positive (scale drawing) ------ $\boxed{D}$

Question 8.

$-1 \leq \frac{x-a}{b} \leq 1$

$\therefore -b \leq x - a \leq b$

$\therefore a - b \leq x \leq a + b$ -------- $\boxed{A}$

Question 9.

$\angle BAO + \angle AOC = \angle ABC + \angle OCB$ (exterior angle)

$\angle AOC = 2\angle ABC$ ($\angle$ at centre = twice $\angle$ at circum)

$\therefore 2x + 20 = x + 52 \rightarrow x = 32^0$ -------- $\boxed{B}$

Question 10.

$$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{d}{dx}\left(\frac{1}{2}(\sin^{-1}x)^2\right)$$

$$= \frac{1}{2} \times 2\sin^{-1}x \times \frac{1}{\sqrt{1-x^2}}$$ -------- $\boxed{A}$

Question 11

a) i) $2cis\left(\frac{\pi}{6}\right)$

ii) $\left[2cis\left(\frac{\pi}{6}\right)\right]^4 = 16cis\left(\frac{2\pi}{3}\right)$ $[DMT]$

$$= 16\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -8 + i8\sqrt{3}$$

b) Let $z = x + iy$.

$\therefore \quad |(x+2) + i(y-3)| = |(x-1) + iy|$

$\therefore \quad x^2 - 2x + 1 + y^2 = x^2 + 4x + 4 + y^2 - 6y + 9$

$\therefore \quad x - y + 2 = 0$

c) $\int \sin\theta \cos^5\theta \, d\theta$

Let $u = \cos\theta \rightarrow du = -\sin\theta d\theta$

$\therefore \quad I = -\int u^5 \, du = -\frac{u^6}{6}$

$\therefore \quad I = -\frac{\cos^6\theta}{6} + c$

d) i)

Let $5 = A(x^2+4) + (Bx+C)(x+1) ------ \boxed{*}$

Let $x = -1$ in $*: 5 = 5A \rightarrow A = 1$
Let $x = 0$ in $*: 5 = 4A + C \rightarrow C = 1$
Let $x = 1$ in $*: 5 = 5A + 2B + 2C \rightarrow B = -1$

ii) $\int \frac{5}{(x+1)(x^2+4)} \, dx = \int \frac{1}{x+1} + \frac{1-x}{x^2+4} \, dx$

$$= \ln(x+1) - \frac{1}{2}\ln(x^2+4) + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$$

$$= \ln\left(\frac{x+1}{\sqrt{x^2+4}}\right) + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$$

e) $P(x) = x^3 + 3x^2 - 24x + k$

$P'(x) = 3x^2 + 6x - 24$

$= 3(x^2 + 2x - 8)$

$= 3(x+4)(x-2)$

$\therefore \quad P'(x) = 0 \quad \rightarrow \quad x = -4$ or 2

$P(-4) = 80 + k = 0 \rightarrow k = -80$

$P(2) = k - 28 = 0 \rightarrow k = 28$

Question 12.

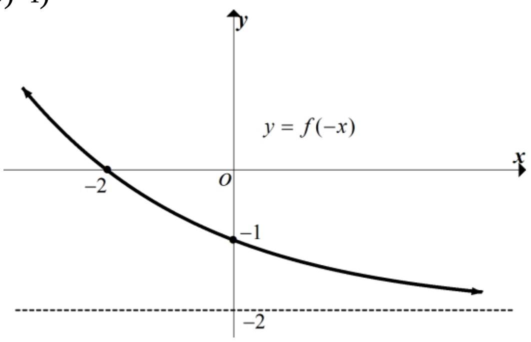
a) $\int \frac{x^2}{x+1} \, dx = \int \frac{x^2 - 1 + 1}{x+1} \, dx$

$$= \int \frac{(x-1)(x+1) + 1}{x+1} \, dx$$

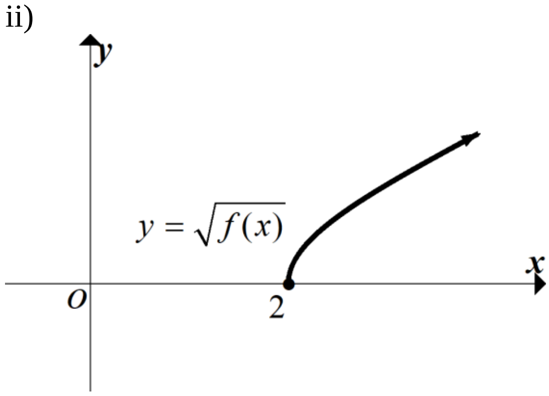
$$= \int x - 1 + \frac{1}{x+1} \, dx$$

$$= \frac{x^2}{2} - x + \ln(x+1) + c$$

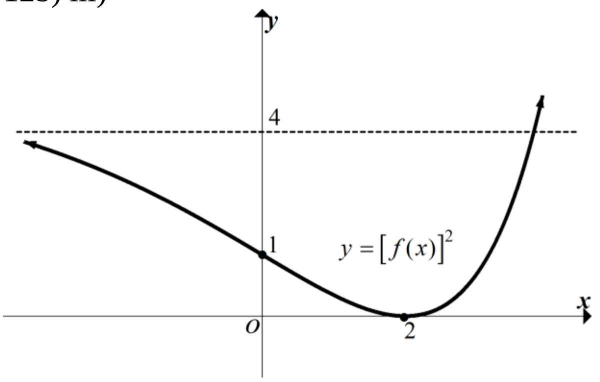
b) i)



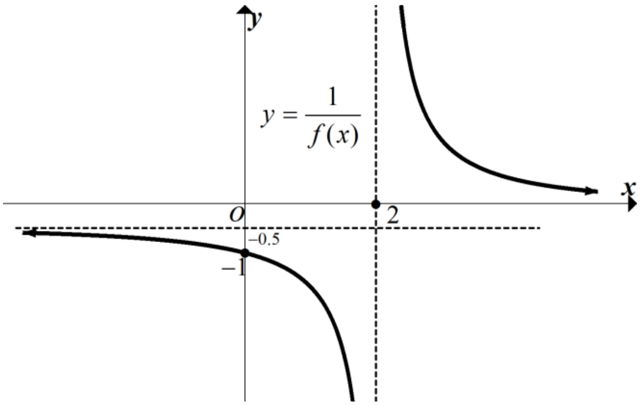
ii)



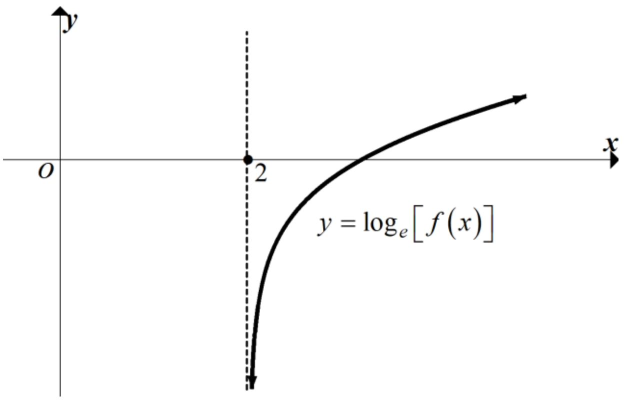
12b) iii)



iv)



v)



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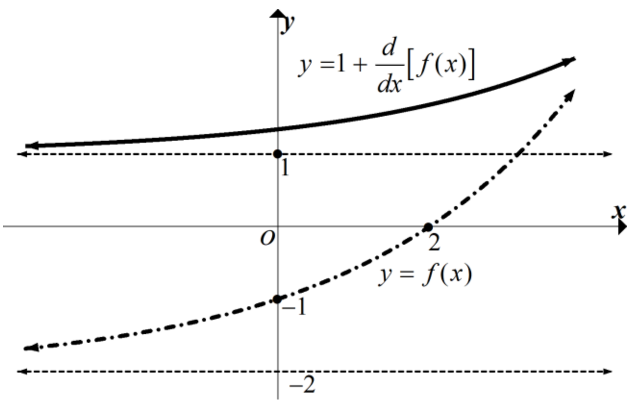

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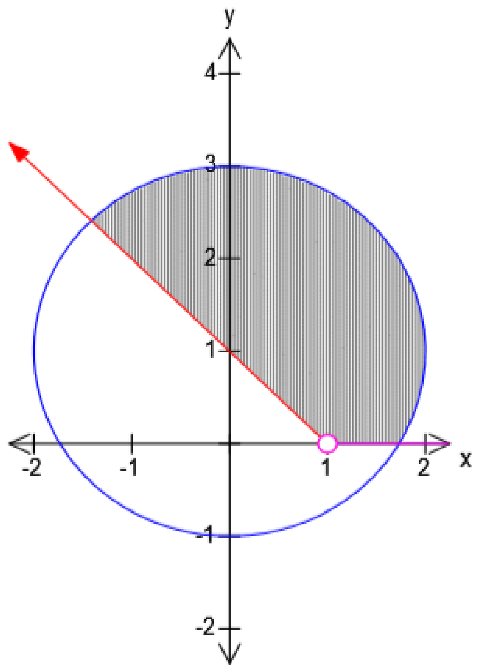


Question 13.

a) $x^2-9y^2=1 \quad \rightarrow \quad x^2-9y^2=1$

i) $b^2=a^2\left(e^2-1\right) \quad \rightarrow \quad \frac{1}{9}=e^2-1$

$\therefore \quad e=\frac{\sqrt{10}}{3}$

ii) Foci $\left(\pm\frac{\sqrt{10}}{3},0\right)$

iii) directrices $\therefore \quad x=\pm\frac{3}{\sqrt{10}}$

b i) $\frac{x^2}{9}+\frac{y^2}{4}=1 \quad \rightarrow \quad \frac{2x}{9}+\frac{2yy'}{4}=0$

$\therefore \quad y'=-\frac{4x}{9y}$ [gradient of tangent]

hence gradient of the normal is $\frac{9y}{4x}$

$\therefore$ at $P: m_N=\frac{9\times 2\sin\alpha}{4\times 3\cos\alpha}=\frac{3\sin\alpha}{2\cos\alpha}$

ii) Similarly the gradient of the normal at Q is

$$m_N=\frac{3\sin\beta}{2\cos\beta}$$

and as the normals are at right angles then:

$$\frac{3\sin\alpha}{2\cos\alpha}\times\frac{3\sin\beta}{2\cos\beta}=-1$$

$$\therefore\quad \frac{3}{2}\tan\alpha\times\frac{3}{2}\tan\beta=-1$$

$$\therefore\quad \cot\alpha\cot\beta=-\frac{9}{4}$$

c) The required roots are $\alpha\beta,\beta\gamma$ and $\alpha\gamma$

which is the same as $\frac{\alpha\beta\gamma}{\gamma},\frac{\alpha\beta\gamma}{\alpha}$ and $\frac{\alpha\beta\gamma}{\beta}$.

Using sum of the roots $\alpha\beta\gamma=-1$, hence the required

roots are $\frac{-1}{\alpha},\frac{-1}{\beta}$ and $\frac{-1}{\gamma}$.

Let $x=\frac{-1}{\alpha}\;\rightarrow\;\alpha=\frac{-1}{x}$

$\therefore$ the required equation is given by

$$\therefore\quad \left(\frac{-1}{x}\right)^3+6\left(\frac{-1}{x}\right)+1=0$$

$$\therefore\quad \frac{-1}{x^3}-\frac{6}{x}+1=0$$

$$\therefore\quad x^3-6x^2-1=0$$

c) $f(x)=x-\ln\left(1+x^2\right)$

i) $f'(x)=1-\frac{2x}{1+x^2}$

$$f'(x)=\frac{1+x^2-2x}{1+x^2}=\frac{(x-1)^2}{1+x^2}$$

hence $f'(x)=\frac{(x-1)^2}{1+x^2}\geq 0$ thus is an increasing function for all x except $x=1$ where it is stationary.

ii) Since $f(0)=0-\ln 1=0$

then $x-\ln\left(1+x^2\right)\geq 0$ for $x\geq 0$.

$$\therefore\quad x\geq \ln\left(1+x^2\right)$$

$$\therefore\quad e^x\geq e^{\ln\left(1+x^2\right)}$$

$$e^x\geq 1+x^2$$

Question 14.

a) i) Since ω is a non-real cube root of unity then it satisfies $z^3=1$ but $\omega\neq 1$.

$$\therefore\quad \omega^3=1\quad\rightarrow\quad(\omega-1)\left(\omega^2+\omega+1\right)=0$$

$$\therefore\quad \omega^2+\omega+1=0$$

ii)

$$\left(1-3\omega+\omega^2\right)\left(1+\omega-8\omega^2\right)=(-\omega-3\omega)\left(-\omega^2-8\omega^2\right)$$

$$=(-4\omega)\left(-9\omega^2\right)$$

$$=36\omega^3=36.$$

b) i) $(a-b)^2\geq 0$

$$\therefore\quad a^2-2ab+b^2\geq 0$$

$$\therefore\quad a^2+b^2\geq 2ab$$

similarly

$\therefore\; b^2+c^2\geq 2bc$ and $\therefore\; c^2+a^2\geq 2ca$

hence on adding

$$\therefore\quad 2\left(a^2+b^2+c^2\right)\geq 2(ab+bc+ca)$$

$$\therefore\quad a^2+b^2+c^2\geq ab+bc+ca.$$

ii) Using (i)

$$2\left(a^2+b^2+c^2\right)\geq 2ab+2bc+2ca.$$

$$\therefore\quad 3\left(a^2+b^2+c^2\right)\geq a^2+b^2+c^2+2ab+2bc+2ca.$$

and since

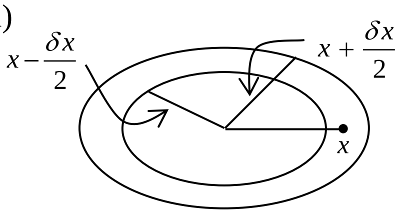
$$a^2+b^2+c^2+2ab+2bc+2ca=(a+b+c)^2$$

then $3\left(a^2+b^2+c^2\right)\geq(a+b+c)^2$.

c) i) $\frac{d}{dx}\left[\ln\left(\cos x^2\right)\right] = \frac{-2x\sin\left(x^2\right)}{\cos\left(x^2\right)}$

$$= -2x\tan\left(x^2\right)$$

ii)



Take a typical shell at a distance x units from O.

Thus the area of the annulus is given by

$$A = \pi\left(x+\frac{\delta x}{2}+x-\frac{\delta x}{2}\right)\left(x+\frac{\delta x}{2}-\left(x-\frac{\delta x}{2}\right)\right)$$

$$= \pi\left(2x\right)\left(\delta x\right)$$

$$= 2\pi x\,\delta x$$

$\therefore$ Volume of the shell $= 2\pi x\,\delta x\,y$

Total Volume $= \lim_{\delta x\to 0}\sum_{x=0}^{\frac{\sqrt{\pi}}{2}} 2\pi xy\,\delta x$

$$= 2\pi\int_0^{\frac{\sqrt{\pi}}{2}} x\tan\left(x^2\right)dx$$

$$= -\pi\left[\ln\left(\cos x^2\right)\right]_0^{\frac{\sqrt{\pi}}{2}}$$

$$= -\pi\left[\ln\left(\cos\frac{\pi}{4}\right)-\ln\left(\cos 0\right)\right]$$

$$= -\pi\left(\ln\frac{1}{\sqrt{2}}-\ln 1\right)$$

$$= \pi\ln\left(\sqrt{2}\right) = \frac{\pi}{2}\ln 2\ u^3$$

d) i) $I_n = \int_0^1\left(1-x^2\right)^n dx$ for $n\geq 1$.

Using *IBP*: Let $u = \left(1-x^2\right)^n$, $\quad v' = 1$

$u' = -2nx\left(1-x^2\right)^{n-1}$, $v = x$

$$\therefore\ I_n = x\left(1-x^2\right)^n\Big]_0^1 + \int_0^1 2nx^2\left(1-x^2\right)^{n-1}dx$$

$$= 0 - 2n\int_0^1\left(\left(1-x^2\right)-1\right)\left(1-x^2\right)^{n-1}dx$$

$$= -2n\int_0^1\left(1-x^2\right)^n - \left(1-x^2\right)^{n-1}dx$$

$$\therefore\ I_n = -2n\left[I_n - I_{n-1}\right]$$

$$\therefore\ I_n + 2nI_n = 2nI_{n-1}$$

$$\therefore\ I_n\left(1+2n\right) = 2nI_{n-1}$$

$$I_n = \frac{2n}{2n+1}I_{n-1}$$

ii) $I_2 = \frac{4}{5}I_1$

$$I_1 = \int_0^1 1-x^2\,dx$$

$$= x-\frac{x^3}{3}\Big]_0^1 = \frac{2}{3}$$

$$\therefore\ I_2 = \frac{4}{5}\times\frac{2}{3} = \frac{8}{15}$$

Question 15.

a) i) Let $I = \int_0^a f\left(x\right)dx$ and let $u = a-x$

$\therefore\ x = a-u \quad\rightarrow\quad dx = -du$

When $x=0,\ u=a;\ \ x=a,\ u=0$

$$\therefore\ I = \int_a^0 f\left(a-u\right)\bullet -du$$

$$= \int_0^a f\left(a-u\right)du = \int_0^a f\left(a-x\right)dx$$

ii) Using (i)

$$\int_0^{\frac{\pi}{4}} \frac{1-\sin 2x}{1+\sin 2x}\,dx = \int_0^{\frac{\pi}{4}} \frac{1-\sin 2\left(\frac{\pi}{4}-x\right)}{1+\sin 2\left(\frac{\pi}{4}-x\right)}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1-\sin\left(\frac{\pi}{2}-2x\right)}{1+\sin\left(\frac{\pi}{2}-2x\right)}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1-\cos 2x}{1+\cos 2x}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1-\left(1-2\sin^2 x\right)}{1+\left(2\cos^2 x-1\right)}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{2\sin^2 x}{2\cos^2 x}\,dx = \int_0^{\frac{\pi}{4}} \tan^2 x\,dx$$

iii) $\displaystyle\int_0^{\frac{\pi}{4}} \tan^2 x\,dx = \int_0^{\frac{\pi}{4}} \sec^2 x - 1\,dx$

$$= \left[\tan x - x\right]_0^{\frac{\pi}{4}} = 1 - \frac{\pi}{4}$$

b) i) $y = \dfrac{c^2}{x} \quad \rightarrow \quad \dfrac{dy}{dx} = -\dfrac{c^2}{x^2}$

at $x = cp,\ \dfrac{dy}{dx} = m = -\dfrac{1}{p^2}$

$$\therefore\ y - \frac{c}{p} = -\frac{1}{p^2}\left(x - cp\right)$$

$\therefore\ p^2 y - cp = -x + cp$

$\therefore\ x + p^2 y = 2cp$ ----- $\boxed{1}$

ii) Similiarly $x + q^2 y = 2cq$ ----- $\boxed{2}$

$\boxed{2} - \boxed{1}$: $\left(p^2 - q^2\right)y = 2c\left(p-q\right)$

And since $p \neq q$

$$\left(p+q\right)y = 2c \quad \rightarrow \quad y = \frac{2c}{p+q}$$

Subbing into $\boxed{1}$ gives $x = \dfrac{2cpq}{p+q}$

$$\therefore\ T\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$$

iii) $m_{PQ} = \dfrac{\frac{c}{q} - \frac{c}{p}}{cq - cp}$

$$= \frac{\frac{c\left(p-q\right)}{pq}}{c\left(q-p\right)} = -\frac{1}{pq}$$

$$\therefore\ y - \frac{c}{p} = -\frac{1}{pq}\left(x - cp\right)$$

$pqy - cq = -x + cp$

$\therefore\ x + pqy = c\left(p+q\right)$

iv) Using (iii) $2c + 0 = c\left(p+q\right)$

$\therefore\ p + q = 2$

v) Using (iv) T becomes $\left(\dfrac{2cpq}{2}, \dfrac{2c}{2}\right) \rightarrow T\left(cpq, c\right)$

Hence the locus of T is $y = c$ for $0 < x < c$

Note: $p + q = 2$, both $p, q > 0$ and $p \neq q$

$\therefore\ pq < 1$

Also the tangents must meet outside of H in the first quadrant.

Also when $y = c,\ x = c$.

Question 16.

a) i) Since the coefficients are real, the roots occur in complex conjugate pairs.

Using sum of the roots:

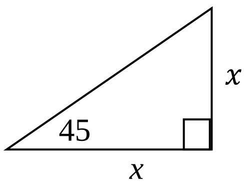
$$1 + 2i + 1 - 2i + \alpha = 0 \quad \to \quad \alpha = \frac{1}{2}$$

Hence the roots are $z = 1 + 2i, \quad z = 1 - 2i, \quad z = \frac{1}{2}$

ii) Since $z = \frac{1}{2}$ is a solution then $P\left(\frac{1}{2}\right) = 0.$

$$\therefore \ \frac{1}{4} - \frac{5}{4} + \frac{q}{2} - 5 = 0 \quad \to \quad q = 12$$

b) i)



Area of $ABCD = x \times 2y$

$= x \times 2y$

The base is half of the circle centre $(0, 0)$, radius 4
Hence has equation
$x^2 + y^2 = 4^2 \quad \to \quad y = \sqrt{16 - x^2}$

$$\therefore \quad \text{Area of } ABCD = x \times 2\sqrt{16 - x^2}$$
$$= 2x\sqrt{16 - x^2}$$

ii)

$$V_s = x \times 2\sqrt{16 - x^2} \ \delta x$$

$$\therefore \quad V = \lim_{\delta x \to 0} \sum_{x=-4}^{4} 2x\sqrt{16 - x^2}\,\delta x$$

$$= 2\int_{-4}^{4} x\sqrt{16 - x^2} \ dx$$

Let $u = 16 - x^2 \quad \to \quad du = -2xdx$

When $x = 0, \ u = 16; \quad x = 4, \ u = 0$

$$\therefore \quad V = -\int_{16}^{0} u^{\frac{1}{2}} \ du$$

$$= \int_{0}^{16} u^{\frac{1}{2}} \ du$$

$$= \left[\frac{2}{3}u^{\frac{3}{2}}\right]_0^{16}$$

$$= \frac{128}{3} u^3$$

c) i) $A_1 = \alpha^1 + \beta^1 = \text{sum of roots}$

$$= -\frac{b}{a} = -\left(\frac{-1}{1}\right) = 1$$

$$A_2 = \alpha^2 + \beta^2$$

$$= (\alpha + \beta)^2 - 2\alpha\beta$$

$$= 1^2 - 2 \times 1 = -1$$

ii) Since α, β satisfy $x^2 - x + 1 = 0$
then $\alpha^2 - \alpha + 1 = 0 ----\boxed{1}$
$\beta^2 - \beta + 1 = 0 ----\boxed{2}$

$\boxed{1} \times \alpha^{n-2} \quad \to \quad \alpha^n - \alpha^{n-1} + \alpha^{n-2} = 0$
$\therefore \ \alpha^n = \alpha^{n-1} - \alpha^{n-2} -----\boxed{3}$

$\boxed{2} \times \beta^{n-2} \quad \to \quad \beta^n = \beta^{n-1} - \beta^{n-2} -----\boxed{4}$

$\boxed{3} + \boxed{4}$ gives:

$$\alpha^n + \beta^n = \left(\alpha^{n-1} + \beta^{n-1}\right) - \left(\alpha^{n-2} + \beta^{n-2}\right)$$

$$\therefore \ A_n = A_{n-1} - A_{n-2}$$

c) $A_1 = 2\cos\frac{\pi}{3} = 2 \times \frac{1}{2} = 1$

$$A_1 = 2\cos\frac{2\pi}{3} = 2 \times -\frac{1}{2} = -1$$

$\therefore$ true for $n = 1$ and $n = 2.$

Assume true for $n = k$ *i.e.* $A_k = 2\cos\frac{k\pi}{3}$

and $n = k - 1$ *i.e.* $A_{k-1} = 2\cos(k-1)\frac{\pi}{3}$

$\therefore\ A_{k+1} = A_k - A_{k-1}$ { using (b) }

$$= 2\cos\frac{k\pi}{3} - 2\cos(k-1)\frac{\pi}{3}$$

$$= 2\cos\frac{k\pi}{3} - 2\cos\frac{k\pi}{3}\cos\frac{\pi}{3} - 2\sin\frac{k\pi}{3}\sin\frac{\pi}{3}$$

$$= \cos\frac{k\pi}{3} - 2\sin\frac{k\pi}{3}\sin\frac{\pi}{3}$$

$$= 2\cos\frac{k\pi}{3}\cos\frac{\pi}{3} - 2\sin\frac{k\pi}{3}\sin\frac{\pi}{3}$$

$$= 2\cos(k+1)\frac{\pi}{3}$$

Therefore if true when $n = k,\ \ k - 1$ then it is also true when $n = k + 1$.

Hence by induction is true for all $n \geq 1$.