Cheltenham Girls' High School

4 unit mathematics

Trial HSC Examination 1986

1. (i) Sketch each of the following graphs on separate diagrams, showing their main features (do not use graph paper).
(a) $y = e^{-x^2}$ **(b)** $y = \sin|x|$, $-2\pi \le x \le 2\pi$ **(c)** $(x+3)(y+1) = 1$
(ii) (a) Show that the curve $y = \frac{x^2}{x^3+8}$ has two turning points and consists of two branches. Find the coordinates of the turning points, determine their nature and sketch the curve.
(b) Use this sketch to determine the number of real roots of the equation $x^3 - 5x^2 + 8 = 0$ (N.B. You are not asked to find the roots).

2. (i) Simplify $\frac{(\cos 3\theta - i\sin 3\theta)^8(\cos 2\theta + i\sin 2\theta)^7}{(\cos 5\theta + i\sin 5\theta)^4}$
(ii) (a) Express $\sqrt{24 - 70i}$ in the form $a + ib$, $(a > 0)$.
(b) Hence solve for z: $z^2 - (1-i)z - 6 + 17i = 0$
(iii) For the complex number $z = x + iy$, find the locus of z if
(a) $\arg(z-4) = \frac{\pi}{4}$ **(b)** $|z| = z + \overline{z} + 1$
(iv) The complex numbers $z = x + iy$ and $w = X + iY$ are such that $w = z + \frac{1}{z}$
(a) Show that $X = x + \frac{x}{x^2+y^2}$, $Y = y - \frac{y}{x^2+y^2}$
(b) Find and sketch the equation of the locus in the X-Y plane of a point which in the x-y plane traces out the circle $|z| = 2$.

3. (i) Evaluate **(a)** $\int_0^1 xe^{-2x}\,dx$ **(b)** $\int_{-1}^1 \frac{dx}{(x+2)(x+5)}$
(ii) Find the following indefinite integrals
(a) $\int x\sqrt{8x-1}\,dx$ **(b)** $\int \frac{\cos^2 x}{4+5\sin^2 x}\,dx$ **(c)** $\int \sin(\ln x)\,dx$

4. (i) If $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, show that $\frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}$
(ii) The pressure p kilopascals on a mass of gas of volume $V\text{cm}^3$ is given by the formula $pV = 1500$. If the volume increases at the rate of $60\text{cm}^3/\text{min}$, find the rate at which the pressure is decreasing at the instant when the volume is 30cm^3.
(iii) (a) Find the seven seventh roots of unity in the form $r(\cos\theta + i\sin\theta)$ and show them on an Argand diagram.
(b) If α is a complex root of $z^7 - 1 = 0$, find the cubic equation whose roots are $\alpha + \alpha^{-1}, \alpha^2 + \alpha^{-2}, \alpha^3 + \alpha^{-3}$.

5. (i) State the definition of a conic section.
(ii) Determine the (real) values of λ for which the equation $\frac{x^2}{4-\lambda} + \frac{y^2}{2-\lambda} = 1$ defines respectively an ellipse and an hyperbola. Sketch the curve corresponding to the

value $\lambda = 1$. Describe how the shape of this curve changes as λ increases from 1 towards 2. What is the limiting position of the curve as 2 is approached?
(iii) Find the equation of the conic with focus $(-1, 1)$ directrix $x + y + 1 = 0$ and eccentricity $\frac{4}{3}$. Sketch the curve.

6. **(i)** Prove that 1 and -1 are both zeros of multiplicity 2 of the polynomial $P(x) = x^6 - 3x^2 + 2$. Express $P(x)$ as the product of irreducible factors over the fields of:
(a) rational numbers.
(b) complex numbers.
(ii) Prove that if A, B, C are polynomials over a field $\mathbb{F}$ and if $A|B$ and $A|C$ then $A|(B - C)$.
(iii) The equation $x^4 + px^3 + qx^2 + rx + t = 0$ has roots $x = a, b, c$ and d. Obtain the monic, quartic (degree 4) equation which has roots $x = 2a, 2b, 2c, 2d$ in terms of x, p, q, r, t.

7. **(i)** Show that $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$. Hence, use the principle of mathematical induction to establish the result:
$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\cdot\cdots\cdot(\cos\theta_n + i\sin\theta_n) =$
$\cos(\theta_1 + \theta_2 + \cdots + \theta_n) + i\sin(\theta_1 + \theta_2 + \cdots + \theta_n)$
(ii) If $p + q = 1$ and $p^2 + q^2 = 2$, determine the values of $p^3 + q^3$ and $p^4 + q^4$ without finding the values of p and q.
(iii) Find all x such that $\cos 2x - \sin 2x = \cos x - \sin x$ and $0 \le x \le 2\pi$.

8. **(i)** The positive integers are bracketed as follows:
(1), (2,3), (4,5,6), ...
where there are r integers in the rth bracket. Prove that the sum of the integers in the rth bracket is $\frac{1}{2}r(r^2 + 1)$.
(ii) The curve in the sketch is the tractrix $x = \ln\cot\frac{\theta}{2} - \cos\theta$, $y = \sin\theta$. Find the equation of the tangent at P in terms of θ, and the length of PT.