CHELTENHAM GIRLS 2007 EXT II

Question 1 **(15 marks)** **Start a NEW page**

(a) Let $\alpha = 3 + 4i$ and $\beta = 1 - i$

Express in the form $x + iy$ where x and y are real

(i) $\alpha\beta$ **1**

(ii) $\dfrac{\alpha}{\beta}$ **2**

(iii) $(\overline{\beta})^2$ **2**

(b) Consider the equation $z^2 + \gamma z + (2 - i) = 0$ **2**

Find the complex number γ, given that $2i$ is a root of the given equation.

(c) Let $\beta = -1 - i\sqrt{3}$

(i) Express β in modulus-argument form. **2**

(ii) Express β^{-10} in modulus-argument form. **2**

(iii) Hence express β^{-10} in the form $x+iy$ **1**

(d) Shade the region in the number plane described by **3**

$\dfrac{\pi}{3} < \arg z < \pi$ and $1 \le |z| \le 3$

Question 2 **(15 marks)** **Start a NEW page**

(a) (i) Find real numbers a and b such that **2**

$$\frac{1}{(3-x)(1+x)} \equiv \frac{a}{(3-x)} + \frac{b}{(1+x)}$$

(ii) Hence find $\displaystyle\int \frac{1}{(3-x)(1+x)}\,dx$ **2**

(b) Find $\displaystyle\int \frac{1}{e^x + e^{-x}}\,dx$ **3**

(c) By completing the square, find **2**

$$\int \frac{3}{\sqrt{x^2 + 6x + 13}}\,dx$$

(d) (i) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1 + \cos x + \sin x}\,dx$, **3**

using the substitution $t = \tan\dfrac{x}{2}$

(ii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{x}{1 + \cos x + \sin x}\,dx$,

using the substitution $u = \dfrac{\pi}{2} - x$ **3**

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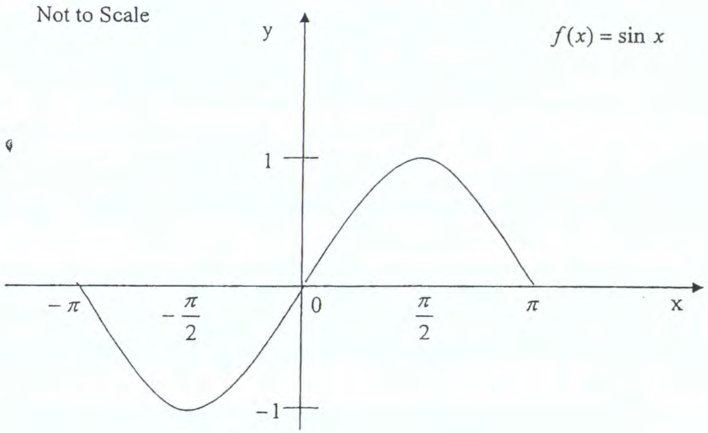
Question 3 (15 marks) **Start a NEW page**

(a) The diagram shows the graph of $f(x)=\sin x$, for $-\pi \le x \le \pi$



Draw separate one-third page sketches of the graphs of the following:

(i) $y=\dfrac{1}{f(x)}$ **1**

(ii) $y=(f(x))^2$ **2**

(iii) $|y|=f(x)$ **2**

(iv) $y=\sin^{-1} f(x)$ **2**

Question 3 continues on page 4

(b) (i) Sketch on the same number plane: **2**

$y=|x|-2$ and

$y=-x^2+3x$

(ii) Hence or otherwise solve $\dfrac{|x|-2}{-x^2+3x}>0$ **2**

(c) α,β,γ are the roots of the equation $x^3+bx^2+12x+4=0$, where b is a real constant.

(i) Find an equation with $\alpha^{-1},\beta^{-1},\gamma^{-1}$ as roots. **2**

(ii) Hence, or otherwise, find the value of b, given that **2**

$\alpha^{-1},\beta^{-1},\gamma^{-1}$ are in arithmetic progression.

End of Question 3

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Question 4 (15 marks) **Start a NEW page**

(a) The area bounded by the curve $y = e^{-\frac{1}{2}x^2}$ and the x-axis, between $x = 0$ and $x = 2$ is rotated about the y-axis. Using the method of cylindrical shells, find the volume of the solid formed. (Leave your answer in exact form) **3**

$$y = e^{-\frac{1}{2}x^2}$$

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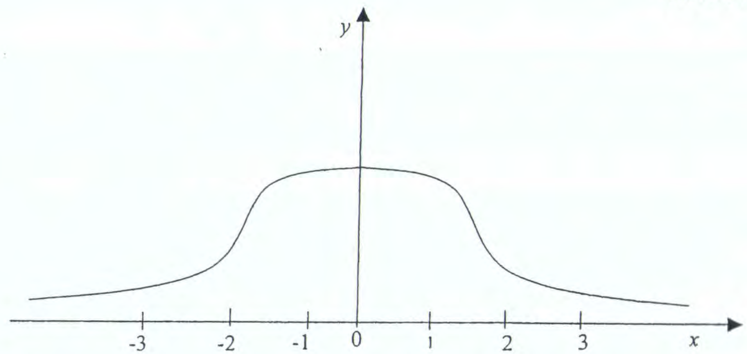

(b) Consider the conic defined by the equation $\frac{x^2}{19-l} + \frac{y^2}{7-l} = 1$

(i) Determine the real value of l for which the equation defines: **2**

(α) an ellipse

(β) an hyperbola

(ii) Sketch the curve corresponding to $l = 3$. Show the foci and the x and y intercepts. You are NOT required to show the directrices. **2**

(iii) Describe how the shape of this curve changes as l increases from 3 towards 7. **1**

(iv) Describe the limiting shape of the curve as l approaches 7. **1**

Question 4 continues on page 6

(c) A circle is drawn to pass through the vertices of ΔABC. AM bisects $\angle BAC$ and meets BC at P as shown in the diagram.

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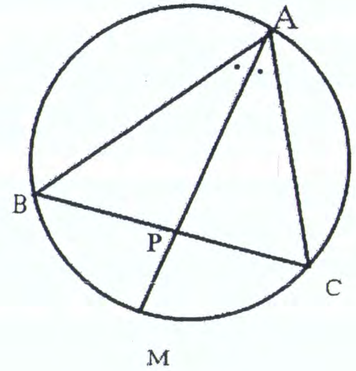

Copy or trace this diagram onto your answer sheet.

(i) Prove that ΔABM and ΔACP are similar. **2**

(ii) Prove that $AB.AC = AP.AM$ **1**

(iii) Hence prove that $AB.AC - BP.PC = AP^2$ **3**

End of Question 4

Question 5 (15 marks) **Start a NEW page**

(a) (i) Show that $2\cos A \sin B = \sin(A+B) - \sin(A-B)$ **1**

(ii) Hence, or otherwise, show that

$$(1 + 2\cos\theta + 2\cos 2\theta + 2\cos 3\theta)\sin\frac{\theta}{2} = \sin\frac{7\theta}{2}$$ **2**

(iii) Let $A = 1 + 2\cos\theta + 2\cos 2\theta + 2\cos 3\theta$ **1**

Show that if $\theta = \dfrac{2\pi}{7}$, then $A = 0$

(iv) Express A in terms of $\cos\theta$ **1**

You may assume $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. Do NOT prove this.

(v) Hence or otherwise prove that $\cos\dfrac{2\pi}{7}$ **1**

is a solution of the polynomial equation $8x^3 + 4x^2 - 4x - 1 = 0$

Question 5 continues on page 8

(b) (i) Let $I_n = \int_1^e x(\ln x)^n \, dx, \quad n = 0,1,2,.....$ **2**

Use integration by parts to show that

$$I_n = \frac{e^2}{2} - \frac{n}{2} I_{n-1} \quad n = 1,2,3,.....$$

(ii) The area bounded by the curve $y = \sqrt{x}\,(\ln x)$, $x \geq 1$, **3**
the x-axis and the line $x = e$ is rotated about the x-axis.
Find the exact volume of the solid of revolution so formed.

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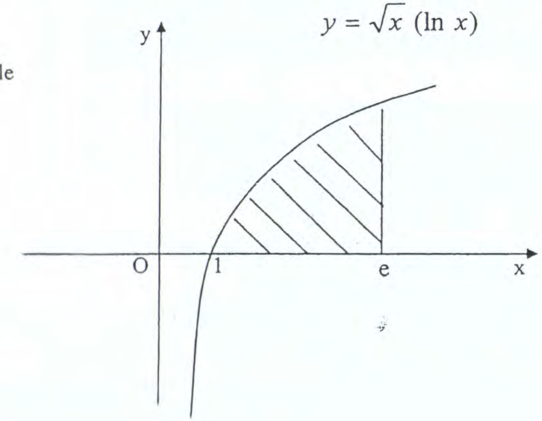

(c) 20 teachers volunteer to be considered for a special 50th anniversary celebrations committee. These consist of 4 Maths teachers, 4 Art teachers, 6 English teachers and 6 History teachers. 9 teachers are to be randomly chosen to form the committee.

(i) If there are no restrictions, in how many ways can the 9 teachers be chosen? **1**

(ii) In how many ways can the 9 teachers be chosen if no Art or Maths teachers are included? **1**

(iii) What is the probability that no more than 1 Maths teacher and no more than 1 Art teacher are on the committee? **2**

End of Question 5

45

Question 6 (15 marks) **Start a NEW page**

(a) A body of mass m is falling from rest and experiences air resistance of kv^2 per unit mass. k is a constant, g is gravity under acceleration and v is the velocity of the body.

(i) Show that the equation of motion of the body is given by **1**

$$\ddot{x} = g - kv^2$$

(ii) If V is the terminal velocity, show that $V = \sqrt{\frac{g}{k}}$ **1**

(iii) Show that $x = \frac{1}{2k}\ln\left(\frac{g}{g - kv^2}\right)$, where x is the distance travelled. **4**

(iv) If W is the velocity of the body when it reaches the ground, show that the distance, S, fallen is given by **1**

$$S = \frac{1}{2k}\ln\left(\frac{V^2}{V^2 - W^2}\right)$$

Question 6 continues on page 10

(b) The Bundeena Voluntary Firefighters have a massive water storage tank. The tank has a rectangular base with sides 16 metres and 10 metres. Its top is also rectangular with dimensions 20 metres and 12 metres. The tank has a depth of 4 metres and each of its four side faces is an isosceles trapezium. Each horizontal cross section parallel to the base of the tank is a rectangle.

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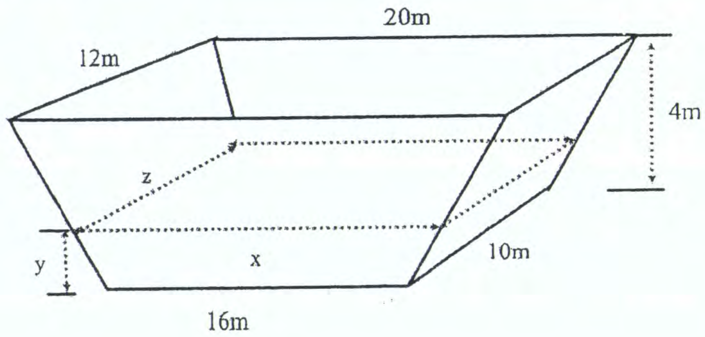


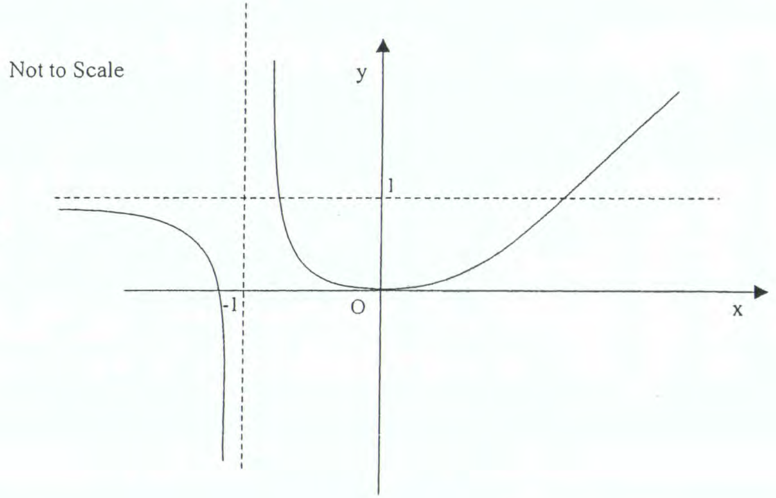
(i) Consider a cross section of the tank x metres by z metres and height y above the base. Show that the area of this cross section is given by **3**

$$\frac{1}{2}y^2 + 18y + 160$$

(ii) Hence find the volume of the tank. **2**

Question 6 continues on page 11

(c) 3



The diagram shows the graph of $y = f'(x)$, the gradient function of $y = f(x)$

Copy or trace this diagram onto your answer sheet.

Sketch, on a separate number plane and underneath your copy, the graph of $y = f(x)$, given that $f(0) = 1$ and $f(x) < 0$ for $x < -1$. Show all relevant asymptotes.

End of Question 6

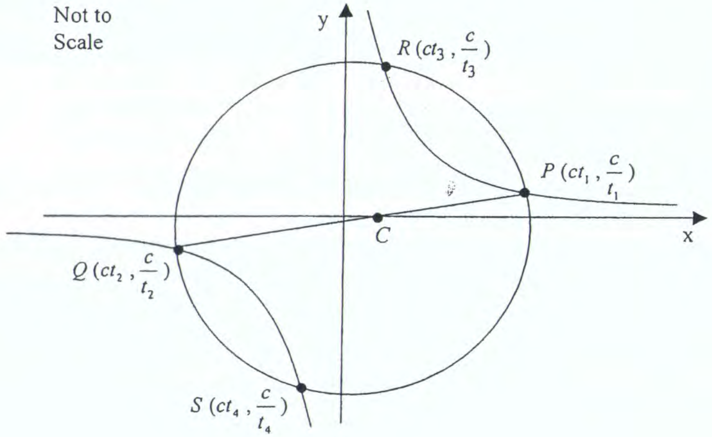
Question 7 (15 marks) Start a NEW page

(a) Let P, Q, R and S be four points with parameters t_1, t_2, t_3 and t_4 on the hyperbola $x = ct$, $y = \frac{c}{t}$, as shown in the diagram.

The hyperbola is intersected by a circle whose equation is

$$(x-g)^2 + y^2 = r^2$$

P, Q, R and S lie on the circle, as shown and **C is the centre of the circle.**



(i) Show that $t_1 + t_2 + t_3 + t_4 = \dfrac{2g}{c}$ 2

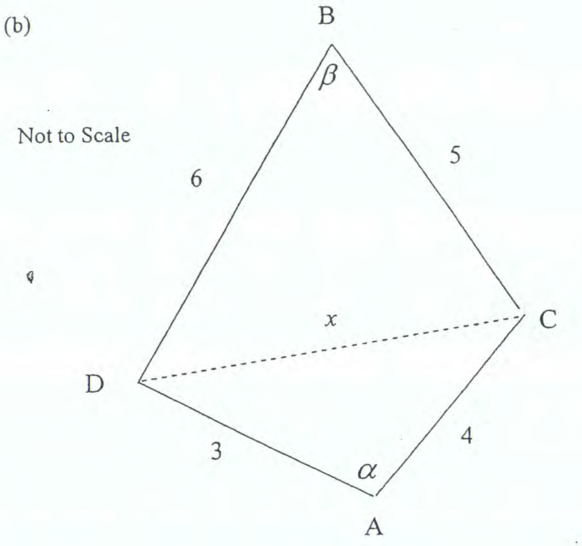
(ii) If C is the midpoint of PQ, show that 2

$$t_1 + t_2 = \frac{2g}{c}$$

(iii) Hence, or otherwise, show that the origin is the midpoint of RS. 2

Question 7 continues on page 13

(b)



The quadrilateral $ACBD$ has sides $AC = 4$, $CB = 5$, $BD = 6$ and $DA = 3$.
$\angle DAC = \alpha$ and $\angle DBC = \beta$. $DC = x$, as shown.

(i) Show that the area of $ACBD$ is given by **1**

$$A = 6 \sin\alpha + 15 \sin\beta$$

(ii) By equating expressions for x^2, show that **2**

$$2\cos\alpha - 5\cos\beta = -3$$

(iii) Differentiate the identity in part (ii) implicitly, with respect to α, **1**

to find an expression for $\dfrac{d\beta}{d\alpha}$.

(iv) Prove that the area of the quadrilateral is a maximum when the quadrilateral is cyclic. You need NOT prove the relevant stationary point is a maximum. **3**

(v) If the quadrilateral is cyclic, find the exact area. **2**

End of Question 7

Question 8 (15 marks) Start a NEW page

(a) (i) Find a general solution for the equation **1**

$\cos A = \cos B$

Write the answer in terms of A and B.

(ii) Hence, or otherwise, solve **3**

$\cos 2\theta = \sin 3\theta$, for $0 \le \theta \le \pi$

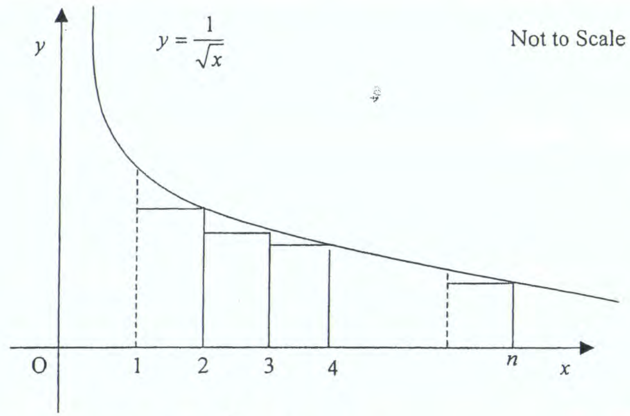
(b) (i) Use Mathematical Induction to show that **3**

$$\sum_{r=1}^{n} \frac{1}{\sqrt{r}} > 2\left(\sqrt{n+1} - 1\right), \text{ where } n = 1, 2, 3, \ldots..$$

You may assume the inequality:

$2k + 3 > 2\sqrt{(k+1)(k+2)}$ Do NOT prove this.

(ii)



Use the graph of $y = \dfrac{1}{\sqrt{x}}$ to show that **2**

$$\sum_{r=1}^{n} \frac{1}{\sqrt{r}} < 1 + \int_{1}^{n} \frac{1}{\sqrt{x}}\,dx$$

(iii) Hence show that **3**

$$198 < \sum_{r=1}^{10\,000} \frac{1}{\sqrt{r}} < 199$$

Question 8 continues on page 15

(c) The points Z_1, Z_2, Z_3 represent the complex numbers z_1, z_2, z_3 and the points W_1, W_2, W_3 represent the complex numbers w_1, w_2, w_3. **3**

If $\dfrac{z_2 - z_1}{z_3 - z_1} = \dfrac{w_2 - w_1}{w_3 - w_1}$ prove that $\Delta Z_1Z_2Z_3$ and $\Delta W_1W_2W_3$ are similar.

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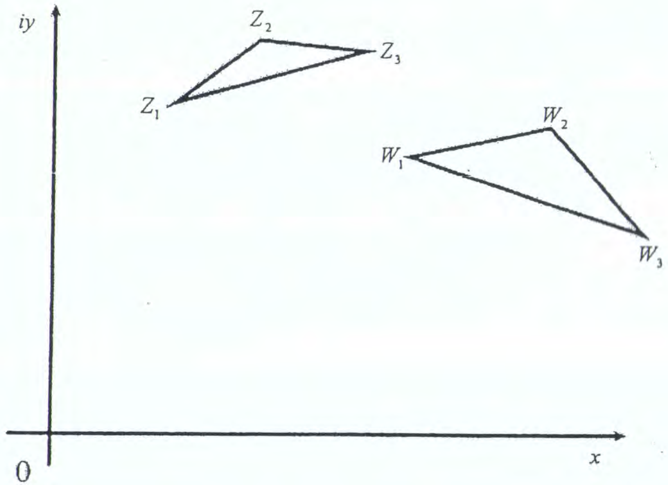


End of Paper

2007 Solutions.

1. $\alpha = 3 + 4i$, $\beta = 1 - i$

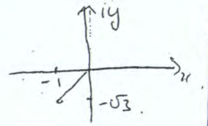
(i) $\alpha\beta = (3+4i)(1-i)$
$= 3 - 3i + 4i + 4$
$= 7 + i.$ (1)

(ii) $\dfrac{\alpha}{\beta} = \dfrac{3+4i}{1-i} \times \dfrac{1+i}{1+i}$
$= \dfrac{3+3i+4i-4}{1+1}$
$= \dfrac{-1+7i}{2}$
$= -\frac{1}{2} + \frac{7}{2}i$ (2)

(iii) $(\bar{\beta})^2 = (1+i)^2$
$= 1 + 2i - 1$
$= 2i$ (2)

(b) $z^2 + \gamma z + (2-i) = 0$
Sub $z = 2i$
$(2i)^2 + \gamma . 2i + 2 - i = 0.$
$-4 + 2i\gamma + 2 - i = 0$
$-2 + 2i\gamma - i = 0$
× bs by i
$-2i - 2\gamma + 1 = 0.$
$+2\gamma = 1 - 2i$
$2(x + iy) = 1 - 2i$
$2x = 1$, $2y = -2$
$x = \frac{1}{2}$, $y = -1$
ie $\gamma = \frac{1}{2} - i$

(c) $\beta = -1 - i\sqrt{3}.$

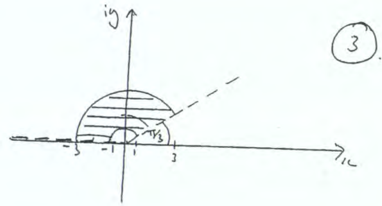


(i) $|\beta| = \sqrt{1+3}$
$= 2.$
$\tan\theta = \frac{1}{\sqrt{3}}.$
$\Rightarrow \theta = \frac{\pi}{3}$
$\arg\beta = -\frac{2\pi}{3}.$
$\beta = 2\,\text{cis}\left(-\frac{2\pi}{3}\right)$ (2)

(ii) $\beta^{-10} = 2^{-10}\,\text{cis}\left(\frac{20\pi}{3}\right)$
$= 2^{-10}\,\text{cis}\left(\frac{2\pi}{3}\right)$ (2)

(iii) $\beta^{-10} = 2^{-10}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$
$= 2^{-10}\left(\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right)$ (1)
$= -2^{-11} + i(\sqrt{3})2^{-11}$

1 (d) $\arg z > \frac{\pi}{3}$, $1 \leq \text{Re}\,z \leq 3.$



(3)

p2

(2)

(a) (i) $\frac{1}{(3-x)(1+x)} \equiv \frac{a}{3-x} + \frac{b}{(1+x)}$

$= \frac{a(1+x) + b(3-x)}{(3-x)(1+x)}$

$1 = a(1+x) + b(3-x)$

let $x=-1$: $1 = 4b$

$b =$ [redacted]

let $x=3$: $1 = 4a$

$a =$ [redacted]

(ii) $\int \frac{1}{(3-x)(1+x)}\,dx = \frac{1}{4}\int \frac{1}{3-x} + \frac{1}{1+x}\,dx$

$= -\frac{1}{4}\ln|3-x| + \frac{1}{4}\ln|1+x| + c$

$=$ [redacted] $+ c$

(b) $\int \frac{1}{e^x + e^{-x}}\,dx$

$= \int \frac{1}{e^x + \frac{1}{e^x}}\,dx$

$= \int \frac{e^x}{e^{2x}+1}\,dx$ let $u = e^x$, $\frac{du}{dx} = e^x$, $du = e^x\,dx$

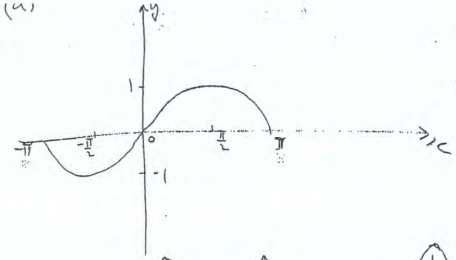
$= \int \frac{du}{u^2+1}$

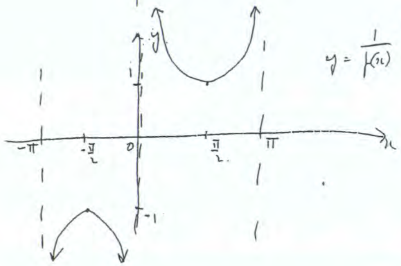
$= \tan^{-1} u + c$

$=$ [redacted] $+ c$

(3)(a)

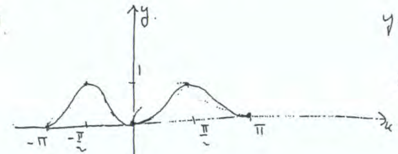
(i)





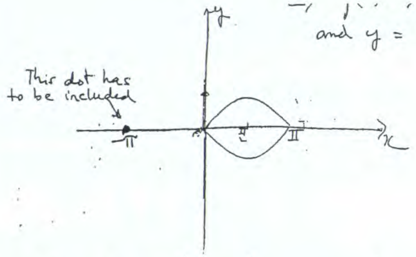
$y = \frac{1}{f(x)}$

(ii)

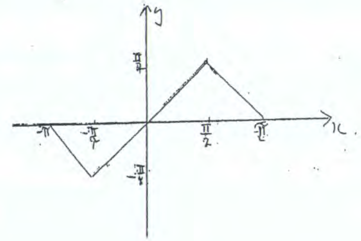


$y = (f(x))^2$

and $y = \pm f$



(iv) $y = \sin^{-1}(\sin x)$



p6

(c) $\int \frac{3}{\sqrt{x^2+6x+13}}\,dx$

$= \int \frac{3}{\sqrt{x^2+6x+9-9+13}}\,dx$

$= \int \frac{3}{\sqrt{(x+3)^2+4}}\,dx$

$x^2+6x+13$

$=$ [redacted]

(d) (i) $\int_0^{\pi/2} \frac{1}{1+\cos x+\sin x}\,dx = \int_0^1 \frac{1}{1+\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}} \cdot \frac{2dt}{(1+t^2)}$

[where $t = \tan\frac{x}{2}$, $\frac{dx}{dt} = \frac{2}{1+t^2}$]

$= \int_0^1 \frac{1+t^2}{1+\cancel{t^2}+1-\cancel{t^2}+2t} \times \frac{2dt}{1+t^2}$

$= \int_0^1 \frac{\cancel{1+t^2}}{2(1+t)} \times \frac{2dt}{\cancel{1+t^2}}$

$= \int_0^1 \frac{dt}{1+t}$

$= [\ln|1+t|]_0^1$

$=$ [redacted]

p4

(ii) $I = \int_0^{\pi/2} \frac{x}{1+\cos x+\sin x}\,dx$

let $u = \frac{\pi}{2} - x$

$\frac{du}{dx} = -1$

$I = \int_{\frac{\pi}{2}}^{0} \frac{\frac{\pi}{2}-u}{1+\cos(\frac{\pi}{2}-u)+\sin(\frac{\pi}{2}-u)}\,(-1)du$

$= \int_0^{\pi/2} \frac{\frac{\pi}{2}-u}{1+\sin u+\cos u}\,du$

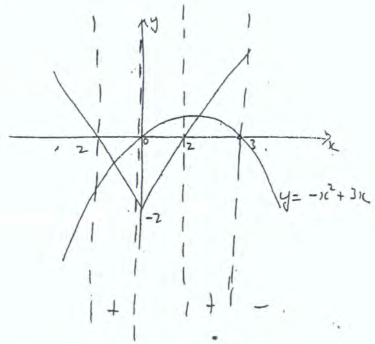
$= \int_0^{\pi/2} \frac{\frac{\pi}{2}\,du}{1+\sin u+\cos u} - \int_0^{\pi/2} \frac{u\,du}{1+\sin u+\cos u}$

$\therefore 2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{du}{1+\sin u+\cos u}$

$= \frac{\pi}{2} \times \ln 2$, from (i)

$I =$ [redacted]

b) (i)



(ii) $-2 < x < 0$, $2 < x < 3$

(c) $x^3 + bx^2 + 12x + 4 = 0$ — ①

(i) Equat with roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ will be:

$(\frac{1}{x})^3 + b(\frac{1}{x})^2 + 12(\frac{1}{x}) + 4 = 0$

$\times$ by x^3

$1 + bx + 12x^2 + 4x^3 = 0$

ie $4x^3 + 12x^2 + bx + 1 = 0$

(ii) If $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ are in A.P.

let $\frac{1}{\alpha} = \frac{1}{\beta} - d$, $\frac{1}{\gamma} = \frac{1}{\beta} + d$

$\therefore \Sigma_1 = \frac{3}{\beta} = -\frac{12}{4} = -3$

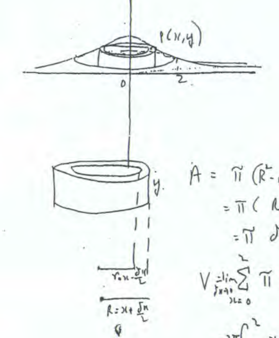
$\therefore \beta = -1$

Sub. for $x = -1$ in ①

$\therefore -1 + b - 12 + 4 = 0$

ie $b = 9$

③



$A = \pi(R^2 - r^2)$

$= \pi(R-r)(R+r)$

$= \pi\, \delta x \cdot 2x$

$V = \lim_{\delta x \to 0} \sum_{x=0}^{2} \pi\, 2xy\, \delta x$

$= 2\pi \int_0^2 xy\, dx$

$= 2\pi \int_0^2 x \cdot e^{-\frac{1}{2}x^2}\, dx$

let $u = x^2$

$\frac{du}{dx} = 2x$

$\therefore x\,dx = \frac{du}{2}$

$= \frac{2\pi}{2} \int_0^4 e^{-\frac{1}{2}u}\, du$

$= \pi \left[\frac{e^{-\frac{1}{2}u}}{-\frac{1}{2}} \right]_0^4$

$= -2\pi \left[e^{-2} - 1 \right]$

$= 2\pi(1 - e^{-2})$ units^3

$19 > l$ | $l < 7$
and $7 - l > 0$
$7 > l$

(β) $19 - l > 0$ ②
$19 > l$
and $7 - l < 0$
$l > 7$
$\therefore 7 < l < 19$

(ii) $\frac{x^2}{16} + \frac{y^2}{4} = 0$ ②

$b^2 = a^2(1 - e^2)$

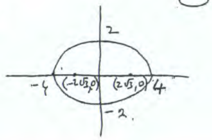
$4 = 16(1 - e^2)$

$\frac{1}{4} = 1 - e^2$

$e^2 = \frac{3}{4}$

$e = \frac{\sqrt{3}}{2}$

foci $x = \pm 2\sqrt{3}$



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5)
a) (i) $\sin(A+B) - \sin(A-B)$

$= \sin A \cos B + \cos A \sin B - \sin A \cos B + \cos A \sin B$ ①

$= 2 \cos A \sin B$

(ii) $(1 + 2\cos\theta + 2\cos 2\theta + 2\cos 3\theta)\sin\frac{\theta}{2}$

$= \sin\frac{\theta}{2} + 2\cos\theta \sin\frac{\theta}{2} + 2\cos 2\theta \sin\frac{\theta}{2} + 2\cos 3\theta \sin\frac{\theta}{2}$ ②

$= \sin\frac{\theta}{2} + \sin\frac{3\theta}{2} - \sin\frac{\theta}{2} + \sin\frac{5\theta}{2} - \sin\frac{3\theta}{2} + \sin\frac{7\theta}{2} - \sin\frac{5\theta}{2}$

$= \sin\frac{7\theta}{2}$

(iii) From (ii):

$A \sin\frac{\theta}{2} = \sin\frac{7\theta}{2}$ ①

if $\theta = \frac{2\pi}{7}$

$A \sin \frac{1}{2} \times \frac{2\pi}{7} = \sin \frac{7}{2} \times \frac{2\pi}{7}$

$A \sin\frac{\pi}{7} = \sin \pi = 0$

$\Rightarrow A = 0$ since $\sin\frac{\pi}{7} \neq 0$

(iv) $A = 1 + 2\cos\theta + 2\cos 2\theta + 2\cos 3\theta$

$= 1 + 2\cos\theta + 2(2\cos^2\theta - 1) + 2(4\cos^3\theta - 3\cos\theta)$

$= 1 + 2\cos\theta + 4\cos^2\theta - 2 + 8\cos^3\theta - 6\cos\theta$ ①

$= 8\cos^3\theta + 4\cos^2\theta - 4\cos\theta - 1$ — ①

(v) If $\theta = \frac{2\pi}{7}$, $A = 0$.

let $x = \cos\theta$ in ① ①

$\therefore A = 8x^3 + 4x^2 - 4x - 1 = 0$ when $\theta = \frac{2\pi}{7}$

ie $\cos\frac{2\pi}{7}$ is a root of the given cubic

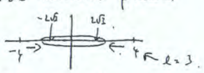
(iii) As l increases from 3 to 7

$a^2 \to 19 - 7 = 12$

$b^2 \to 7 - 7 = 0$ ①

ie $a \to 2\sqrt{3}$, $b \to 0$

so the curve becomes flatter and narrower.



(iv) A straight line on the x axis between $x = -2\sqrt{3}$ & $x = +2\sqrt{3}$, — ①
ie between the original focii.

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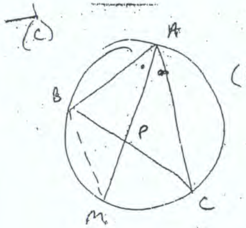
p8

(c)



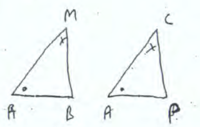
(i) In Δs ABM & ABC

$B\hat{A}M = C\hat{A}P$ (given) ②

$B\hat{M}A = B\hat{C}A$ ($\angle$s subtended at the circumference by the same arc)

$\therefore \Delta ABM \parallel\!\mid \Delta ABC$ (equiangular)

(ii)



$\frac{AB}{AP} = \frac{AM}{AC}$ (corresponding sides in $\parallel\!\mid \Delta$s)

$\therefore AB.AC = AP.AM$ ①

(iii) $BP.PC = AP.PM$ (intersecting chords, in same ratio)

$AB.AC - BP.PC = AP.AM - AP.PM$, from (ii),

$= AP(AM - PM)$

$= AP \times AP$

$= AP^2$ ③

(b)

(i) $I_n = \int_1^e x(\ln x)^n\, dx$

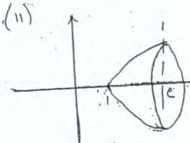
let $u = (\ln x)^n$, $v' = x \Rightarrow v = \frac{1}{2}x^2$ ②

$I_n = \left[\frac{1}{2}x^2(\ln x)^n\right]_1^e - \int_1^e \frac{1}{2}x^2 \cdot n(\ln x)^{n-1} \times \frac{1}{x}\, dx$

$= \frac{1}{2}e^2 - \frac{n}{2}\int_1^e x(\ln x)^{n-1}\, dx$

$= \frac{1}{2}e^2 - \frac{n}{2} I_{n-1}$

(ii)



③

$V = \pi \int_1^e x(\ln x)^2\, dx$

$= \pi I_2 = \left(\frac{1}{2}e^2 - \frac{2}{2} I_1\right)\pi$

$= \left(\frac{1}{2}e^2 - \left(\frac{1}{2}e^2 - \frac{1}{2}I_0\right)\right)\pi$

$I_0 = \int_1^e x\, dx$

$= \left[\frac{1}{2}x^2\right]_1^e = \frac{1}{2}(e^2 - 1)$

$\therefore V = \pi\left[\frac{1}{2}e^2 - \frac{1}{2}e^2 + \frac{1}{2} \times \frac{1}{2}(e^2 - 1)\right]$

$= \frac{\pi}{4}\left[e^2 - 1\right]$ units^3

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p10

(c) 20 = 4 M, 4 A, 6 E, 6 H

(i) ${}^{20}C_9 = 167960$ ①

(ii) ${}^{12}C_9 = 220$ ①

(iii)

no maths	no A	${}^{12}C_9$	220
no maths	1 A	${}^4C_1 \times {}^{12}C_8$	1980
1 maths	no A	${}^4C_1 \times {}^{12}C_8$	1980
1 maths	1 A	${}^4C_1 \times {}^4C_1 \times {}^{12}C_7$	7920
			12100

$\text{Prob} = \frac{12100}{167960}$ ②

$= \frac{605}{8398}$

51

(6)

(a) $\downarrow x=0$ $\uparrow kv^2m$ $\downarrow g$ +ve.

(i) $m\ddot{x} = mg - kv^2 m$ ①

$\therefore \ddot{x} = g - kv^2$

(ii) Terminal vel approached when $\ddot{x} = 0$ ①

$\therefore 0 = g - kV^2$

$V = \sqrt{\frac{g}{k}}$ (+ve $\because$ particle going down)

(iii) $v\frac{d}{dx}v = g - kv^2$

$\frac{dv}{dx} = \frac{g}{v} - kv = \frac{g-kv^2}{v}$ ④

$\frac{dx}{dv} = \frac{v}{g-kv^2}$

$x = \int \frac{v}{g-kv^2}\,dv$ let $v^2 = m$, $\frac{dm}{dv} = 2v$

$= \frac{1}{2}\int \frac{dm}{g-km}$

$= \frac{-1}{2k}\ln(g-km) + c$

$= \frac{-1}{2k}\ln(g-kv^2) + c$

when $x=0$, $v=0$

$0 = -\frac{1}{2k}\ln g + c$

$c = \frac{1}{2k}\ln g$

$x = -\frac{1}{2k}\ln(g-kv^2) + \frac{1}{2k}\ln g$

$= \frac{1}{2k}\ln\left(\frac{g}{g-kv^2}\right)$

(iv) when $v = W$

$S = \frac{1}{2k}\ln\left(\frac{g}{g-kW^2}\right)$ ①

From (ii), $V = \sqrt{\frac{g}{k}}$

$\therefore g = kV^2$

$\therefore S = \frac{1}{2k}\ln\left(\frac{kV^2}{kV^2-kW^2}\right)$

$= \frac{1}{2k}\ln\left(\frac{V^2}{V^2-W^2}\right)$

(i) t_1, t_2, t_3, t_4 are roots of the equation coming from sub $x = ct$ & $y = \frac{c}{t}$ in $(x-g)^2 + y^2 = r^2$.

$(ct-g)^2 + \left(\frac{c}{t}\right)^2 = r^2$

$c^2t^2 - 2gct + g^2 + \frac{c^2}{t^2} = r^2$ ②

$c^2t^4 - 2gct^3 + g^2t^2 + c^2 = r^2t^2$

$c^2t^4 - 2gct^3 + g^2t^2 - r^2t^2 + c^2 = 0$

$t_1 + t_2 + t_3 + t_4 = \frac{2gc}{c^2} = \frac{2g}{c}$

(ii) $C = (g, 0)$

$\therefore g = \frac{ct_1 + ct_2}{2}$ ②

ie $t_1 + t_2 = \frac{2g}{c}$

(iii) Midpt of RS $= M = \left[\frac{c(t_3+t_4)}{2}, \frac{1}{2}\left(\frac{c}{t_3} + \frac{c}{t_4}\right)\right]$

$= (x_1, x_2)$ ②

$x_1 = \frac{c}{2}\left(\frac{2g}{c} - t_1 - t_2\right)$ $y_1 = \frac{c}{2}\left(\frac{t_4+t_3}{t_3t_4}\right)$

$= \frac{c}{2}\left(\frac{2g}{c} - \frac{2g}{c}\right)$ $= 0$, since $t_3 + t_4 = \frac{2g}{c} - (t_1+t_2)$

$= 0$ $= 0$ from (ii)

$= 15\sin\beta + 6\sin\alpha$ ①

(ii) $x^2 = 36 + 25 - 2\times30\cos\beta = 9 + 16 - 2\times12\cos\alpha$

$24\cos\alpha - 60\cos\beta = -36$

ie $2\cos\alpha - 5\cos\beta = -3$

(iii) $-2\sin\alpha + 5\sin\beta \cdot \frac{d\beta}{d\alpha} = 0$

$\frac{d\beta}{d\alpha} = \frac{2\sin\alpha}{5\sin\beta}$

(iv) $\frac{dA}{d\alpha} = 6\cos\alpha + 15\cos\beta\,\frac{d\beta}{d\alpha} = 0$ for a max

ie $0 = 6\cos\alpha + 15\cos\beta \cdot \frac{2\sin\alpha}{5\sin\beta}$

$0 = \frac{6}{\sin\beta}(\sin\beta\cos\alpha + \sin\alpha\cos\beta)$ ③

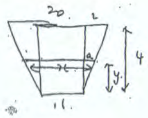
$0 = \sin(\alpha+\beta)$

$\Rightarrow \alpha + \beta = 0, \pi, 2\pi$ → 0 impossible since α, β ...; 2π impossible since sum of 4 $\angle$s $= 2\pi$

$\therefore \alpha + \beta = \pi$

$\therefore$ the quad is cyclic (opp $\angle$s supp)

b) (i)



$\frac{2}{4} = \frac{a}{y}$ $x - 2a = 16$

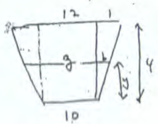
$a = \frac{x-16}{2}$

$= \frac{x}{2} - 8$

$\frac{1}{2} = \frac{\frac{x}{2}-8}{y}$ ③

$y = x - 16$

$x = (y+16)$



$\frac{1}{4} = \frac{b}{y}$ $z - 2b = 10$

$4b = y$ $2b = z - 10$

$2z - 20 = y$

$z = \frac{y}{2} + 10$

Area $= (y+16)\left(\frac{y}{2} + 10\right)$ ②

$= \frac{1}{2}y^2 + 18y + 160$

(ii) Vol $= \lim_{\delta y \to 0} \sum_{y=0}^{4} \frac{1}{2}y^2 + 18y + 160\ \delta y$

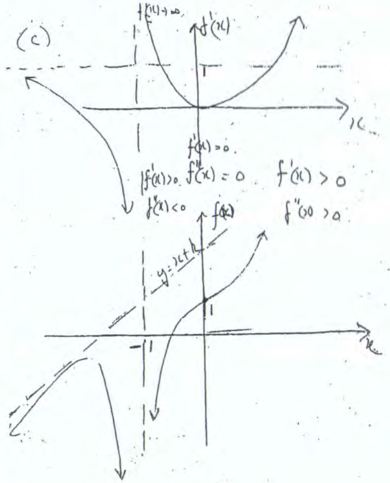
$= \int_0^4 \frac{1}{2}y^2 + 18y + 160\,dy$

$= \left[\frac{1}{6}y^3 + 9y^2 + 160y\right]_0^4$

$= \frac{1}{6}\times 64 + 9\times16 + 160\times4$

$= 794\frac{2}{3}$ m^3

(c)



③

p12

(v) $2\cos\alpha - 5\cos\beta = -3$

$\therefore 2\cos\alpha - 5\cos(\pi - \alpha) = -3$

$2\cos\alpha + 5\cos\alpha = -3$

$\cos\alpha = \frac{-3}{7}$

$\therefore A = 6\sin\alpha + 15\sin(\pi - \alpha)$ ②

$= 6\times\frac{\sqrt{40}}{7} + 15\times\frac{\sqrt{40}}{7}$

$= \left(\frac{12}{7} + \frac{30}{7}\right)\sqrt{10}$

$= 6\sqrt{10}$ units^2

(8)

(a) (i) $\cos A = \cos B$

$\therefore A = 2n\pi \pm \cos^{-1}(\cos B)$ ①

$= 2n\pi \pm B$, n is an integer.

(ii) $\cos 2\theta = \sin 3\theta$. $0 \le \theta \le \pi$

$= \cos\left(\frac{\pi}{2} - 3\theta\right)$ ③

$\therefore 2\theta = 2n\pi \pm \left(\frac{\pi}{2} - 3\theta\right)$

$2\theta = 2n\pi + \frac{\pi}{2} - 3\theta$ or $2\theta = 2n\pi - \frac{\pi}{2} + 3\theta$

$5\theta = 2n\pi + \frac{\pi}{2}$ $-\theta = 2n\pi - \frac{\pi}{2}$

$n=0$ $5\theta = \frac{\pi}{2}$ $n=0$ $\theta = \frac{\pi}{2}$

$\theta = \frac{\pi}{10}$

$n=1$ $5\theta = 2\pi + \frac{\pi}{2}$ $n=-1$ $-\theta = -2\pi - \frac{\pi}{2}$

$\theta = \frac{\pi}{2}$ $\theta = \frac{5\pi}{2}$ X

$n=2$ $5\theta = 4\pi + \frac{\pi}{2}$

$\theta = \frac{9\pi}{10}$

Final solutions $\theta = \frac{\pi}{10}, \frac{\pi}{2}, \frac{9\pi}{10}$

p14

(b) (i) Show $\sum_{r=1}^{n} \frac{1}{\sqrt{r}} > 2(\sqrt{n+1}-1)$

test $r=1$: LHS $=1$ RHS $=2(\sqrt{2}-1)$
$= 0.828$

$\therefore$ True for $r=1$ (3)

assume true for $n=k$

ie $\sum_{r=1}^{k} \frac{1}{\sqrt{r}} > 2(\sqrt{k+1}-1)$ — (1)

prove true for $n=k+1$

ie $\boxed{S_{k+1} > 2(\sqrt{k+2}-1)}$

proof : $S_{k+1} = S_k + \frac{1}{\sqrt{k+1}}$

$> 2(\sqrt{k+1}-1) + \frac{1}{\sqrt{k+1}}$ from (1)

$= \frac{2(k+1) - 2\sqrt{k+1} + 1}{\sqrt{k+1}}$

$= \frac{2k+3 - 2\sqrt{k+1}}{\sqrt{k+1}}$

$> \frac{2\sqrt{(k+1)(k+2)} - 2\sqrt{k+1}}{\sqrt{k+1}}$, given property

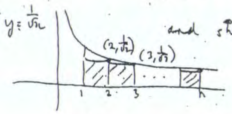
$= 2(\sqrt{k+2}-1)$

as required

$\therefore$ true for $n=1,2,\ldots k, k+1$, $\therefore$ True $\forall$ integers n by induction.

(ii) $\sum_{1}^{n} \frac{1}{\sqrt{r}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}}$

$= 1 + \Sigma$ shaded rectangles



and shaded rectangles $< \int_1^n \frac{1}{\sqrt{x}}\,dx$

(2)

$\therefore \sum_{1}^{n} \frac{1}{\sqrt{r}} < 1 + \int_1^n \frac{1}{\sqrt{x}}\,dx$

(iii) $\sum_{1}^{10\,000} \frac{1}{\sqrt{r}} < 1 + \int_1^{10\,000} \frac{1}{\sqrt{x}}\,dx$

$= 1 + \left[2x^{1/2}\right]_1^{10\,000}$

$= 1 + 2\times 100 - 2$

$= 199$ (3)

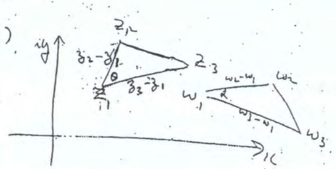
also $\sum_{1}^{10\,000} \frac{1}{\sqrt{r}} > 2(\sqrt{10\,000+1}-1)$, from (i)

$= 198.00\ldots$

$\therefore 198 < \sum_{1}^{10\,000} \frac{1}{\sqrt{r}} < 199$

p16

:)



(3)

...en $\frac{z_2-z_1}{z_3-z_1} = \frac{w_2-w_1}{w_3-w_1}$, let θ + α be as shown

then $\left|\frac{z_2-z_1}{z_3-z_1}\right| = \left|\frac{w_2-w_1}{w_3-w_1}\right|$

$\therefore \frac{|z_2-z_1|}{|z_3-z_1|} = \frac{|w_2-w_1|}{|w_3-w_1|}$, ie corr. sides in same ratio.

also $\text{Arg}\left(\frac{z_2-z_1}{z_3-z_1}\right) = \text{Arg}\left(\frac{w_2-w_1}{w_3-w_1}\right)$, since $\frac{z_2-z_1}{z_3-z_1} = \frac{w_2-w_1}{w_3-w_1}$

ie $\text{Arg}(z_2-z_1) - \text{Arg}(z_3-z_1)$
$= \text{Arg}(w_2-w_1) - \text{Arg}(w_3-w_1)$

ie $\theta = \alpha$

$\therefore \Delta z_1 z_2 z_3 \;|||\; \Delta w_1 w_2 w_3$
(SAS similarity)