CHELTENHAM G.S. EXT II
2008

Total marks – 120.
Attempt Questions 1 – 8.
All questions are of equal value.

Answer each question on a NEW page. Extra paper is available.

Question 1 (15 marks) Use a NEW sheet of paper,

a) $\int xe^{x^2}\,dx$ **1**

b) Evaluate $\int_1^3 x^2 \log_e x\,dx$ **3**

c) $\int \frac{dx}{5+4\cos x}$ **3**

d) True or false:

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i) $\int_{-1}^{1} \sin^7 x\,dx = 0$ **1**

ii) $\int_{-a}^{a} x\cos x\,dx = 2\int_0^a x\cos x\,dx$ **1**

iii) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ **1**

e) i) Write $\frac{x^2+2x+3}{(x+1)(x^2+1)}$ in the form $\frac{A}{x+1}+\frac{Bx+C}{x^2+1}$. **2**

ii) Hence find $\int \frac{x^2+2x+3}{(x+1)(x^2+1)}\cancel{dx}$ **3**

Question 4 (15 marks) Use a NEW sheet of paper

a) The polynomial $x^3 - 4x^2 + x = 0$ has roots α, β and γ. Find a polynomial with integer coefficients that has roots α^2, β^2 and γ^2. **3**

b) Use the method of cylindrical shells to find the volume of the solid generated when the area between $y = \sqrt{1-x^2}$ and $y = x$ from $x = 0$ to $x = \frac{1}{\sqrt{2}}$ is rotated about the y-axis. **3**

c) i) Let n be a positive integer and $I_n = \int_1^2 (\log_e x)^n \, dx$.

Show that $I_n = 2(\log_e 2)^n - nI_{n-1}$. **3**

ii) Hence evaluate $\int_1^2 (\log_e x)^3 \, dx$. **2**

d) $P(x)$ is a polynomial where $P(x) = A(x)(x-k)^n$ where $A(x)$ is a polynomial and n is an integer greater than 1.

i) Show that $P(k) = P'(k) = 0$. **2**

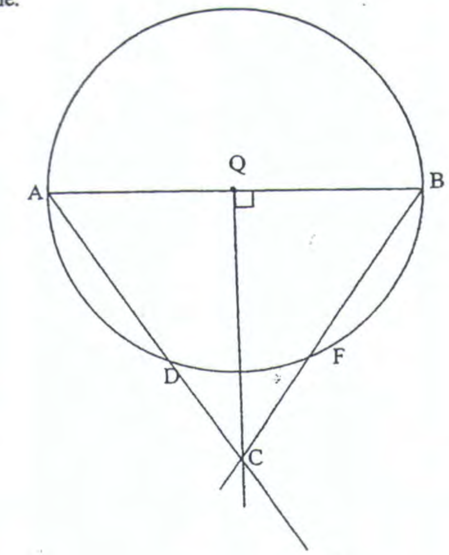
ii) Hence or otherwise find the values of u and v if $x^4 + ux^3 + 9x^2 + vx + 2 = 0$ has a triple root at $x = 3$. **2**

Question 5 (15 marks) Use a NEW sheet of paper

a) AB is the diameter of the circle, centre Q. CQ is perpendicular to AB. AC meets the circle at D, BC meets the circle at F.

Show that CDQB is a cyclic quadrilateral. **3**

Not to scale.



Question 5 is continued on the next page

Question 2 (15 marks) Use a NEW sheet of paper.

a) If $z = 2+3i$ find in $x+iy$ form

i) $2z+3\bar{z}$ **1**

ii) $\frac{1}{z}$ **1**

iii) z^2 **1**

b) i) Write in $(1+i)$ in modulus-argument form. **2**

ii) Hence determine $(1+i)^{10}$. Write your answer in $x+iy$ form. **2**

c) Sketch on an Argand diagram the complex numbers that satisfy both

$$|z-2+i|<3 \text{ and } \frac{-\pi}{4} \le \arg(z) \le \frac{\pi}{4}$$

2

d) If $z_2 = iz_1$ show that $z_2 - z_1 = i(z_1+z_2)$. Interpret this geometrically. **3**

e) The complex number z is a function of the real number t where $z = \frac{t-i}{t+i}$, $0 \le t \le 1$. Evaluate $|z|$ and hence describe the locus of z. **3**

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Question 3 (15 marks) Use a NEW sheet of paper.

(a) i) Sketch $f(x) = x^3 - 3x^2$ showing x and y intercepts and all stationary points. **2**

ii) Hence sketch:

α) $y = (f(x))^2$ **2**

β) $y = \frac{1}{f(x)}$ **2**

γ) $y = \sqrt{f(x)}$ **2**

δ) $y = f(|x|)$ **2**

b) Draw a clear sketch of $\frac{x^2}{4} - \frac{y^2}{9} = 1$ showing asymptotes, foci and directrices. **3**

c) Find the gradient of the tangent to $xy + 2x = 4$ at (1, 2). **2**

Question 6 (cont)

b) Using Mathematical induction, show that for each positive integer n, there are unique positive integers p_n and q_n such that $(1+\sqrt{2})^n = p_n + q_n\sqrt{2}$ and that $p_n^2 - 2q_n^2 = (-1)^n$. **5**

c) A particle of mass m is projected vertically upwards with initial velocity u, in a medium whose resistance to motion varies as the velocity of the particle ie $\ddot{x} = -g - kv$.

i) Show that the time taken to reach the highest point is $\frac{1}{k}\log_e(1+\frac{ku}{g})$. **3**

ii) Find the greatest height the particle will reach. **3**

Question 7 (15 marks) Use a NEW sheet of paper

a) The horizontal interval LM through the point $(0, b)$, where $0 < b < a$, divides the area between the curve $x^2 + y^2 = a^2$ and the coordinate axes into 2 equal parts.

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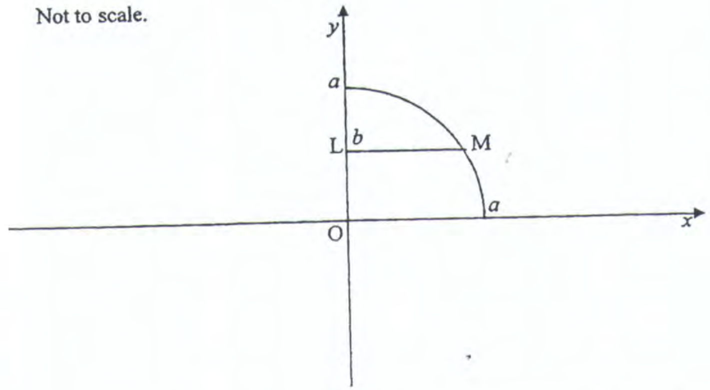

i) By finding the area between LM, the coordinate axes and the circle, show that $\sin^{-1}\frac{b}{a} + \frac{b\sqrt{a^2-b^2}}{a^2} = \frac{\pi}{4}$. **4**

ii) If the radius of the circle is 1 unit, show that b can be found by solving $\sin 2\theta = \frac{\pi}{2} - 2\theta$, where $\theta = \sin^{-1} b$. **2**

iii) Describe how θ and hence b could be approximated. **1**

Question 7 is continued on the next page

Question 5 (cont)

b) If $z = \cos\theta + i\sin\theta$:

i) Show that $z^n + z^{-n} = 2\cos n\theta$ where n is a positive integer. **1**

ii) Given that $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ find an expression for $(z+\frac{1}{z})^4$ in the form $a\cos 4\theta + b\cos 2\theta + c$. **3**

iii) Hence evaluate $\int_0^{\pi} \cos^4\theta \, d\theta$. **2**

c) $P(a\sec\theta, b\tan\theta)$ is a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. $S(ae,0)$ is a focus with corresponding directrix $x = \frac{a}{e}$.

i) Using the focus- directrix definition of an hyperbola or otherwise, prove that $PS = a(e\sec\theta - 1)$. **2**

ii) Show that the normal at P has the equation

$$y - b\tan\theta = \frac{-a\tan\theta}{b\sec\theta}(x - a\sec\theta).$$

2

iii) The normal at P meets the x – axis at G. Show that $\frac{SG}{SP} = e$. **2**

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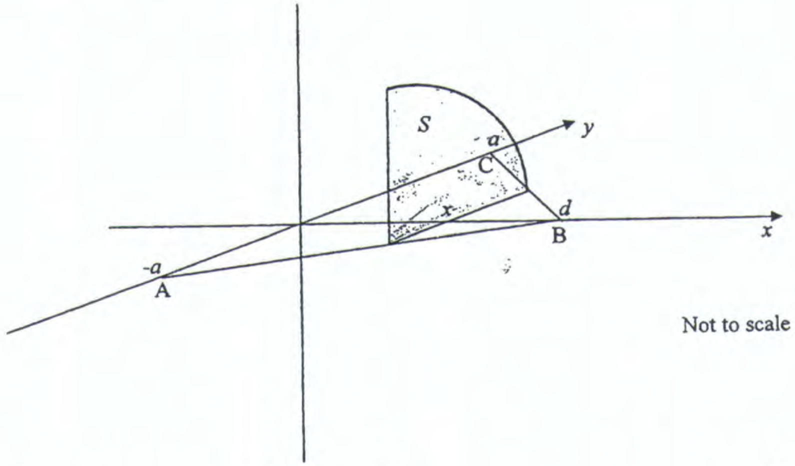
Question 6 (15 marks) Use a NEW sheet of paper

a) The base of a solid is the isosceles triangle ΔABC. The triangle has perpendicular height d and has a base $2a$. Each cross section perpendicular to the x-axis is a quarter circle with the radius in the x-y plane and parallel to the y-axis. A typical cross section S is shown positioned at x on the x-axis.



Not to scale

i) Show that the area of S is $\frac{\pi a^2}{d^2}(d-x)^2$. **2**

ii) Hence find the volume of the solid. **2**

Question 6 is continued on the next page

Question 8 (cont)

b) ABC is a triangle inscribed in a circle of radius r. The centre of the circle is in the interior of the triangle. Let a = BC, b = AC and c = AB.

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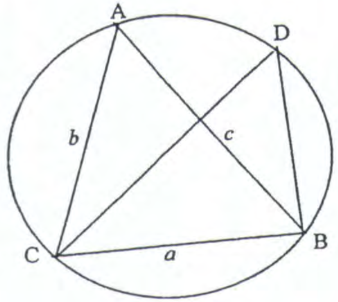


i) Let CD be a diameter. Use the sine rule to show that $\dfrac{a}{\sin A} = 2r$. **2**

ii) Show that the area of the triangle ABC = $\dfrac{1}{2}r^2\,(\sin 2A + \sin 2B + \sin 2C)$. **2**

iii) Show that for angles A,B and C in a triangle, **4**

$$\sin 2A + \sin 2B + \sin 2C \;=\; 4\sin A\,\sin B\,\sin C\,.$$

(iv)Hence show that the area of triangle ABC is $\dfrac{abc}{4r}$ **2**

END OF EXAMINATION

Question 1

(a) $\int x e^{x^2} dx = \frac{1}{2}\int 2x\, e^{x^2} dx$

$= \frac{1}{2}e^{x^2} + C$ (or Sub $u = x^2$)

(b) By Parts $u = \ln x$ $v' = x^2$

$\int_1^3 x^2 \ln x\, dx = \left[\frac{1}{3}x^3 \ln x\right]_1^3 - \int_1^3 \frac{1}{3}x^3 \cdot \frac{1}{x} dx$

$= \frac{1}{3}3^3\ln 3 - 0 - \frac{1}{3}\int_1^3 x^2\, dx$

$= 9\ln 3 - \frac{1}{3}\left[\frac{x^3}{3}\right]_1^3 = 9\ln 3 - \frac{26}{9}$

(c) Let $t = \tan\frac{x}{2}$

$dt = \frac{1}{2}\sec^2\frac{x}{2}\, dx = \frac{1}{2}(1+t^2)\, d$

$\therefore dx = \dfrac{2\,dt}{1+t^2}$

$\int \dfrac{dx}{5+4\cos x} = \int \dfrac{1}{5+\frac{4(1-t^2)}{1+t^2}} \cdot \dfrac{2\,dt}{1+t^2}$

$= 2\int \dfrac{dt}{5(1+t^2)+4(1-t^2)} = 2\int \dfrac{dt}{9+t^2}$

$= \frac{2}{3}\tan^{-1}\left(\frac{t}{3}\right) = \frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\frac{x}{2}\right) + c$

(d) (i) TRUE (odd function)

(ii) FALSE (odd function)

(iii) TRUE (standard result)

(e) (i) $x^2+2x+3 \equiv A(x^2+1) + (B_1x + C)(x+1)$

let $x = -1$ $2A = 2$ $\therefore A = 1$

Equate x^2 $1 = A + B$ $\therefore B = 0$

Equate constant $3 = A + C$ $\therefore C = 2$

$\int \dfrac{x^2+2x+3}{(x+1)(x^2+1)} dx = \dfrac{1}{x+1} + \dfrac{2}{x^2+1}$

$= \ln|x+1| + 2\tan^{-1}x + C$

Question 2. (a)

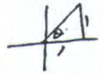
(i) $10 - 3i$

(ii) $\dfrac{1}{2+3i}\cdot\dfrac{2-3i}{2-3i} = \dfrac{2-3i}{13}$

(iii) $(2+3i)^2 = 4 + 12i - 9$

$= -5 + 12i$

(b) (i) $|1+i| = \sqrt{2}$



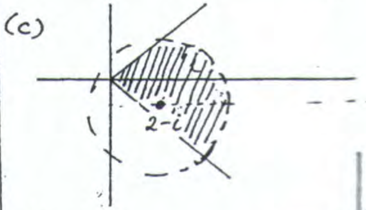
$\arg(1+i) = \pi/4$

$\therefore 1 + i = \sqrt{2}\,\text{cis}\,\frac{\pi}{4}$

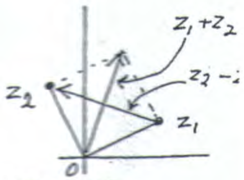
(ii) $(1+i)^{10} = (\sqrt{2})^{10}\,\text{cis}\,\frac{10\pi}{4}$

$= 2^5\,\text{cis}\,\frac{\pi}{2} = 32i$

(c)



(d)



z_2 is a rotation of z_1 by $\pi/2$.

This creates a square.

$z_2 - z_1$ and $z_1 + z_2$ are diagonals.

If $z_2 - z_1 = i(z_1 + z_2)$ Then this states that the diagonals are perpendicular.

Take modulus of each side

$|z_2 - z_1| = |z_1 + z_2|$

This states diagonals are equal.

OR algebraically

$z_2 - z_1 = iz_1 - z_1 = iz_1 + i^2 z_1$

$= i(z_1 + iz_1)$

Geometrical interpretation as above.

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Question 7 (cont)

b)

i) Show that $y^2 = (1-x)^2(4-x)$ has 2 stationary points at x = 3 and determine their nature. **3**

ii) Draw a labelled sketch of $y^2 = (1-x)^2(4-x)$ indicating any relevant points. **2**

c) Ten cards are numbered from 1 to 10. They are dealt randomly to 9 people, with eight people receiving 1 card and 1 person receiving 2 cards.

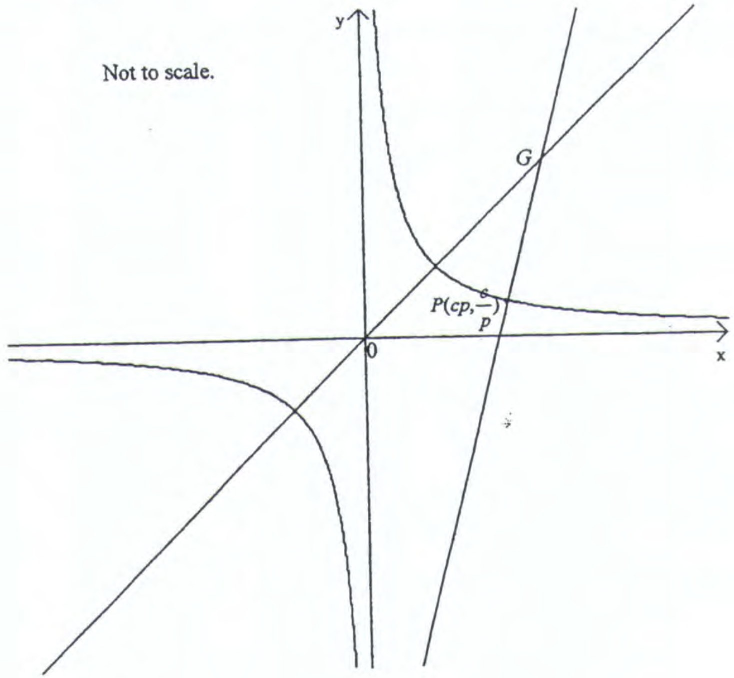
i) What is the probability that the person receiving 2 cards is dealt two odd cards? **2**

ii) What is the probability that this person is dealt one odd card and even card ? **1**

Question 8 (15 marks) Use a NEW sheet of paper

a) The normal at $P(cp, \frac{c}{p})$, where $p^2 \neq 1$, on the hyperbola $xy = c^2$ meets $y = x$ at G.

Not to scale.



ii) Show that the equation of the normal at P is $p^3x - py = c(p^4 - 1)$. **1**

iii) Find the length PG in terms of p. **2**

iv) Show that the distance PG is greater than $c\sqrt{2}$. **2**

Question 8 is continued on the next page

(b) (i) $z^n + z^{-n} = \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$ by de Moivre

$= \cos n\theta + \cos n\theta + i\sin n\theta - i\sin n\theta$

$= 2\cos n\theta$

(ii) $(z+\frac{1}{z})^4 = z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4}$

$= (z^4 + z^{-4}) + 4(z^2 + z^{-2}) + 6$

$= 2\cos 4\theta + 8\cos 2\theta + 6$

Also $(z+\frac{1}{z})^4 = (2\cos\theta)^4$

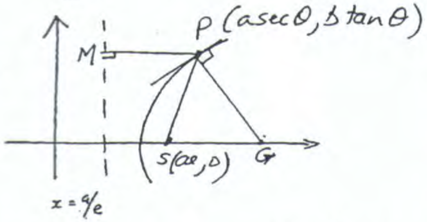
$= 16\cos^4\theta$

$\therefore \cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$

(iii) $\int_0^{\pi} \cos^4\theta \, d\theta$

$= \left[\frac{1}{32}\sin 4\theta + \frac{1}{4}\sin 2\theta + \frac{3\theta}{8}\right]_0^{\pi}$

$= \frac{3\pi}{8}$



from diagram & definition

$PS = e \cdot PM$

$= e[a\sec\theta - a]$

$= a[e\sec\theta - 1]$

(ii) $\frac{2x}{a^2} - \frac{2yy'}{b^2} = 0$ (differentiating)

$\therefore y' = \frac{b^2 x}{a^2 y}$

gradient at P $= \frac{b^2 a\sec\theta}{a^2 b\tan\theta}$

$= \frac{b\sec\theta}{a\tan\theta}$

$\therefore$ gradient of normal $= \frac{-a\tan\theta}{b\sec\theta}$

Equation of normal

$y - b\tan\theta = \frac{-a\tan\theta}{b\sec\theta}(x - a\sec$

(iii) G is where normal meets x-axis, i.e. $y = 0$

$\therefore -b\tan\theta = -\frac{a\tan\theta}{b\sec\theta}(x - a$

$\therefore \frac{a}{b\sec\theta}(x - a\sec\theta) = b$

$a(x - a\sec\theta) = b^2\sec\theta$

$\therefore ax = (a^2 + b^2)\sec\theta$

$= (a^2 + a^2(e^2-1))\sec\theta$

$\therefore x = ae^2\sec\theta$

$\therefore SG = ae^2\sec\theta - ae$

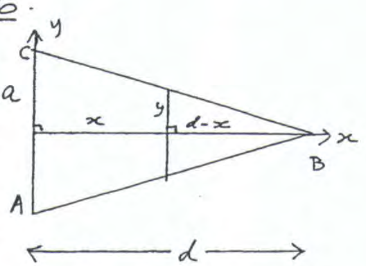
$= ae(e\sec\theta - 1)$

$\therefore \frac{SG}{SP} = \frac{ae(e\sec\theta - 1)}{a(e\sec\theta - 1)}$

$= e$

Question 6.

(a) (i)



By similar Δs $\frac{y}{d-x} = \frac{a}{d}$

$\therefore y = \frac{a}{d}(d-x)$

OR

$\therefore$ gradient of line BC $= -\frac{a}{d}$

y-intercept $= a$

Equation of BC $= y = -\frac{a}{d}x + a$

$\therefore y = \frac{a}{d}(d-x)$

Area $S = \frac{1}{4}\pi(2y)^2 = \pi y^2$

$= \frac{\pi a^2}{d^2}(d-x)^2$

(ii) Vol $= \int_0^d \frac{\pi a^2}{d^2}(d-x)^2 \, dx$

$= \frac{\pi a^2}{d^2}\left[-\frac{(d-x)^3}{3}\right]_0^d$

$= \frac{\pi a^2}{d^2}\left[0 - -\frac{d^3}{3}\right] = \frac{\pi a^2 d}{3}$

(b) When $n=1$

$(1+\sqrt{2})^1 = 1+\sqrt{2}$ $\therefore p_1 = 1$ and $q_1 = 1$

$(p_1)^2 - 2(q_1)^2 = 1 - 2 = (-1)^1$

$\therefore$ True when $n=1$.

Assume true for $n=k$

$(1+\sqrt{2})^{k+1} = (1+\sqrt{2})^k(1+\sqrt{2})$

$= (p_k + q_k\sqrt{2})(1+\sqrt{2})$ by assumption

$= p_k + p_k\sqrt{2} + q_k\sqrt{2} + 2q_k$

$= (p_k + 2q_k) + (p_k + q_k)\sqrt{2}$

Hence $p_{k+1} = p_k + 2q_k$ and

$q_{k+1} = p_k + q_k$

(These are uniquely determined since p_k and q_k are)

$p_{k+1}^2 - 2q_{k+1}^2 =$

$= (p_k + 2q_k)^2 - 2(p_k + q_k)^2$

$= p_k^2 + 4p_kq_k + 4q_k^2 - 2p_k^2 - 4p_kq_k - 2q_k^2$

$= -p_k^2 + 2q_k^2 = -(p_k^2 - 2q_k^2)$

$= -(-1)^k$ by hypothesis

$= (-1)^{k+1}$

(c) (i) $\frac{dv}{dt} = -g - kv$

$\frac{dt}{dv} = -\frac{1}{g+kv}$ $\therefore t = -\int \frac{1}{g+kv} \, dv$

$= -\frac{1}{k}\ln(g+kv) + C$

When $t=0$ $v=u$

$0 = -\frac{1}{k}\ln(g+ku) + C$

$\therefore t = \frac{1}{k}\ln(g+ku) - \frac{1}{k}\ln(g+kv)$

$= \frac{1}{k}\ln\left(\frac{g+ku}{g+kv}\right)$

When $v=0$ (for max. height,

$t = \frac{1}{k}\ln\left(\frac{g+ku}{g}\right) = \frac{1}{k}\ln\left(1+\frac{ku}{g}\right)$

(ii) Easier to start from equation of motion

$v\frac{dv}{dx} = -g - kv = -(g+kv)$

$\frac{dv}{dx} = -\left(\frac{g+kv}{v}\right)$ $\frac{dx}{dv} = -\frac{v}{g+kv}$

$\therefore x = -\int \frac{v}{g+kv} \, dv = -\frac{1}{k}\int \frac{g+kv-g}{g+kv} \, d$

$= -\frac{1}{k}\int 1 - \frac{g}{g+kv} \, dv$

$= -\frac{1}{k}\left[v - \frac{g}{k}\ln(g+kv)\right] + C_1$

When $x=0$ $v=u$

$\therefore 0 = -\frac{1}{k}\left[u - \frac{g}{k}\ln(g+ku)\right] + C_1$

$\therefore x = \left(\frac{u}{k} - \frac{v}{k}\right) + \frac{g}{k^2}\ln\left(\frac{g+kv}{g+ku}\right)$

for max height let $v=0$

$\therefore x = \frac{u}{k} + \frac{g}{k^2}\ln\left(\frac{g}{g+ku}\right)$

OR $\frac{u}{k} - \frac{g}{k^2}\ln\left(1 + \frac{k}{g}u\right)$

$|t+i| \quad \sqrt{x+1}$

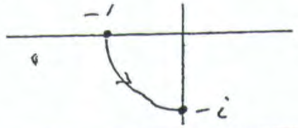
since t is real

when $t=0$ $\quad z = \frac{-i}{i} = -1$

when $t=1$ $\quad z = \frac{1-i}{1+i} \cdot \frac{1-i}{1-i}$

$= \frac{1^2 - 2i + -1}{2} = -i$

Locus is $\frac{1}{4}$ circle



Question 3.

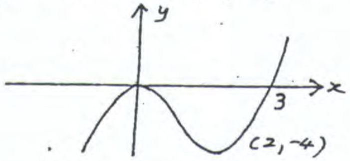
(a) (i) $y = x^3 - 3x^2$ $\quad y' = 3x^2 - 6x$

$y'' = 6x - 6$

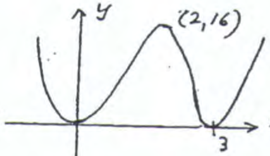
St. Pts $y' = 0 \Rightarrow x = 0$ or 2

$y''(0) = -6 \quad \therefore (0,0)$ is MAX

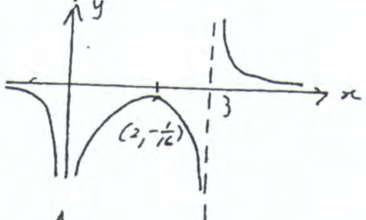
$y''(2) = 6 \quad \therefore (2,-4)$ is MIN



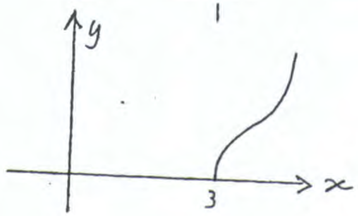
(ii) (α)

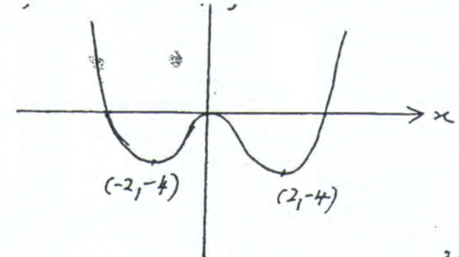


(β)

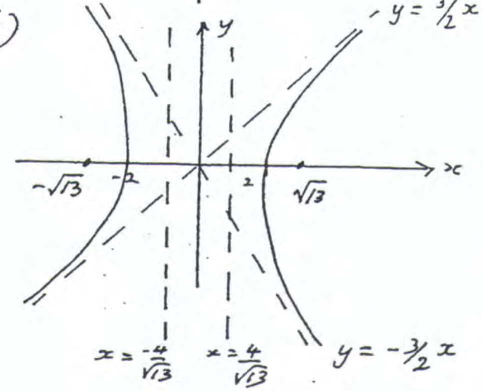


(γ)





(b)



$a = 2 \qquad b = 3$

$b^2 = a^2(e^2-1) \qquad \therefore 9 = 4(e^2-1)$

$\therefore e^2 - 1 = 9/4 \qquad e^2 = \frac{13}{4} \qquad e = \frac{\sqrt{13}}{2}$

$ae = \sqrt{13}$

$\frac{a}{e} = \frac{4}{\sqrt{13}}$

(c) $xy' + y + 2 = 0$

$y' = -\left(\frac{y+2}{x}\right)$

At $(1,2)$ $\quad y' = -\left(\frac{2+2}{1}\right) = -4$

Question 4

(a) Replace x by $\sqrt{x}$

$(\sqrt{x})^3 - 4x + \sqrt{x} = 0$

$x\sqrt{x} - 4x + \sqrt{x}$

$\sqrt{x}(x+1) = 4x$

Square both sides

$x(x^2+2x+1) = 16x^2$

$x^3 - 14x^2 + x = 0$

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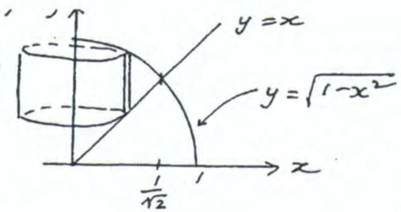


$\delta V = 2\pi r h \; \delta x$

$= 2\pi x\left[\sqrt{1-x^2} - x\right]\delta x$

Vol $= \lim \sum \delta V$

$= 2\pi \int_0^{1/\sqrt{2}} \left(x\sqrt{1-x^2} - x^2\right) dx$

$= 2\pi \left[-\frac{1}{2}\cdot\frac{2}{3}(1-x^2)^{3/2} - \frac{x^3}{3}\right]_0^{1/\sqrt{2}}$

$= 2\pi\left[-\frac{1}{3}\cdot\frac{1}{2\sqrt{2}} - \frac{1}{6\sqrt{2}} - \left(-\frac{1}{3} - 0\right)\right]$

$= 2\pi\left[\frac{1}{3} - \frac{1}{3\sqrt{2}}\right]$

(c)(i) $I_n = \int_1^2 (\log_e x)^n \, dx$

[By parts $\quad u = (\log_e x)^n \quad v' = 1$]

$\therefore I_n = \left[x(\log_e x)^n\right]_1^2 - \int_1^2 x \cdot n(\log_e x)^{n-1} \cdot \frac{1}{x} \, dx$

$= 2(\log_e 2)^n - 0 - n\int_1^2 (\log_e x)^{n-1} dx$

$= 2(\log_e 2)^n - n I_{n-1}$

(ii) $I_0 = \int_1^2 dx = 1$

$I_1 = 2\log_e 2 - 1$

$I_2 = 2(\log_e 2)^2 - 2\left[2\log_e 2 - 1\right]$

$= 2(\log_e 2)^2 - 4\log_e 2 + 2$

$I_3 = 2(\log_e 2)^3 - 3\left[2(\log_e 2)^2 - 4\log_e 2 + 2\right]$

$= 2(\log_e 2)^3 - 6(\log_e 2)^2 + 12\log_e 2 - 6$

(d) (i) $P(x) = A(x)(x-k)^n$

$P'(x) = A(x)\,n(x-k)^{n-1} + A'(x)(x$

$= (x-k)^{n-1}\left[nA(x) + A'(x)(\right.$

$P(k) = A(k) \quad (k-k)^n = 0$

$P'(k) = (k-k)^{n-1}\left[nA(k) + \right.$

$= 0$

(ii) Triple root at $x = 3$

$\therefore P(3) = P'(3) = P''(3) = 0$

$P(x) = x^4 + ux^3 + 9x^2 + vx +$

$P'(x) = 4x^3 + 3ux^2 + 18x + v$

$P''(x) = 12x^2 + 6ux + 18$

$P''(3) = 0 \Rightarrow 108 + 18u + 18$

$\therefore 18u = -126$

$\therefore u = -7$

$P'(3) = 0 \Rightarrow 108 - 189 + 54 +$

$\therefore v = 27$

Question 5

(a) $\angle ADB = 90°$ ($\angle$ in semicircle

$\angle AQC = 90°$ (given)

$\therefore CDQB$ is cyclic

(Converse of $\angle$'s in same seg

OTHER WAYS ARE POSSI

We can prove $\triangle AQC \equiv \triangle BQC$ (

let $\angle QAC = \theta$, then $\angle ABC = \theta$

(corr sides congruent

Also since $\triangle AQD$ is isosceles

$\angle ADQ = \theta$

Hence $\angle ADQ = \angle QBC = \theta$

$\therefore CDQB$ is cyclic

(exterior $\angle$ = opp. interior $\angle$

(iii) Since A, B, C are angles in a triangle

$C = \pi - (A+B)$

$\therefore \sin 2A + \sin 2B + \sin 2C$

$= \sin 2A + \sin 2B + \sin 2\left(\pi - (A+B)\right)$

$= \sin 2A + \sin 2B - \sin(2A + 2B)$

$(\sin(2\pi - \theta) = -\sin\theta)$

$= \sin 2A + \sin 2B - \sin 2A \cos 2B - \cos 2A \sin 2B$

$= \sin 2A\left(1 - \cos 2B\right) + \sin 2B\left(1 - \cos 2A\right)$

$= \sin 2A . 2\sin^2 B + \sin 2B \sin^2 A$

$(\cos 2\theta = 1 - 2\sin^2\theta)$

$= 4\sin A \cos A \sin^2 B + 4\sin B \cos B \sin^2 A$

$= 4\sin A \sin B\left(\cos A \sin B + \sin A \cos B\right)$

$= 4\sin A \sin B \sin(A+B)$

$= 4\sin A \sin B \sin(\pi - C)$

$= 4\sin A \sin B \sin C$

(iv) Area ΔABC

$= \frac{1}{2}r^2(\sin 2A + \sin 2B + \sin 2C)$

from (ii)

$= \frac{1}{2}r^2 . 4\sin A \sin B \sin C$

from (iii)

From (i) $\sin A = \frac{a}{2r}$

$\sin B = \frac{b}{2r}$ $\sin C = \frac{c}{2r}$

$\therefore$ Area $\Delta ABC = \frac{1}{2}r^2 . 4 \frac{a}{2r} . \frac{b}{2r} . \frac{c}{2r}$

$= \frac{abc}{4r}$

OR Area $= \frac{1}{2}bc \sin A$

$= \frac{1}{2}bc . \frac{a}{2r}$

from (i)

$= \frac{abc}{4r}$

It is possible to prove (iii) from (ii) and (iv) since (iv) can be proven independently from (iii):

from (iv)

Area $\Delta ABC = \frac{abc}{4r}$

from (i) $a = 2r\sin A$
$b = 2r \sin B$
$c = 2r \sin C$

$\therefore$ Area $\Delta ABC = \frac{(2r)^3 \sin A \sin B \sin}{4r}$

$= 2r^2 \sin A \sin B \sin$

From (ii)

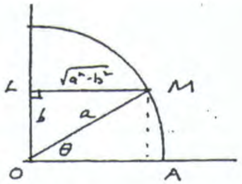
Area $\Delta ABC = \frac{1}{2}r^2(\sin 2A + \sin 2B + \sin$

Equating those gives

$\sin 2A + \sin 2B + \sin 2C$

$= 4\sin A \sin B \sin C.$

...stion 7.



a of OLMA

ea Δ OLM

Area sector OMA

$\frac{1}{2}OL.LM + \frac{1}{2}a^2\theta$

$\frac{1}{2}b\sqrt{a^2-b^2} + \frac{1}{2}a^2\sin^{-1}\left(\frac{b}{a}\right)$

OR

ea OLMA $= \int_0^b \sqrt{a^2-y^2}\,dy$

ange variable: Let $y = a\sin\theta$

$\therefore \frac{dy}{d\theta} = a\cos\theta$.

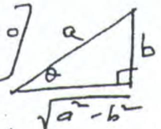
$\int_0^b \sqrt{a^2-y^2}\,dy = \int_0^{\sin^{-1}(b/a)} \sqrt{a^2-a^2\sin^2\theta}\,.\,a\cos\theta\,d\theta$

$\int_0^{\sin^{-1}(b/a)} \cos^2\theta\,d\theta = \frac{a^2}{2}\int_0^{\sin^{-1}(b/a)}(\cos 2\theta + 1)\,d\theta$

$\frac{x^2}{2}\left[\frac{1}{2}\sin 2\theta + \theta\right]_0^{\sin^{-1}(b/a)}$

$\frac{a^2}{2}\left[\sin\theta\cos\theta + \theta\right]_0^{\sin^{-1}(b/a)}$

$\left[\frac{b}{a}\frac{\sqrt{a^2-b^2}}{a} + \sin^{-1}\left(\frac{b}{a}\right) - 0\right]$



$\sin\theta = b/a$

$\therefore \cos\theta = \frac{\sqrt{a^2-b^2}}{a}$

$\frac{1}{2}b\sqrt{a^2-b^2} + \frac{1}{2}a^2\sin^{-1}\left(\frac{b}{a}\right)$

OLMA $= \frac{1}{2}$ area of $\frac{1}{4}$ circle

$= \frac{\pi a^2}{8}$

ating + multiplying by $\frac{2}{a^2}$:

$\sqrt{a^2-b^2} + \sin^{-1}\left(\frac{b}{a}\right) = \frac{\pi}{4}$

(ii) Letting

$b\sqrt{1-b^2} + \sin^{-1}b = \pi/4$

if $\sin^{-1}b = \theta$

Then $\sin\theta =$

$\cos\theta = \sqrt{1}$

$\therefore \sin\theta\cos\theta + \theta = \pi/4$

$\therefore 2\sin\theta\cos\theta + 2\theta = \pi/2$

$\therefore \sin 2\theta = \pi/2 - 2\theta$

(iii) Various answers: e.g. graphically or Newton's Method

(b) $y^2 = (1-x)^2(4-x)$

$\therefore 2yy' = (1-x)^2(-1) + (4-x)(-2(1-x)$

$= (1-x)(3x-9)$

$\therefore y' = \frac{3(1-x)(x-3)}{y}$

[Note y' is undefined at $x=1$.

When $x = 3$ $y' = 0$

when $x = 3$ $y^2 = 4$

$\therefore y = \pm 2$

When $y = 2$

x	2	3	4
y'	+	0	−

MAX

When $y = -2$

x	2	3	4
y'	−	0	+

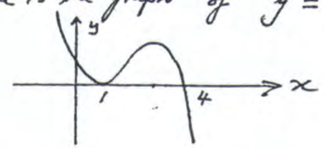
MIN

$\therefore$ Two St Pts

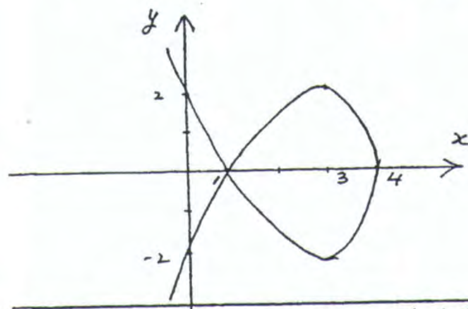
(3, 2) is a max.

(3, −2) is a min.

Here is the graph of $y = (1-x)^2(4-x)$



$\Rightarrow y = \pm\sqrt{(1-x)^2(4-x)}$



(c) It is irrelevant who gets the 2 cards

(i) # ways someone can be dealt 2 cards $= {}^{10}C_2$

ways someone can be dealt 2 odd cards $= {}^{5}C_2$

$\therefore$ Prob $= \frac{{}^5C_2}{{}^{10}C_2} = \frac{2}{9}$

(ii) Prob (2 even cards) = Prob (2 odd) $= \frac{2}{9}$

$\therefore$ Prob (1 of each) $= 1 - 2\times\frac{2}{9} = \frac{5}{9}$

Question 8:

(a) (i) $xy = c^2$ $\therefore xy' + y = 0$ $\therefore$

$\therefore y' = -\frac{y}{x}$ $\therefore m_\perp = \frac{x}{y}$

At P $m_\perp = \frac{cp}{c/p} = p^2$

Equation of normal:

$y - \frac{c}{p} = p^2(x - cp)$

or $p^3x - py = c(p^4 - 1)$ on rearranging.

(ii) At G $y = x$

$\therefore p^3x - px = c(p^4-1)$

$x(p^2-1)p = c(p^2-1)(p^2+1)$

$p^2 \neq 1)$ $\therefore x = \frac{c(p^2+1)}{p} = cp + \frac{c}{p}$

dist PG $= \sqrt{\left(cp + \frac{c}{p} - cp\right)^2 + \left(cp + \frac{c}{p} - \frac{c}{p}\right)^2}$

$= \sqrt{\left(\frac{c}{p}\right)^2 + (cp)^2}$

$= c\sqrt{p^2 + \frac{1}{p^2}}$

Find min value of $p^2 + \frac{1}{p^2}$

Let $y = p^2 + p^{-2}$ $y' = 2p - 2p^{-3}$

$= 0$ when $p = \frac{1}{p^3}$ $\therefore p^4 = 1$

$\therefore p = \pm 1$

When $p = \pm 1$ $y = 1 + 1 = 2$

But $p \neq \pm 1$ $\therefore y > 2$

Hence distance $> c\sqrt{2}$.

[It is interesting to note that when $p = 1$ Then $P = (c, c)$ $\therefore$ P lies on the line $y = x$ $\therefore y = x$ is the normal].

(b) (i) $\angle A = \angle D$ angles in same segment

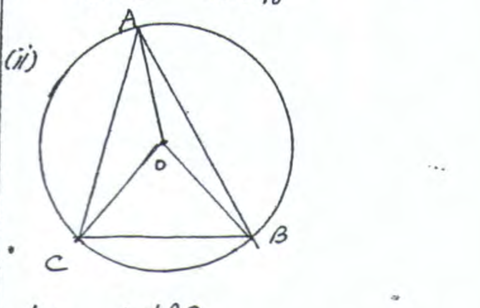
$\therefore \frac{a}{\sin A} = \frac{a}{\sin D} = \frac{2r}{\sin\angle DBC}$

using sine rule in Δ BCD

$\angle DBC = 90°$ ($\angle$ in semicircle)

$\therefore \frac{a}{\sin A} = \frac{2r}{\sin 90°} = 2r$

(ii)



Area Δ ABC

$= \frac{1}{2}r^2\sin\angle BOC + \frac{1}{2}r^2\sin\angle AOC + \frac{1}{2}r^2\sin\angle AOB$

But $\angle BOC = 2\angle A$ ($\angle$ at centre $= 2\angle$ at circum.)

etc

$\therefore \Delta ABC = \frac{1}{2}r^2(\sin 2A + \sin 2B + \sin 2C$