QUESTION 1 **MARKS**

(a) Find $\int \frac{dx}{\sqrt{9-4x^2}}$. 1

(b) (i) Find real constants A, B and C such that 3

$$\frac{x^2+5x+2}{(x^2+1)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1}.$$

(ii) Hence find $\int \frac{x^2+5x+2}{(x^2+1)(x+1)}\,dx$. 2

(c) Evaluate $\int_1^5 x\sqrt{2x-1}\,dx$. 3

(d) Evaluate $\int_0^1 x^5 e^{x^3}\,dx$. 3

(e) (i) Simplify $\sin(A-B)+\sin(A+B)$. 1

(ii) Hence find $\int \sin 5x \cos 3x\,dx$. 2

QUESTION 2 **BEGIN A NEW PAGE** **MARKS**

(a) Let $z = \frac{2-4i}{1+i}$.

(i) Find $\bar{z}$, giving your answer in the form $a+bi$, where a and b are real. 2

(ii) Find iz. 1

(b) Find a and b if $(a+ib)^2 = 3-4i$, where a and b are real and $a>0$. 2

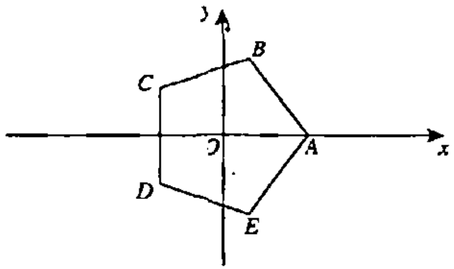
(c) Consider the region defined by $|z-4i| \le 3$.

(i) Sketch the region. 1

(ii) Determine the maximum value of $|z|$. 1

(iii) Determine the maximum value of $\arg z$, where $-\pi < \arg z \le \pi$. 2

(d)



In the diagram above, the complex numbers z_0, z_1, z_2, z_3, z_4 are represented by the vertices of a regular polygon with centre O and vertices A, B, C, D, E respectively.

Given that $z_0 = 2$:

(i) Express z_2 in modulus-argument form. 2

(ii) Find the value of $z_2^{\,5}$. 2

(iii) Show that the perimeter of the pentagon is $20\sin\frac{\pi}{5}$. 2

QUESTION 3 BEGIN A NEW PAGE

MARKS

(a) Let α, β and γ be the roots of

$$x^3 - 7x^2 + 18x - 7 = 0$$

(i) Find a cubic equation that has roots, $1+\alpha^2$, $1+\beta^2$ and $1+\gamma^2$. **2**

(ii) Hence or otherwise, find the value of $(1+\alpha^2)(1+\beta^2)(1+\gamma^2)$. **1**

(b) (i) The polynomial equation $p(x) = 0$ has a root α of multiplicity 3. Show that α is a root of $p'(x) = 0$ and is of multiplicity 2. **2**

(ii) The polynomial $q(x) = x^6 + ax^5 + bx^4 - x^2 - 2x - 1$ has a quadratic factor of $x^2 + 2x + 1$. Find a and b. **2**

(iii) Consider the polynomial **3**

$$r'(x) = 1 + x + \frac{x^2}{2!} + \ldots\ldots + \frac{x^n}{n!} \quad \text{where } r(0) = 1.$$

Show that $r(x)$ has no double roots.

(c) The acceleration of a particle moving in a straight line, starting from a position 2 metres on the positive side of the origin, with a velocity of $1\cdot5\ ms^{-1}$ is given by

$$\frac{dv}{dt} = \frac{9 - x^2}{x^4}.$$

(i) Show that the velocity in ms^{-1} of the particle can be expressed as **3**

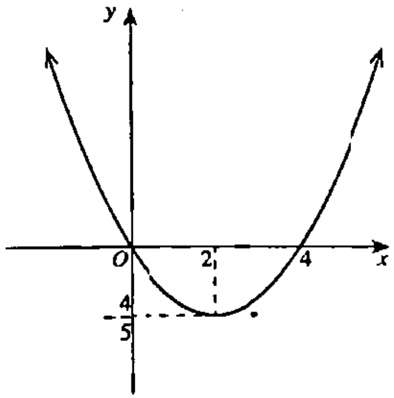
$$v = \frac{\sqrt{2x(x^3 + x^2 - 3)}}{x^2}.$$

(ii) Describe the behaviour of the velocity of the particle after it passes $x = 3$. **2**

QUESTION 4 BEGIN A NEW PAGE

MARKS

(a)



The sketch above shows the parabolic curve $y = f(x)$ where

$$f(x) = \frac{x^2 - 4x}{5}.$$

Without the use of calculus, draw sketches of the following, showing intercepts, asymptotes and turning points:

(i) $y = |f(x)|$, **1**

(ii) $y = \frac{1}{f(x)}$, **2**

(iii) $y = \frac{x}{5}|x - 4|$, **2**

(iv) $y = \tan^{-1}(f(x))$. **2**

Question 4 continued on page 5

(b) The circle $x^2 + y^2 = 4$ is revolved about the y-axis to generate a solid sphere. A cylindrical hole whose diameter is 2 units and whose axis is the y-axis is then removed from the sphere, leaving a solid S. 4

Figure 1 below shows a three-dimensional perspective and Figure 2 shows a cross-sectional view.

Using the method of cylindrical shells, find the volume of S.

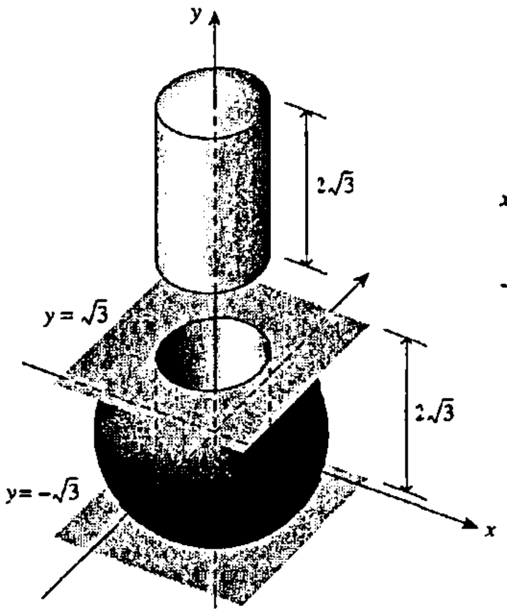


Figure 1

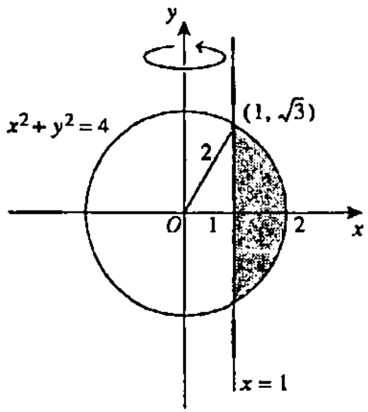


Figure 2

(c) The length of an arc joining $P(a,c)$ and $Q(b,d)$ on a smooth, continuous curve $y=f(x)$ is given by

$$\text{arc length} = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

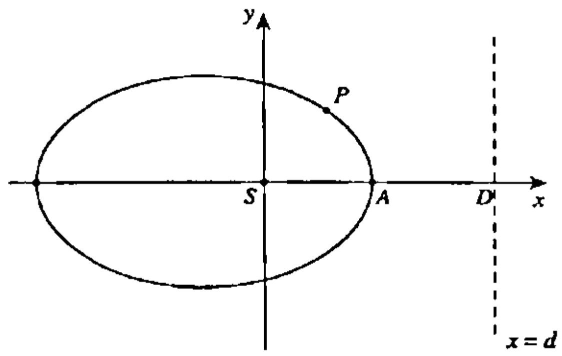
Consider the curve defined by $y = \frac{x^2}{4} - \frac{\ln x}{2}$.

(i) Show that $1+\left(\frac{dy}{dx}\right)^2 = \frac{1}{4}\left(x+\frac{1}{x}\right)^2$. 2

(ii) Find the length of the arc between $x=1$ and $x=e$. 2

End of Question 4

(a)



Consider the ellipse sketched above of eccentricity e with one focus S at the origin and its corresponding directrix at $x=d$.

(i) If P corresponds to the complex number z, where $z = r(\cos\theta + i\sin\theta)$, use the focus-directrix definition of an ellipse to show that the ellipse can be expressed as 2

$$r = \frac{ed}{1+e\cos\theta}.$$

(ii) Hence draw the ellipse represented by 3

$$r = \frac{33}{5+3\cos\theta}$$

showing the coordinates of the points A and D.
[There is no need to find the coordinates of any other point, or to write the Cartesian equation of the ellipse.]

(b) Consider the curve $x^2 - xy + y^2 = 3$.

(i) Show that $\frac{dy}{dx} = \frac{2x-y}{x-2y}$. 2

(ii) Hence find the two stationary points on the curve. 2

(iii) Find the values of x where there are vertical tangents. 1

Question 5 continued on page 7

(c) Consider the complex number $z = \cos\theta + i\sin\theta$.

(i) Using de Moivre's theorem, show that $z^n + \frac{1}{z^n} = 2\cos n\theta$, for any integer n. **1**

(ii) Hence or otherwise express $\left(z + \frac{1}{z}\right)^6$ in the form $A\cos 6\theta + B\cos 4\theta + C\cos 2\theta + D$, where A, B, C and D are real constants. **2**

(iii) Hence evaluate $\int_0^{\frac{\pi}{4}} \cos^6\theta \, d\theta$. **2**

End of Question 5

(a) (i) On the same set of axes, sketch the graphs of **3**

$$y = \cos^{-1}\left(\frac{x-2}{2}\right) \text{ and } y = \frac{\pi}{2} + \tan^{-1}(x-2)$$

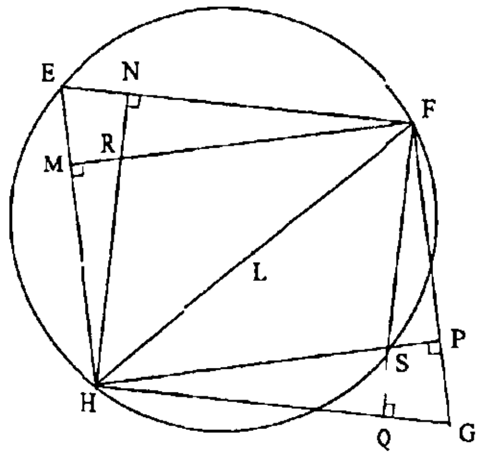
(ii) From your graph, or otherwise, solve the inequation **1**

$$\cos^{-1}\left(\frac{x-2}{2}\right) - \tan^{-1}(x-2) \le \frac{\pi}{2}$$

(b) Find the smallest positive integer p such that **4**

$$\left(\frac{\sqrt{3}+i}{\sqrt{3}-i}\right)^p = \frac{1}{2}(-1 + i\sqrt{3})$$

(c) The vertices E, F and H of the parallelogram EFGH lie on the circle.
L is the midpoint of the diagonal FH.
R is the point of intersection of the perpendicular heights HN and FM of the triangle EFH.
S is the point of intersection of the perpendicular heights HP and FQ of the triangle FGH.



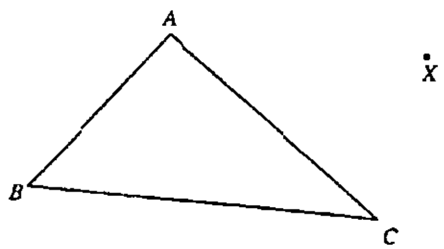
i) Prove that point S lies on the circle. **2**

ii) Prove that the points R, L and S are collinear. **4**

iii) Show that the hexagon MNFPQH is cyclic. **1**

QUESTION 7 BEGIN A NEW PAGE MARKS

(a) 2



The diagram above shows a point X outside a triangle ABC.

Show that $AX + BX + CX > \dfrac{AB + BC + CA}{2}$.

(b) (i) Show that the normal at the point $P\left(cp, \dfrac{c}{p}\right)$ to the rectangular hyperbola $xy = c^2$ is given by 2

$$p^3x - py = c(p^4 - 1).$$

(ii) If this normal meets the hyperbola again at $Q\left(cq, \dfrac{c}{q}\right)$, show that 2

$$p^3q = -1.$$

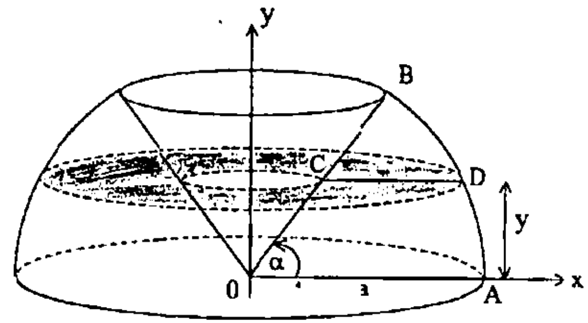
(iii) Hence find the area of the triangle PQR, where R is the point of intersection of the tangent at P with the y-axis. 2

You may assume that the equation of the tangent is given by $x + p^2y = 2cp$.

(iv) What is the value of p that produces a triangle of minimum area? 2

Question 7 continued on page 10

(c) The sector OAB with an angle α at the center (0,0) is rotated about the y axis to form a solid. When the sector is rotated, the line segment CD at height y sweeps out an annulus as shown in the diagram below.



(i) Show that the area of the annulus is 2

$$\pi(a^2 - y^2 \operatorname{cosec}^2\alpha).$$

(ii) Find the volume of the solid. 3

End of Question 7

QUESTION 8 **BEGIN A NEW PAGE** **MARKS**

(a) Let x, y, z and w be positive real numbers.

(i) Prove that $\frac{x}{y}+\frac{y}{x}\geq 2$. **2**

(ii) Deduce that $\frac{x+y+z}{w}+\frac{w+y+z}{x}+\frac{w+x+z}{y}+\frac{w+x+y}{z}\geq 12$. **2**

(iii) Hence prove that if $x+y+z+w=1$, then $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{w}\geq 16$. **2**

(b) Let $J_n=\int_0^1 x^n e^{-x}\,dx$, where $n\geq 0$.

(i) Show that $J_0=1-\frac{1}{e}$. **1**

(ii) Show that $J_n=nJ_{n-1}-\frac{1}{e}$, for $n\geq 1$. **2**

(iii) Show that $J_n\to 0$ as $n\to\infty$. **1**

(iv) Deduce by the principle of mathematical induction that for all $n\geq 0$, **4**

$$J_n=n!-\frac{n!}{e}\sum_{r=0}^{n}\frac{1}{r!}.$$

(v) Conclude that $e=\lim_{n\to\infty}\left(\sum_{r=0}^{n}\frac{1}{r!}\right)$. **1**

End of paper

CTHS 4U Trial 2002

Q1

a) $\int \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{\left(\frac{3}{2}\right)^2 - x^2}}$

$= \frac{1}{2}\sin^{-1}\frac{2x}{3} + C$ $\boxed{1}$

b)(i) $\frac{x^2+5x+2}{(x^2+1)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1}$

$= \frac{(Ax+B)(x+1)+C(x^2+1)}{(x^2+1)(x+1)}$

$\therefore (Ax+B)(x+1) + C(x^2+1) = x^2+5x+2 \quad (1)$ $\boxed{1}$

Put $x=-1$ $\quad 2C = -2$

$C = -1$

Put $x = 0$ $\quad B + C = 2$

$\therefore B = 3$

Equate coeff of x^2

$A + C = 1$

$\therefore A = 2$

$\boxed{2}$ −1 for each mistake

(ii) $\therefore \int \frac{x^2+5x+2}{(x^2+1)(x+1)}dx = \int \frac{2x+3}{x^2+1} - \frac{1}{x+1}\,dx$

$= \int \frac{2x\,dx}{x^2+1} + \int \frac{3}{x^2+1}dx - \int \frac{1}{x+1}dx$ $\boxed{1}$

$= \ln(x^2+1) + 3\tan^{-1}x - \ln(x+1) + C$

$= \ln\frac{x^2+1}{x+1} + 3\tan^{-1}x + C$ $\boxed{1}$

(c) Put $u = \sqrt{2x-1}$

$\therefore u^2 = 2x-1$

$dx = u\,du$

when $x = 5$ $\quad u = 3$

when $x = 1$ $\quad u = 1$

$\boxed{1}$

$\therefore \int_1^5 x\sqrt{2x-1}\,dx = \int_1^3 \frac{u^2+1}{2}\cdot u \cdot u\,du$

$= \int_1^3 \frac{u^4+u^2}{2}\,du$

$= \left[\frac{u^5}{10} + \frac{u^3}{6}\right]_1^3$ $\boxed{1}$

$= \frac{3^5}{10} + \frac{3^3}{6} - \frac{1}{10} - \frac{1}{6}$

$= \frac{428}{15}$ or $28\frac{8}{15}$ $\boxed{1}$

d) $\int_0^1 x^5 e^{x^3}dx$

$= \int_0^1 x^3\, d\left(\frac{e^{x^3}}{3}\right)$ $\boxed{1}$

$= \left[\frac{1}{3}x^3 e^{x^3}\right]_0^1 - \frac{1}{3}\int_0^1 e^{x^3}d(x^3)$ $\boxed{1}$

$= \frac{1}{3}e - \int_0^1 x^2 e^{x^3}dx$

$= \frac{1}{3}e - \left[\frac{1}{3}e^{x^3}\right]_0^1$

$= \frac{1}{3}e - \left(\frac{1}{3}e - \frac{1}{3}\right)$

$= \frac{1}{3}$ $\boxed{1}$

e)(i) $\sin(A+B) + \sin(A-B)$

$= 2\sin A\cos B$ $\boxed{1}$

(ii) $\int \sin 5x \cos 3x\,dx$

$= \frac{1}{2}\int (\sin 8x + \sin 2x)\,dx$ $\boxed{1}$

$= -\frac{1}{16}\cos 8x - \frac{1}{4}\cos 2x + C$ $\boxed{1}$

Q2

a) $z = \frac{2-4i}{1+i}$

(i) $z = \frac{2-4i}{1+i} \times \frac{1-i}{1-i} = \frac{(2-4)-6i}{2}$

$= -1-3i$ $\boxed{1}$

$\therefore \bar{z} = -1+3i$ $\boxed{1}$

(ii) $iz = i(-1-3i)$

$= 3-i$ $\boxed{1}$

(b) $(a+ib)^2 = 3-4i$

$a^2-b^2+2abi = 3-4i$

$\therefore a^2-b^2=3$ (1) $\boxed{1}$

$2ab=-4$

$\therefore b = -\frac{2}{a}$ (2)

Put (2) into (1)

$a^2 - \frac{4}{a^2} = 3$

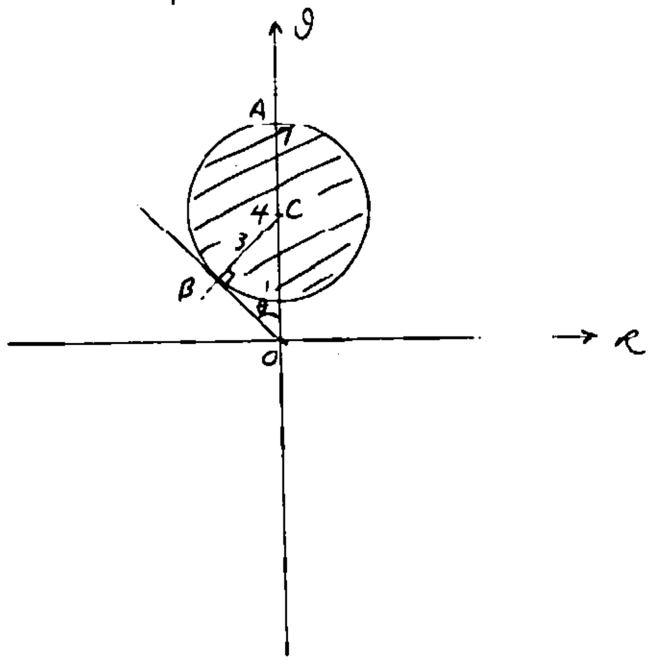
$\therefore a^4-3a^2-4=0$

$(a^2+1)(a^2-4)=0$

$\therefore a=2$ since $a>0$ $\boxed{1}$

$b=-1$

c) (i)



$\boxed{1}$

(ii) Max $|z|$ occurs at A

max $|z| = 7$ $\boxed{1}$

(iii) In $\triangle OBC$

$\sin\theta = \frac{3}{4}$ $\boxed{1}$

$\therefore \theta = \sin^{-1}(\frac{3}{4})$

$\therefore$ max value of $\arg z = \frac{\pi}{2} + \sin^{-1}(\frac{3}{4})$ $\boxed{1}$

d) (i) $\angle COA = \frac{2\pi}{5}$ $\boxed{1}$

$\therefore z_2 = 2\,\text{cis}\frac{2\pi}{5}$ $\boxed{1}$

(ii) $z_2^5 = 2^5(\text{cis}\frac{2\pi}{5})^5$

$= 2^5 \text{cis}\, 2\pi$ $\boxed{1}$

$= 2^5$

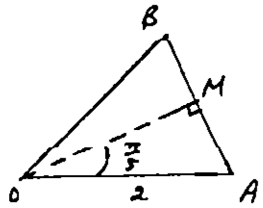
$= 32$ $\boxed{1}$

(iii) $\angle AOM$

$= \frac{1}{2}\angle AOB$

$= \frac{\pi}{5}$

$AM = 2\sin\frac{\pi}{5}$ $\boxed{1}$



$\therefore$ Perimeter

$= 10\,AM$

$= 20\sin\frac{\pi}{5}$ $\boxed{1}$

Q3. (a) (i) Put $y = 1+x^2$ $\therefore$ $x = \sqrt{y-1}$

$\therefore$ The transformed equation is

$(\sqrt{y-1})^3 - 7(y-1) + 18\sqrt{y-1} - 7 = 0$ $\boxed{1}$

$\sqrt{y-1}\,(y-1+18) = 7(y-1+1)$

$\sqrt{y-1}\,(y+17) = 7y$

$\therefore (y-1)(y+17)^2 = 49y^2$

$(y-1)(y^2+34y+289) = 49y^2$

$y^3-16y^2+255y-289=0$

$x^3-16x^2+255x-289=0$ (1) $\boxed{1}$

(ii) $(1+\alpha^2)(1+\beta^2)(1+\gamma^2)$

= product of roots of (1)

$= 289$ $\boxed{1}$

b (i) Let $P(x) = (x-\alpha)^3 Q(x)$ where $Q(x) \neq 0$ $\boxed{1}$

$\therefore P'(x) = 3(x-\alpha)^2 Q(x) + (x-\alpha)^3 Q'(x)$

$= (x-\alpha)^2[3Q(x) + (x-\alpha)Q'(x)]$

$\therefore$ $x=\alpha$ is a double root of $P'(x)$ $\boxed{1}$

(ii) $x^2+2x+1 = (x+1)^2$

$\therefore$ $x=-1$ is a double root of $q(x)$.

Hence by (i) $x=-1$ is a root of $q'(x)$.

$\therefore$ $q(-1) = q'(-1) = 0$ $\boxed{1}$

$1-a+b-1+2-1=0$

$a-b=1$ (1)

$-6+5a-4b+2-2=0$

$5a-4b=6$ (2)

(1)×4 $\quad 4a-4b=4$ (3)

(2) − (3) $\quad a=2$

Put into (1) $\quad b=1$ $\boxed{1}$

(iii) $r'(x) = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots+\frac{x^n}{n!}$

$r(x) = x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\dots+\frac{x^{n+1}}{(n+1)!}+C$

$\because r(0)=1$

$\therefore 1 = C$

$\therefore r(x) = 1+x+\frac{x^2}{2!}+\dots+\frac{x^n}{n!}+\frac{x^{n+1}}{(n+1)!}$ $\boxed{1}$

suppose $x=\alpha$ is a double root of $r(x)=0$

then $r(\alpha) = r'(\alpha) = 0$ $\boxed{1}$

ie $1+\alpha+\frac{\alpha^2}{2!}+\frac{\alpha^3}{3!}+\dots+\frac{\alpha^n}{n!}=0$ (1)

$1+\alpha+\frac{\alpha^2}{2!}+\frac{\alpha^3}{3!}+\dots+\frac{\alpha^n}{n!}+\frac{\alpha^{n+1}}{(n+1)!}=0$ (2)

(2) − (1) $\quad \frac{\alpha^{n+1}}{(n+1)!} = 0$

$\therefore \alpha = 0$

but $r(0)=1$ and $r'(0)=1$

$\therefore$ $\alpha=0$ is not a double root. $\boxed{1}$

Hence $r(x)=0$ does not have any double root.

c (i) $\frac{dv}{dt} = \frac{d}{dx}\left(\frac{v^2}{2}\right) = \frac{9-x^2}{x^4}$

$\therefore \int d\left(\frac{v^2}{2}\right) = \int \left(\frac{9}{x^4} - \frac{1}{x^2}\right)dx$ $\boxed{1}$

$\frac{v^2}{2} = -\frac{3}{x^3}+\frac{1}{x}+C$

since $v=1.5$ when $x=2$

$\frac{1.5^2}{2} = -\frac{3}{2^3}+\frac{1}{2}+C$

$\therefore C=1$

$\therefore v^2 = 2\left(1+\frac{1}{x}-\frac{3}{x^3}\right)$ $\boxed{1}$

$= \frac{2(x^4+x^3-3x)}{x^4}$

$\therefore v = \frac{\sqrt{2x(x^3+x^2-3)}}{x^2}$ $\boxed{1}$

(iii) $a = \frac{9-x^2}{x^4}$

$\therefore a<0$ for all $x<-3$ or $x>3$

$\therefore$ The particle will slow down after it passes $x=3$ towards the right. $\boxed{1}$

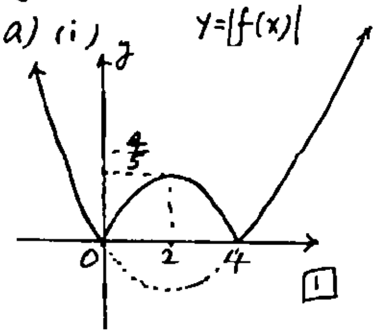
and $\lim_{x\to\infty} v = \lim_{x\to\infty} \sqrt{\frac{2(x^4+x^3-3x)}{x^4}}$

$= \sqrt{2}$

$\therefore$ The speed of the particle will eventually be $\sqrt{2}\ ms^{-1}$, ie it will never stop. $\boxed{1}$

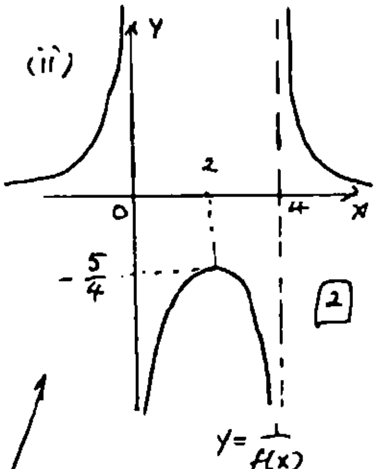
Q4

a) (i)



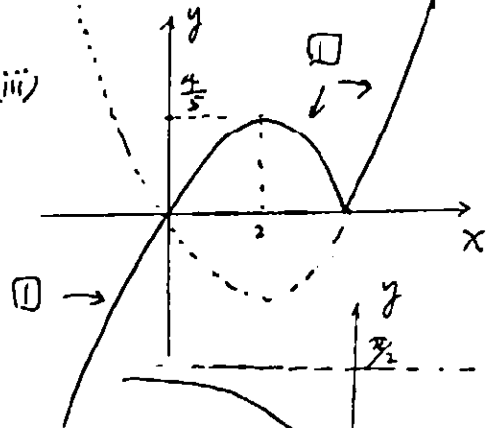
$\boxed{1}$

(ii)



$\boxed{2}$

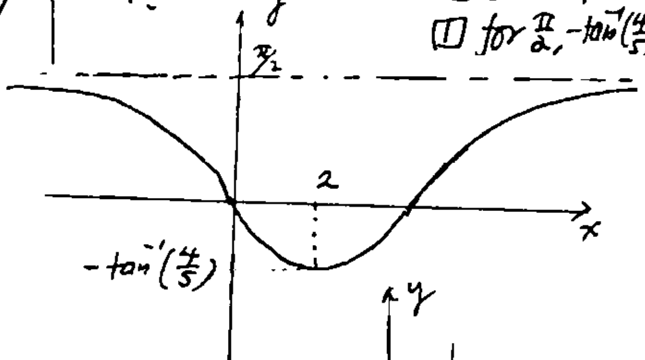
(iii)



$\boxed{1}$ for slope

$\boxed{1}$ for $\frac{\pi}{2}, -\tan^{-1}(\frac{4}{5})$

(iv)

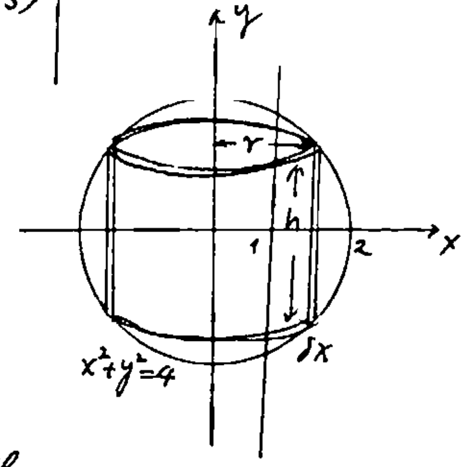


b) $r = x$

and $1 \leq r \leq 2$

$\left(\frac{h}{2}\right)^2 = 4 - r^2$

$\therefore \frac{h}{2} = \sqrt{4-r^2}$



$\therefore$ Volume of the cylindrical shell

$$\delta V = 2\pi r h \, \delta x$$
$$= 4\pi x\sqrt{4-x^2}\,\delta x \qquad \boxed{1}$$

$\therefore$ Volume of $S = 4\pi \int_1^2 x\sqrt{4-x^2}\, dx \qquad \boxed{1}$

Put $u = 4-x^2 \qquad \therefore du = -2x\,dx$

when $x=2, \ u=0$

when $x=1, \ u=3$

$$\therefore V = -\frac{4\pi}{2}\int_3^0 \sqrt{u}\, du \qquad \boxed{1}$$
$$= \left(\frac{2}{3}\right)2\pi\left[u^{\frac{3}{2}}\right]_0^3$$
$$= 4\pi\sqrt{3} \ \text{unit}^3 \qquad \boxed{1}$$

c) $y = \frac{x^2}{4} - \frac{\ln x}{2}$

(i) $\frac{dy}{dx} = \frac{x}{2} - \frac{1}{2x} \qquad \boxed{1}$

$$\therefore 1+\left(\frac{dy}{dx}\right)^2 = 1+\left(\frac{x}{2}-\frac{1}{2x}\right)^2$$
$$= 1+\frac{x^2}{4}-\frac{1}{2}+\frac{1}{4x^2}$$
$$= \frac{x^2}{4}+\frac{1}{2}+\frac{1}{4x^2}$$
$$= \frac{1}{4}\left(x^2+2+\frac{1}{x^2}\right)$$
$$= \frac{1}{4}\left(x+\frac{1}{x}\right)^2 \qquad \boxed{1}$$

(ii) Arc length

$$= \int_1^e \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$
$$= \int_1^e \sqrt{\frac{1}{4}\left(x+\frac{1}{x}\right)^2}\, dx$$
$$= \int_1^e \frac{1}{2}\left(x+\frac{1}{x}\right)dx \qquad \boxed{1}$$
$$= \frac{1}{2}\left[\frac{x^2}{2}+\ln x\right]_1^e$$
$$= \frac{1}{2}\left(\frac{e^2}{2}+\ln e-\frac{1}{2}\right)$$
$$= \frac{1}{4}(e^2+1) \ \text{units} \qquad \boxed{1}$$

Q5

(a)(i) Coordinates of P is $(r\cos\theta, r\sin\theta)$

$\therefore \perp$ distance from P to directrix $x = d$ is

$PN = d - r\cos\theta$

$PS = r$

$\frac{PS}{PN} = e$

$\therefore PS = e\,PN$

$r = e(d - r\cos\theta)$ $\boxed{1}$

$= ed - er\cos\theta$

$r(1 + e\cos\theta) = ed$

$\therefore r = \frac{ed}{1+e\cos\theta}$ $\boxed{1}$

(ii) By comparing $r = \frac{33}{5+3\cos\theta} = \frac{33/5}{1+\frac{3}{5}\cos\theta}$ $\boxed{1}$

to $r = \frac{ed}{1+e\cos\theta}$

$ed = \frac{33}{5}$

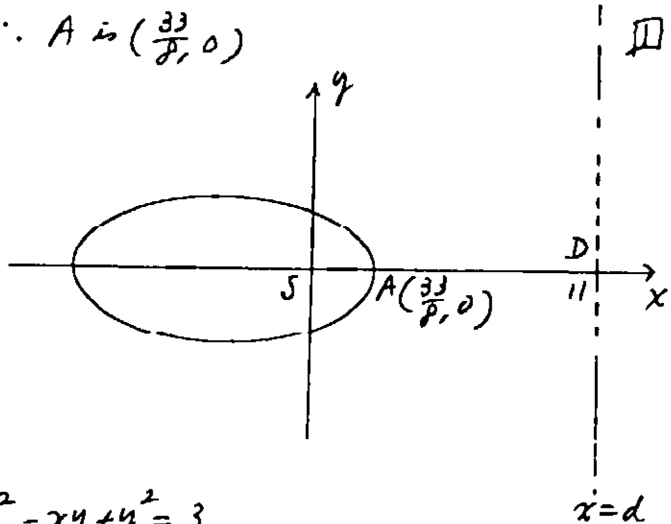
$e = \frac{3}{5}$

$\therefore d = 11$ $\boxed{1}$

At A on the ellipse, $\theta = 0$

$r = \frac{33}{5+3} = \frac{33}{8}$

$\therefore A$ is $(\frac{33}{8}, 0)$ $\boxed{1}$



b) (i) $x^2 - xy + y^2 = 3$

$2x - y - x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$ $\boxed{1}$

$2x - y = (x - 2y)\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{2x-y}{x-2y}$ $\boxed{1}$

(ii) At stationary points, $\frac{dy}{dx} = 0$

$\therefore 2x - y = 0$

$y = 2x$ (1) $\boxed{1}$

Put $y = 2x$ into $x^2 - xy + y^2 = 3$ (2)

$x^2 - x(2x) + (2x)^2 = 3$

$x^2(1 - 2 + 4) = 3$

$x^2 = 1$

$x = \pm 1$

$\therefore$ the two stationary pts are

$(1, 2)$ and $(-1, -2)$ $\boxed{1}$

(iii) When the tangent is vertical,

$x - 2y = 0$

$x = 2y$

Put $y = \frac{x}{2}$ into (2)

$x^2 - x(\frac{x}{2}) + (\frac{x}{2})^2 = 3$

$4x^2 - 2x^2 + x^2 = 12$

$x^2 = 4$

$x = \pm 2$ $\boxed{1}$

c)(i) $z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$\frac{1}{z^n} = (\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$

$\therefore z^n + \frac{1}{z^n} = 2\cos n\theta$ $\boxed{1}$

(ii) $(z + \frac{1}{z})^6 = z^6 + 6z^4 + 15z^2 + 20 + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6}$ $\boxed{1}$

$= (z^6 + \frac{1}{z^6}) + 6(z^4 + \frac{1}{z^4}) + 15(z^2 + \frac{1}{z^2}) + 20$

$= 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$ $\boxed{1}$

(iii) From (ii) $64\cos^6\theta = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$

$\therefore \int_0^{\frac{\pi}{4}} \cos^6\theta\, d\theta = \frac{1}{32}\int_0^{\frac{\pi}{4}} (\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10)\, d\theta$

$= \frac{1}{32}\left[\frac{\sin 6\theta}{6} + \frac{3}{2}\sin 4\theta + \frac{15}{2}\sin 2\theta + 10\theta\right]_0^{\frac{\pi}{4}}$ $\boxed{1}$

$= \frac{1}{32}\left[\frac{1}{6}\sin\frac{3\pi}{2} + \frac{3}{2}\sin\pi + \frac{15}{2}\sin\frac{\pi}{2} + \frac{5\pi}{2}\right]$

$= \frac{1}{32}(-\frac{1}{6} + \frac{15}{2} + \frac{5\pi}{2})$

$= \frac{1}{32}(\frac{22}{3} + \frac{5\pi}{2})$

$= \frac{44 + 15\pi}{192}$ $\boxed{1}$

Q6.

a(i) For $y = \cos^{-1}\left(\frac{x-2}{2}\right)$

Domain $-1 \le \frac{x-2}{2} \le 1$

$\therefore \quad 0 \le x \le 4$

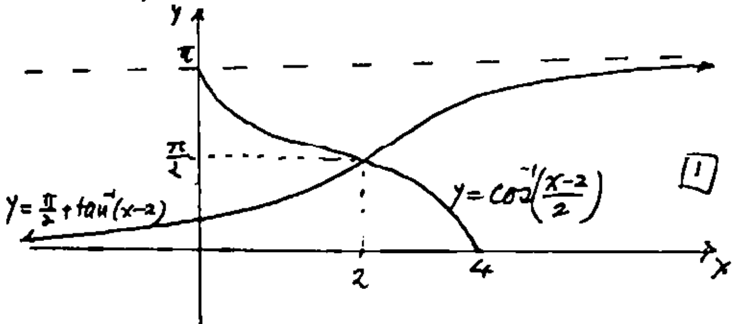
Range $0 \le y \le \pi$ $\boxed{1}$

or showing clearly on the graph

For $y = \frac{\pi}{2} + \tan^{-1}(x-2)$

Domain : all real x

Range : $0 < y < \pi$ $\boxed{1}$



$\boxed{1}$

(ii) Noting that $y = \cos^{-1}\left(\frac{x-2}{2}\right)$ is the translation of $y = \cos^{-1}\frac{x}{2}$ to $x=2$ while $y = \frac{\pi}{2} + \tan^{-1}(x-2)$ is obtained by translating the origin to $(2, \frac{\pi}{2})$

$\therefore$ the intersection of $y = \cos^{-1}\left(\frac{x-2}{2}\right)$ and $y = \frac{\pi}{2} + \tan^{-1}(x-2)$ is $(2, \frac{\pi}{2})$

$\therefore$ solution to $\cos^{-1}\left(\frac{x-2}{2}\right) - \tan^{-1}(x-2) \le \frac{\pi}{2}$

ie $\cos^{-1}\left(\frac{x-2}{2}\right) \le \frac{\pi}{2} + \tan^{-1}(x-2)$

is $x \ge 2$ $\boxed{1}$

(b) $\frac{\sqrt{3}+i}{\sqrt{3}-i} = \frac{(\sqrt{3}+i)(\sqrt{3}+i)}{(\sqrt{3}-i)(\sqrt{3}+i)} = \frac{2+2\sqrt{3}i}{4}$

$= \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$= \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ $\boxed{1}$

$\left(\frac{\sqrt{3}+i}{\sqrt{3}-i}\right)^p = \cos\frac{p\pi}{3} + i\sin\frac{p\pi}{3}$

$\therefore$ If $\left(\frac{\sqrt{3}+i}{\sqrt{3}-i}\right)^p = \frac{1}{2}(-1 + i\sqrt{3})$

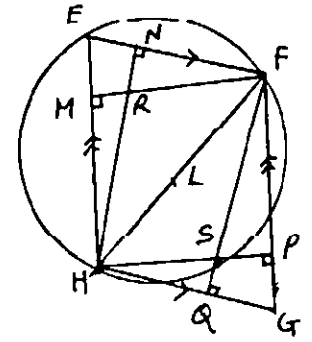
$\cos\frac{p\pi}{3} + i\sin\frac{p\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ $\boxed{1}$

ie $\cos\frac{p\pi}{3} = -\frac{1}{2}$ $\boxed{1}$

$\sin\frac{p\pi}{3} = \frac{\sqrt{3}}{2}$

$\therefore$ The smallest positive integer p is 2 $\boxed{1}$

(c)



(i) In $SPGQ$,

$\angle P + \angle Q = 180°$ ($\angle P = \angle Q = 90°$, given)

$\therefore S, P, G, Q$ are concyclic (opp $\angle$s supplementary)

$\therefore \angle FSP = \angle PGQ$ (ext $\angle$ of cyclic quad) $\boxed{1}$

but $\angle PGQ = \angle FEH$ (opp $\angle$s of //gram)

$\therefore \angle FSP = \angle FEH$

$\therefore E, F, S, H$ are concyclic (ext $\angle$ equals to int opp $\angle$) $\boxed{1}$

$\therefore$ S lies on the circle.

(ii) In $HNFG$

$NF \parallel HQ$ (given)

$\angle NFQ + \angle HQF = 180°$ (co-int $\angle$, $NF \parallel HG$)

$\therefore \angle NFQ = 90°$ ($\angle HQF = 90°$)

$\therefore HNFG$ is a rectangle (all $\angle$s being rt $\angle$s)

$\therefore HR \parallel SF$ (opp sides of rectangle) $\boxed{1}$

Similarly, $HMFP$ is a rectangle

$\therefore RF \parallel HS$ (opp sides of rectangle)

$\therefore HRFQ$ is a //gram. $\boxed{1}$

$\therefore HF, RS$ bisect each other (diagonals of //gram) $\boxed{1}$

L is mid-pt of HF (given)

$\therefore L$ must be mid-pt of RS

$\therefore R, L, S$ are collinear. $\boxed{1}$

(iii) $\angle HMF = \angle HNF$ (rt $\angle$s, given)

$\therefore H, M, N, F$ lie on the semi-circle with HF as diameter.

Similarly, $FPQH$ is a semi-circle on HF as diameter.

$\therefore MNFPQH$ is concyclic. $\boxed{1}$

Q7.

a) Using the triangle inequality

$AX + BX > AB$

$AX + CX > AC$ $\boxed{1}$

$BX + CX > BC$

$\therefore\ 2(AX+BX+CX) > AB+AC+BC$

ie $\quad AX+BX+CX > \dfrac{AB+BC+CA}{2}$ $\boxed{1}$

(b)(i) $\quad xy = c^2$

$y = \dfrac{c^2}{x}$

$\therefore\ \dfrac{dy}{dx} = -\dfrac{c^2}{x^2}$

At P, $\quad \dfrac{dy}{dx} = -\dfrac{c^2}{c^2p^2} = -\dfrac{1}{p^2}$

$\therefore$ gradient of normal at $P = p^2$ $\boxed{1}$

$\therefore$ Equation of normal is

$y - \dfrac{c}{p} = p^2(x - cp)$

$py - c = p^3(x - cp)$

ie $\ p^3x - py = c(p^4 - 1)$ $\boxed{1}$

(ii) If $Q(cq, \frac{c}{q})$ lies on this normal

$p^3(cq) - p(\frac{c}{q}) = c(p^4 - 1)$ $\boxed{1}$

$p^3q^2 - p = p^4q - q$

$p^4q - p^3q^2 + p - q = 0$

$p^3q(p-q) + (p-q) = 0$

$(p-q)(p^3q+1) = 0$

$\therefore\ p^3q + 1 = 0 \quad \because p \neq q$

$\therefore\ p^3q = -1$ $\boxed{1}$

(iii) Equation of tangent at P

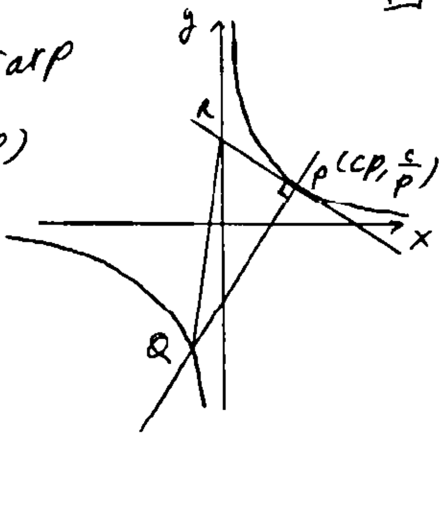
is $\ y - \dfrac{c}{p} = -\dfrac{1}{p^2}(x - cp)$

$p^2y - cp = -x + cp$

ie $\ x + p^2y = 2cp$

$\therefore$ At R, $x = 0$

$y = \dfrac{2c}{p}$



$\therefore R$ is $(0, \frac{2c}{p})$

$\therefore PR = \sqrt{c^2p^2 + \left(\frac{2c}{p} - \frac{c}{p}\right)^2} = \dfrac{c}{p}\sqrt{p^4+1}$

Q is $(cq, \frac{c}{q}) = \left(-\frac{c}{p^3}, -cp^3\right)$ from (ii)

$\therefore\ PQ = \sqrt{c^2\left(p+\frac{1}{p^3}\right)^2 + c^2\left(p^3+\frac{1}{p}\right)^2}$

$= c\sqrt{p^2 + \frac{2}{p^2} + \frac{1}{p^6} + p^6 + 2p^2 + \frac{1}{p^2}}$

$= c\sqrt{p^6 + 3p^2 + \frac{3}{p^2} + \frac{1}{p^6}}$

$= c\left(p^2 + \frac{1}{p^2}\right)^{\frac{3}{2}}$

$\therefore$ Area of ΔPQR

$= \frac{1}{2} PQ \cdot PR$

$= \dfrac{c}{2p}\sqrt{p^4+1} \times c\left(p^2 + \frac{1}{p^2}\right)^{\frac{3}{2}}$

$= \dfrac{c}{2}\sqrt{p^2 + \frac{1}{p^2}} \times c\left(p^2 + \frac{1}{p^2}\right)^{\frac{3}{2}}$

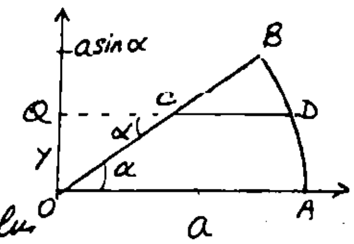
$= \dfrac{c}{2}\left(p^2 + \frac{1}{p^2}\right)^2$ or $\dfrac{c}{p^4}(p^4+1)^2$ unit^2

(iv) Since $\left(p^2 + \frac{1}{p^2}\right) = \left(p - \frac{1}{p}\right)^2 + 2$ $\boxed{1}$

$\therefore\ p^2 + \frac{1}{p^2}$ hence ΔPQR will be minimum when $p - \frac{1}{p} = 0$ ie $p = \pm 1$ $\boxed{1}$

c)(i) $QC = y\cot\alpha$

$\left.\begin{aligned} QD &= \sqrt{OD^2 - y^2} \\ &= \sqrt{a^2 - y^2} \end{aligned}\right\}$ $\boxed{1}$



$\therefore$ Area of the annulus

$= \pi QD^2 - \pi QC^2$

$= \pi\left[(a^2 - y^2) - y^2\cot^2\alpha\right]$

$= \pi\left[a^2 - y^2(1 + \cot^2\alpha)\right]$ $\boxed{1}$

$= \pi(a^2 - y^2\operatorname{cosec}^2\alpha)$

(ii) Volume of the solid

$= \pi\displaystyle\int_0^{a\sin\alpha}(a^2 - y^2\operatorname{cosec}^2\alpha)\,dy$ $\boxed{1}$

$= \pi\left[a^2y - \frac{y^3}{3}\operatorname{cosec}^2\alpha\right]_0^{a\sin\alpha}$ $\boxed{1}$

$= \pi\left[a^3\sin\alpha - \frac{1}{3}a^3\sin^3\alpha\operatorname{cosec}^2\alpha\right]$

$= \dfrac{2\pi}{3}a^3\sin\alpha$ unit^3 $\boxed{1}$

Q8 (a)

(i) $\left(\sqrt{\frac{x}{y}}-\sqrt{\frac{y}{x}}\right)^2 \geq 0$ $\boxed{1}$

$\frac{x}{y}-2+\frac{y}{x} \geq 0$

$\therefore \frac{x}{y}+\frac{y}{x} \geq 2$ $\boxed{1}$

(ii) $\frac{x+y+z}{w}+\frac{w+y+z}{x}+\frac{w+x+z}{y}+\frac{w+x+y}{z}$

$=\left(\frac{x}{w}+\frac{w}{x}\right)+\left(\frac{y}{w}+\frac{w}{y}\right)+\left(\frac{z}{w}+\frac{w}{z}\right)+\left(\frac{y}{x}+\frac{x}{y}\right)$ $\boxed{1}$

$+\left(\frac{z}{x}+\frac{x}{z}\right)+\left(\frac{y}{z}+\frac{z}{y}\right)$

$\geq 2+2+2+2+2+2$

$\boxed{1}$

$\therefore \frac{x+y+z}{w}+\frac{w+y+z}{x}+\frac{w+x+z}{y}+\frac{w+x+y}{z} \geq 12$ (1)

(iii) $\frac{x+y+z}{w}=\frac{w+x+y+z-w}{w}=\frac{1-w}{w}=\frac{1}{w}-1$

Similarly $\frac{w+y+z}{x}=\frac{1}{x}-1$

$\frac{w+x+z}{y}=\frac{1}{y}-1$

$\frac{w+x+y}{z}=\frac{1}{z}-1$ (2)

Substitute (2) into (1)

$\left(\frac{1}{w}-1\right)+\left(\frac{1}{x}-1\right)+\left(\frac{1}{y}-1\right)+\left(\frac{1}{z}-1\right) \geq 12$ $\boxed{1}$

$\therefore \frac{1}{w}+\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq 16$ $\boxed{1}$

(b)(i) $J_0=\int_0^1 e^{-x}dx=\left[-e^{-x}\right]_0^1$

$=1-e^{-1}=1-\frac{1}{e}$ $\boxed{1}$

(ii) $J_n=\int_0^1 x^n e^{-x}dx=\int_0^1 x^n d(-e^{-x})$

$=\left[-x^n e^{-x}\right]_0^1+n\int_0^1 x^{n-1}e^{-x}dx$ $\boxed{1}$

$=-\frac{1}{e}+nJ_{n-1}$

$\therefore J_n=nJ_{n-1}-\frac{1}{e}$ $\boxed{1}$

(iii) Since $0 \leq x^n e^{-x} \leq x^n \quad \forall\, 0 \leq x \leq 1$

$\therefore 0 \leq \int_0^1 x^n e^{-x}dx \leq \int_0^1 x^n dx$

$\therefore 0 \leq \lim_{n\to\infty} J_n \leq \lim_{n\to\infty}\frac{1}{n+1}$ $\boxed{1}$

Hence $\lim_{n\to\infty} J_n = 0$

(iv) when $n=0$

$n!-\frac{n!}{e}\sum_{r=0}^{n}\frac{1}{r!}=0!-\frac{0!}{e}\cdot\frac{1}{0!}$

$=1-\frac{1}{e}$

$=J_0$ from (i)

$\therefore$ it is true for $n=0$ $\boxed{1}$

Assume it is true for $n=k$

ie $J_k=k!-\frac{k!}{e}\sum_{r=0}^{k}\frac{1}{r!}$

then $J_{k+1}=(k+1)J_k-\frac{1}{e}$ from (ii)

$=(k+1)\left[k!-\frac{k!}{e}\sum_{r=0}^{k}\frac{1}{r!}\right]-\frac{1}{e}$ $\boxed{1}$

$=(k+1)!-\frac{(k+1)!}{e}\sum_{r=0}^{k}\frac{1}{r!}-\frac{1}{e}$

$=(k+1)!-\frac{(k+1)!}{e}\left[\sum_{r=0}^{k}\frac{1}{r!}+\frac{1}{(k+1)!}\right]$ $\boxed{1}$

$=(k+1)!-\frac{(k+1)!}{e}\sum_{r=0}^{k+1}\frac{1}{r!}$

$\therefore$ it is true for $n=k+1$ if it is true for $n=k$.

Since it is proved true for $n=0$, $\boxed{1}$

$\therefore$ it will be true for $n=1,2,3,\cdots$

ie true for all integers $n\geq 0$

(v) $J_n=n!-\frac{n!}{e}\sum_{r=0}^{n}\frac{1}{r!}$

$\therefore \frac{J_n}{n!}=1-\frac{1}{e}\sum_{r=0}^{n}\frac{1}{r!}$

Pass the limit $n\to\infty$ on both sides

$0=1-\lim_{n\to\infty}\frac{1}{e}\sum_{r=0}^{n}\frac{1}{r!}$ since $J_n\to 0$ as $n\to\infty$ by (iii)

$\therefore 1-\lim_{n\to\infty}\frac{1}{e}\sum_{r=0}^{n}\frac{1}{r!}=0$

or $\frac{1}{e}\lim_{n\to\infty}\sum_{r=0}^{n}\frac{1}{r!}=1$

ie $\lim_{n\to\infty}\sum_{r=0}^{n}\frac{1}{r!}=e$ $\boxed{1}$