HB.

Name: ____________________

Class: 12MTZ1

Teacher: ____________________

CHERRYBROOK TECHNOLOGY HIGH SCHOOL

2003 AP4

YEAR 12 TRIAL HSC EXAMINATION

MATHEMATICS EXTENSION 2

Time allowed - 3 HOURS
(Plus 5 minutes' reading time)

DIRECTIONS TO CANDIDATES:

- Attempt all questions.
- Each question is to be commenced on a new page clearly marked Question 1, Question 2, etc on the top of the page. **
- Each question is to be returned in a separate bundle.
- All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used.
- A Table of Standard Integrals is provided.

Each page must show your name and your class. *

Question 1

Marks

(a) Find $\int \frac{1}{\sqrt{4-(1+x)^2}}dx$ **2**

(b) Use integration by parts to evaluate **3**

$$\int_0^1 \tan^{-1} x\,dx$$

(c) (i) Find real numbers a, b and c such that **3**

$$\frac{x^2-11}{(3x-1)(x+2)^2} \equiv \frac{a}{3x-1}+\frac{b}{x+2}+\frac{c}{(x+2)^2}\ .$$

(ii) Find $\int \frac{x^2-11}{(3x-1)(x+2)^2}dx$ **2**

(d) Using the substitution $u^2 = 4x-3$, show that **5**

$$\int 8x\sqrt[3]{4x-3}\ dx = \cancel{\frac{1}{8}(4x-3)^{\frac{4}{3}}(32x-15)+c}$$

$$= \frac{3}{56}(4x-3)^{\frac{4}{3}}(16x+9)+C$$

Question 2 **BEGIN A NEW PAGE** **Marks**

(a) If $z = 7 - 3i$ and $w = 29 + 29i$, find $\dfrac{w}{z}$ in simplest form. **3**

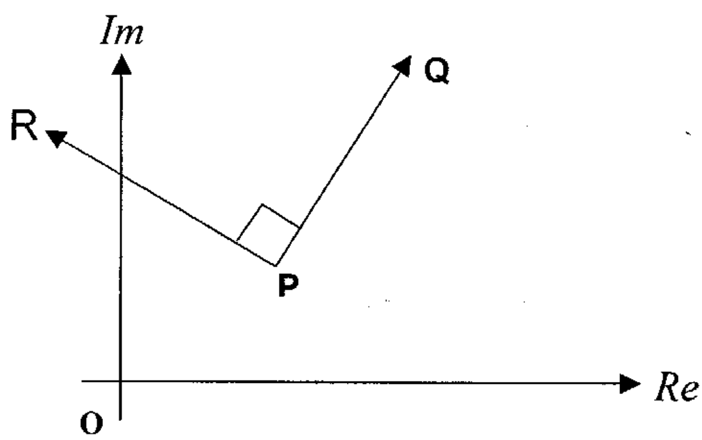
(b) (i) Express $-1 + i\sqrt{3}$ in modulus-argument form. **2**

(ii) Hence find $(-1 + i\sqrt{3})^5$, giving your answer in the form $a + ib$. **3**

(c) Given that $|z| = 1$, show that $z^{-1} = \overline{z}$. **2**

(d) Sketch in the Argand diagram the locus of a complex number z that satisfies $0 \le \arg(z - i) \le \frac{2\pi}{3}$. **2**

(e)



In the above diagram, P represents the complex number $3 + 2i$ and Q represents $7 + 8i$.

(i) What complex number is represented by the vector PQ? **1**

(ii) Suppose that R is the image of Q under an anticlockwise rotation of $\frac{\pi}{2}$ about P. Find the complex number represented by the point R. **2**

Question 3 **BEGIN A NEW PAGE** **Marks**

(a) Consider the product P_n for $n = 3,4,5,\ldots$ where

$$P_n = \frac{3(3-1)}{(3-2)(3+2)} \cdot \frac{4(4-1)}{(4-2)(4+2)} \cdot \frac{5(5-1)}{(5-2)(5+2)} \cdots \frac{n(n-1)}{(n-2)(n+2)}$$

(i) Find the maximum value of P_n and the value(s) of n for which this occurs. **2**

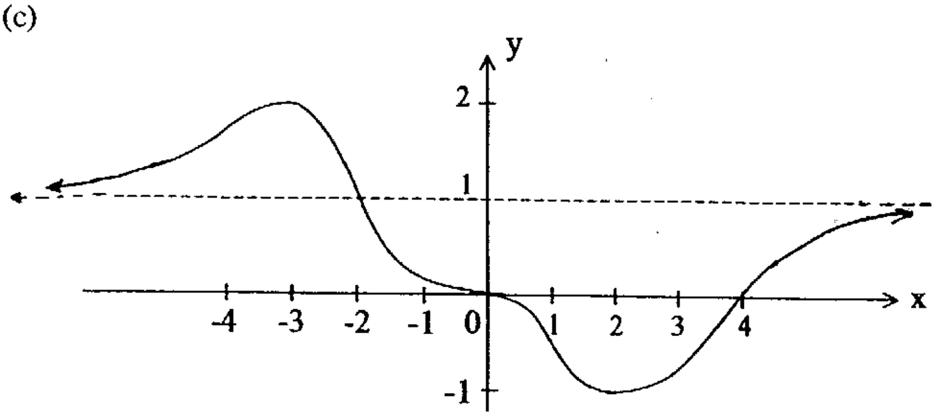
(ii) Show by induction that **2**

$$P_n = \frac{12(n-1)}{(n+1)(n+2)} \text{ for } n = 3,4,5\ldots$$

(b) (i) Find the domain and range of the function $f(x) = \tan^{-1} e^x$. Sketch the curve $y = f(x)$ showing any intercepts on the coordinate axes and the equations of any asymptotes. **3**

(ii) Show that $f'(x) = \frac{1}{2}\sin 2y$ **2**

(c)



The above diagram shows the graph of $y = f(x)$. Sketch on separate diagrams the following curves, indicating clearly any turning points and asymptotes

(i) $y = \dfrac{1}{|f(x)|}$ **2**

(ii) $y = [f(x)]^2$ **2**

(iii) $y = \ln[f(x)]$ **2**

Question 4 **BEGIN A NEW PAGE** **Marks**

(a) The ellipse $\boldsymbol{E}$ has equation $4x^2 + 9y^2 = 16$.

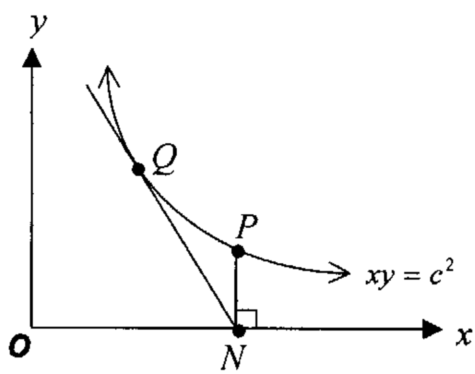
(i) Show that $\boldsymbol{E}$ has eccentricity $\frac{\sqrt{5}}{3}$. **1**

(ii) Find the coordinates of the foci of $\boldsymbol{E}$ and the equations of the directrices of $\boldsymbol{E}$. **2**

(iii) Show that the tangent at the variable point $P(2\cos\theta, \frac{4}{3}\sin\theta)$ on $\boldsymbol{E}$ has gradient $\dfrac{-2\cos\theta}{3\sin\theta}$. **2**

(iv) Hence, or otherwise, find the coordinates of the two points on $\boldsymbol{E}$ at which the gradient of the tangent is $\frac{1}{2}$. **3**

(b)



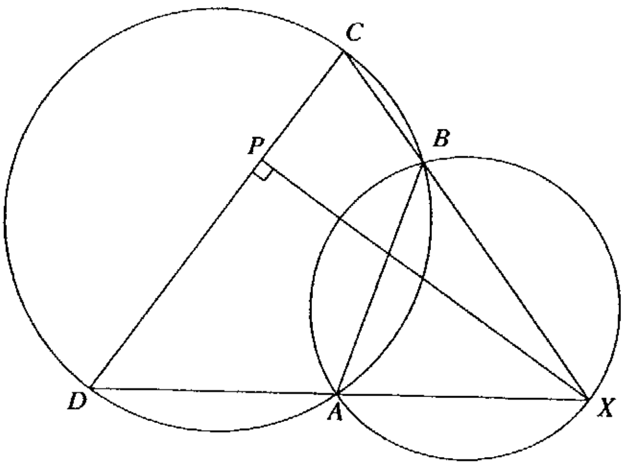
In the diagram above, $P\left(ct_1, \dfrac{c}{t_1}\right)$ and $Q\left(ct_2, \dfrac{c}{t_2}\right)$ (where $t_1, t_2 > 0$) are distinct variable points on the rectangular hyperbola $xy = c^2$. PN is the perpendicular from P to the x-axis and the tangent at Q passes through N.

(i) Show that $t_1 = 2t_2$. **3**

(ii) Find the Cartesian equation of the locus of T, the point of intersection of the tangents at P and Q. **4**

Question 5 BEGIN A NEW PAGE **Marks**

(a)



In the diagram above, $AB = AD = AX$ and $XP \perp DC$

(i) Prove that $\angle DBX = 90^0$ **2**

(ii) Hence, or otherwise, prove that $AB = AP$ **3**

(b) The arc of the curve $y = x\left(2 - x^2\right)$ from $x = 0$ to $x = 1$ is rotated about the y-axis. Find the volume of the solid of revolution, using the method of cylindrical shells. **4**

(c) A group of married couples are seated around a circular table. The position of each person is chosen at random, so that partners are not necessarily seated together.
The distance between a husband and wife is defined to be equal to the number of people sitting between them, measured either clockwise or anticlockwise, whichever gives the smaller result.

(i) Considering all possible arrangements for two married couples, show that the average distance between the members of a particular couple is $\dfrac{1}{3}$ **2**

(ii) Considering all possible seating arrangements for n married couples, show that the average distance between the members of a particular couple is $\dfrac{(n-1)^2}{2n-1}$ **4**

Question 6 **BEGIN A NEW PAGE** **Marks**

(a) (i) If α is a root of $P(x)$ of multiplicity n, show that α is also a root of $P'(x)$ with multiplicity of $n-1$. **2**

(ii) If $P(x)=2x^4+9x^3+6x^2-20x-24$ has a triple root, then factorise $P(x)$ into its linear factors. **3**

(b) Two possible forms of a cubic polynomial function $y=f(x)$ are sketched below, where a and b are the x-coordinates of the turning points.

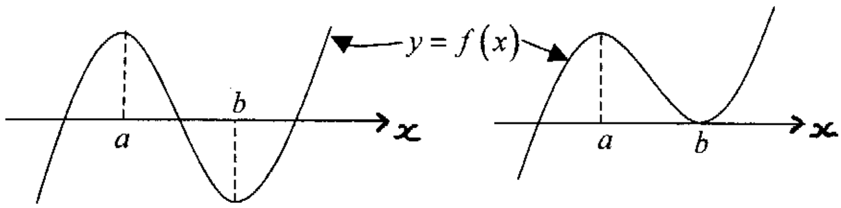


Fig. 1 Fig. 2

(i) Comment on the possible values of $f(a)\times f(b)$ in figures 1 and 2. **1**

(ii) For what range of values of k will the equation **3**

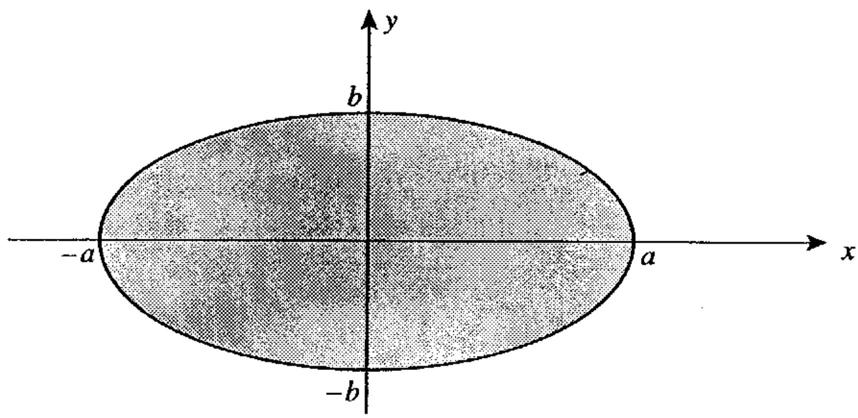
$2x^3-3x^2-36x+3k=0$ have three real roots, not all necessarily distinct. (Hint: Find the stationary points for this function)

(iii) For $P(x)=x^3-3m^2x+n$ where $m,n>0$ show that the roots are real and distinct if $n<2m^3$. **3**

(iv) If the roots of $P(x)$ in part (iii) are in the ratio $2:-3:5$, show that $90\sqrt{3}m^3=11\sqrt{11}n$. **3**
[**Hint:** Let the roots be $2\alpha,-3\alpha$ and 5α.]

Question 7 **BEGIN A NEW PAGE** **Marks**

(a)



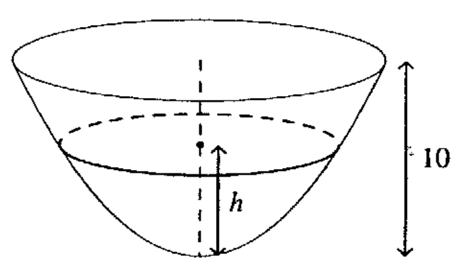
The diagram above shows the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with major diameter $2a$ and minor diameter $2b$, where a and b are positive real numbers.

(i) Show that the shaded area of the ellipse is given by **2**

$$\frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx$$

(ii) Hence show that the shaded area is πab square units. **2**

(iii)



The diagram above shows a solid of height 10cm. At height $h$cm above the vertex, the cross-section of the solid is an ellipse with major diameter $10\sqrt{h}$ cm and minor diameter $8\sqrt{h}$ cm.

(α) Show that the cross-section at height h cm above the vertex has an area $20\pi h\text{ cm}^2$ **2**

(β) Find the volume of the solid. **2**

QUESTION 7 CONTINUED ON PAGE 8

(b) $P(x) = x^6 + x^3 + 1$

(i) Show that the roots of $P(x) = 0$ are amongst the roots of $x^9 - 1 = 0$ **1**

(ii) Hence show the roots of $P(x) = 0$ on the unit circle, centre the origin, on an Argand Diagram **2**

(iii) Show that **2**

$$P(x) = \left(x^2 - 2x\cos\frac{2\pi}{9} + 1\right)\left(x^2 - 2x\cos\frac{4\pi}{9} + 1\right)\left(x^2 - 2x\cos\frac{8\pi}{9} + 1\right)$$

(iv) Evaluate $\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} + \cos\frac{4\pi}{9}\cos\frac{8\pi}{9} + \cos\frac{8\pi}{9}\cos\frac{2\pi}{9}$ **2**

QUESTION 8 IS ON THE NEXT PAGE

Question 8 BEGIN A NEW PAGE **Marks**

(a) A component from a machine is immersed successively in two solutions. The first solution cleans the component and the second treats it.
Whilst it is in the first solution, the mass of the component in grams at time t hours is given by m_1 where

$$\frac{dm_1}{dt} = -km_1,\ 0 \le t < p,\ k > 0 \text{ and } k \text{ is a constant.}$$

Whilst it is in the second solution, the mass of the component in grams at time t hours is given by m_2 where

$$\frac{dm_2}{dt} = k(m_2 + 1),\ t \ge p\,.$$

The component is transferred, without delay, from solution one to solution two at $t = p$ hours. Initially, and then again at $t = \frac{1}{k}\ln 11$, the mass of the component is 10 grams.

(i) Explain why $p < \frac{1}{k}\ln 11$. **1**

(ii) Find an expression for $m_1(t)$. **1**

(iii) Find an expression for $m_2(t)$. **2**

(iv) Show that $p \approx \frac{1}{k}\log_e 3\cdot 7$. **3**

(iv) Explain whether or not the component reaches its original mass if it is in solution two for p hours. **2**

(b) Let $F(x) = 1 + 2\binom{n}{1}x + 3\binom{n}{2}x^2 + \ldots\ldots + (n+1)\binom{n}{n}x^n$.

(i) By integrating both sides of this equation with respect to x, show that **4**

$$F(x) = \frac{d}{dx}\left(x(1+x)^n\right)$$

(ii) Hence or otherwise, show that **2**

$$1^2 + 2^2\binom{n}{1}x + 3^2\binom{n}{2}x^2 + \ldots\ldots + (n+1)^2\binom{n}{n}x^n = F(x) + xF'(x)$$

THE END

QUESTION 1

(a) $\displaystyle\int \frac{1}{\sqrt{4-(1+x)^2}}\,dx$

$\displaystyle = \int \frac{1}{\sqrt{2^2-(1+x)^2}}\,dx$ ✓

$\displaystyle = \sin^{-1}\left(\frac{1+x}{2}\right) + c$ ✓

(b) $\displaystyle\int_0^1 \tan^{-1}x\,dx$

$\displaystyle = \int_0^1 \underset{v'}{1} \times \underset{u}{\tan^{-1}x}\,dx$

(using $\int u.v'\,dx = uv - \int v.u'\,dx$)

$\displaystyle = \left[\tan^{-1}x \cdot x\right]_0^1 - \int_0^1 x \cdot \frac{1}{1+x^2}\,dx$ ✓

$\displaystyle = \left[\tan^{-1}1 - 0\right] - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx$

$\displaystyle = \frac{\pi}{4} - \left[\frac{1}{2}\ln|1+x^2|\right]_0^1$ ✓

$\displaystyle = \frac{\pi}{4} - \left[\frac{1}{2}\ln 2 - \frac{1}{2}\ln 1\right]$

$\displaystyle = \frac{\pi}{4} - \frac{1}{2}\ln 2$ ✓

(c)(i) $\displaystyle\frac{x^2-4}{(3x-1)(x+2)^2} = \frac{a(x+2)^2 + b(3x-1)(x+2) + c(3x-1)}{(3x-1)(x+2)^2}$

$x^2 - 11 = a(x+2)^2 + b(3x-1)(x+2) + c(3x-1)$

put $x = -2$

$(-2)^2 - 11 = c(3\times -2 - 1)$

$c = 1$ ✓

put $x = \frac{1}{3}$

$\left(\frac{1}{3}\right)^2 - 11 = a\left(\frac{1}{3}+2\right)^2$

$\displaystyle -\frac{98}{9} = \frac{49}{9}a$

$\therefore a = -2$ ✓

$\therefore x^2 - 11 = -2(x+2)^2 + b(3x-1)(x+2) + 1(3x-1)$

put $x = 0$

$-11 = -2\times 4 + b(-1)(2) - 1$

$-11 = -8 - 2b - 1$

$b = 1$ ✓

c(ii) $\displaystyle\int \frac{x^2-11}{(3x-1)(x+2)^2}\,dx$

$\displaystyle = \int\left(\frac{-2}{3x-1} + \frac{1}{x+2} + \frac{1}{(x+2)^2}\right)dx$

$\displaystyle = -\frac{2}{3}\ln|3x-1| + \ln|x+2| + \frac{-1}{x+2} + c$ ✓✓

(d) $\displaystyle\int 8x\sqrt[3]{4x-3}\,dx$

let $u^2 = 4x - 3$

$\displaystyle x = \frac{u^2+3}{4}$

$\displaystyle \frac{dx}{du} = \frac{2u}{4} = \frac{u}{2}$ ✓

$\displaystyle \int 8\left(\frac{u^2+3}{4}\right)(u^2)^{1/3}\cdot\frac{u}{2}\,du$ ✓

$\displaystyle = \int u^{5/3}(u^2+3)\,du$

$\displaystyle = \int\left(u^{11/3} + 3u^{5/3}\right)du$

$\displaystyle = \frac{3}{14}u^{\frac{14}{3}} + \frac{9}{8}u^{\frac{8}{3}} + c$ ✓

$\displaystyle = \frac{3u^{\frac{8}{3}}}{56}(4u^2 + 21) + c$

$\displaystyle = \frac{3}{56}(u^2)^{\frac{4}{3}}(4u^2+21) + c$ ✓

$\displaystyle = \frac{3}{56}(4x-3)^{\frac{4}{3}}(4(4x-3)+21) + c$

$\displaystyle = \frac{3}{56}\sqrt[3]{(4x-3)^4}\,(16x+9) + c$ ✓

$$\int \left(u^{11/3} + 3u^{5/3}\right) du$$

$$= \frac{3}{14} u^{14/3} + \frac{9}{8} u^{8/3} + C$$ ✓

(+2)

$$= \frac{3u^{8/3}}{56} \left(4u^{2} + 21\right) + C$$

$$= \frac{3}{56} \left(u^{2}\right)^{4/3} \left(4u^{2} + 21\right) + C$$

$$= \frac{3}{56} \left(4x-3\right)^{4/3} \left(4(4x-3) + 21\right) + C$$

$$= \frac{3}{56} \sqrt[3]{4x-3}^{\,4} \left(16x + 9\right) + C$$

QUESTION 2

(a) $\frac{w}{z} = \frac{29+29i}{7-3i} \times \frac{7+3i}{7+3i}$ ✓

$= \frac{203+87i^2+290i}{49-9i^2}$ ✓

$= \frac{116+290i}{58}$

$= 2+5i$ ✓

(b)(i) $-1+\sqrt{3}i = 2\,\text{cis}\,\frac{2\pi}{3}$ ✓✓

(ii) $(-1+\sqrt{3}i)^5 = \left(2\,\text{cis}\,\frac{2\pi}{3}\right)^5$

$= 2^5\,\text{cis}\,\frac{10\pi}{3}$ ✓

$= 32\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$ ✓

$= 32\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= 16(-1-\sqrt{3}i)$ ✓

(c) let $z = x+iy$

$|z|=1 \quad \therefore x^2+y^2=1$ ✓

LHS $= z^{-1}$

$= \frac{1}{x+iy} \times \frac{x-iy}{x-iy}$

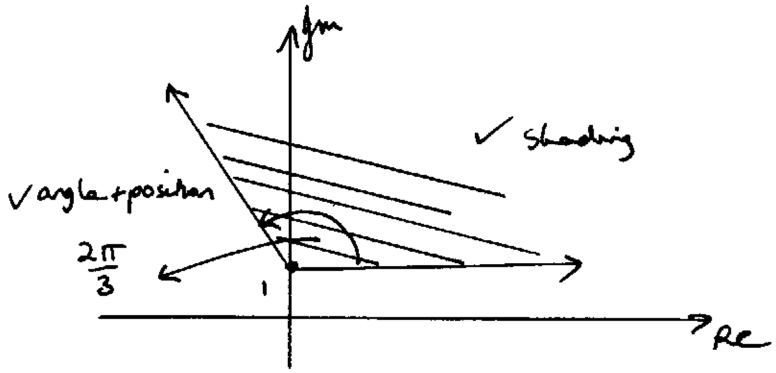
$= \frac{x-iy}{x^2+y^2}$ ✓

$= \frac{x-iy}{1}$

$= \bar{z}$

$=$ RHS $\quad \therefore z^{-1} = \bar{z}$

(d)



(e)(i) $(7+8i)-(3+2i)$

$= 4+6i$ ✓

(ii) Vector PR

$(4+6i)i = -6+4i$ ✓

R is $(3+2i)+(-6+4i)$

$= -3+6i$ ✓

QUESTION 3

(a)(i) $n=3,\ P_3 = \frac{3.2}{1.5} = \frac{6}{5}$

$n=4,\ P_4 = P_3 \times \frac{4\cdot 3}{2\cdot 6}$

$= \frac{6}{5}$

$n=5,\ P_5 = P_4 \times \frac{5.4}{3.7}$

$= \frac{120}{105} = \frac{8}{7}$

for $n \geq 5$ $\quad \frac{n(n-1)}{(n-2)(n+2)} < 1$
which will continue to decrease the result.

So maximum value of P_n is $\frac{6}{5}$
and this occurs when $n=3$ or $n=4$

(ii) Step 1: Prove true for $n=3$

LHS $= P_3$
$= \frac{3(3-1)}{(3-2)(3+2)}$
$= \frac{6}{5}$

RHS $= \frac{12(3-1)}{(3+1)(3+2)}$
$= \frac{24}{20}$
$= \frac{6}{5}$

LHS = RHS
$\therefore$ true for $n=3$

Step 2: Assume true for $n=k$

ie $P_k = \frac{12(k-1)}{(k+1)(k+2)}$

Step 3: Prove true for $n=k+1$

$$P_{k+1} = \frac{3(3-1)}{(3-2)(3+2)} \cdot \frac{4(4-1)}{(4-2)(4+2)} \cdots \frac{k(k-1)}{(k-2)(k+2)} \cdot \frac{(k+1)(k+1-1)}{(k+1-2)(k+1+2)}$$

$$= \frac{12(k-1)}{(k+1)(k+2)} \cdot \frac{(k+1)k}{(k-1)(k+3)} \quad \text{using step 2}$$

$$= \frac{12k}{(k+2)(k+3)}$$

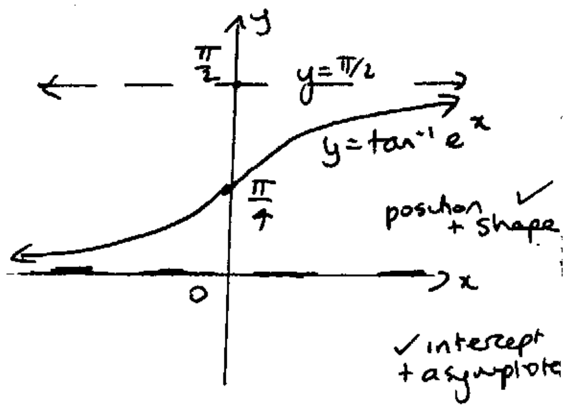
$$= \frac{12((k+1)-1)}{((k+1)+1)((k+1)+2)}$$

$\therefore$ if $P_k = \frac{12(k-1)}{(k+1)(k+2)}$ then $P_{k+1} = \frac{12((k+1)-1)}{((k+1)+1)((k+1)+2)}$

$\therefore$ true for $n=k+1$

Step 4: If it is true for $n=k$, it is true for $n=k+1$. Since P_n is true for $n=3$, it is true for $n=4$, $n=5$ and so on $\forall$ positive integral values of $n \geq 3$.

(b)(i) domain : $x \in \mathbb{R}$
range : $0 < y < \frac{\pi}{2}$



(ii) $y = \tan^{-1} e^x \Rightarrow \tan y = e^x$

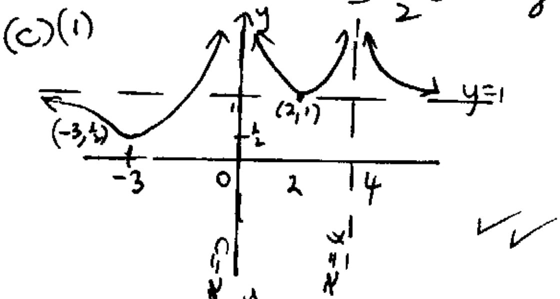
$\frac{dy}{dx} = \frac{e^x}{1+(e^x)^2}$

$= \frac{\tan y}{1+\tan^2 y}$

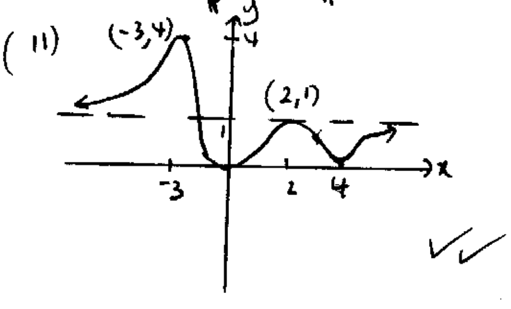
$t = \tan y \Rightarrow \frac{dy}{dx} = \frac{1}{2}\,\frac{2t}{1+t^2}$

$= \frac{1}{2}\sin 2y$

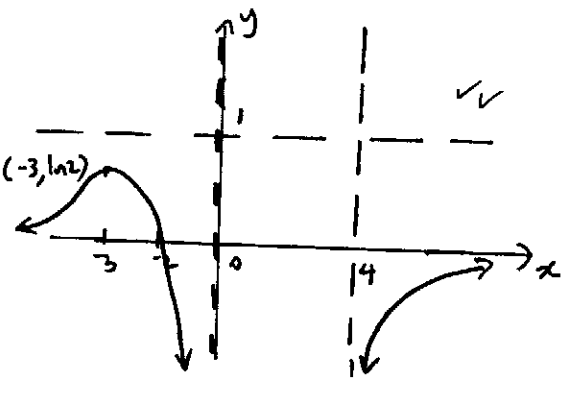
(c)(i)



(ii)



(iii)



QUESTION 4

(a) (i) $4x^2+9y^2=16$

$$\frac{x^2}{4}+\frac{9y^2}{16}=1$$

$\therefore a=2,\ b=\frac{4}{3}$

$b^2=a^2(1-e^2)$

$\frac{16}{9}=4(1-e^2)$

$4e^2=4-\frac{16}{9}$

$e^2=\frac{5}{9}$ ✓

$e=\frac{\sqrt{5}}{3}$

(ii) foci $(\pm ae,0)$

$=\left(\pm\frac{2\sqrt{5}}{3},0\right)$ ✓

directrices $x=\pm\frac{a}{e}$

$x=\pm\frac{6}{\sqrt{5}}$ ✓

(iii) $8x+18y\cdot\frac{dy}{dx}=0$

$\frac{dy}{dx}=\frac{-8x}{18y}$

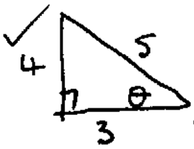
$=\frac{-4x}{9y}$ ✓

at P gradient $=\frac{-4\times2\cos\theta}{9\times\frac{4}{3}\sin\theta}$

$=-\frac{2\cos\theta}{3\sin\theta}$ ✓

(iv) $\frac{-2\cos\theta}{3\sin\theta}=\frac{1}{2}$

$\tan\theta=-\frac{4}{3}$ ✓



quad 2 and quad 4

$x=a\cos\theta,\ y=b\sin\theta$

QUAD 2
when $x=2\times-\frac{3}{5},\ y=\frac{4}{3}\times\frac{4}{5}$

ie $\left(-\frac{6}{5},\frac{16}{15}\right)$ ✓

QUAD 4
or when $x=2\times\frac{3}{5},\ y=\frac{4}{3}\times-\frac{4}{5}$

ie $\left(\frac{6}{5},-\frac{16}{15}\right)$ ✓

(b) (i) $\frac{dy}{dx}=-\frac{c^2}{x^2}$ ✓

$\therefore$ gradient of tangent at $Q=-\frac{1}{t_2^2}$

Eqn of tangent at Q

$y-\frac{c}{t_2}=-\frac{1}{t_2^2}(x-ct_2)$

$t_2^2y-t_2c=-x+ct_2$

$x+t_2^2y=2ct_2$ ✓

let $y=0$, $x=2ct_2$

$\therefore N$ is $(2ct_2,0)$

but $N=(ct_1,0)$ ✓

$\therefore ct_1=2ct_2$

$\therefore t_1=2t_2$

(ii) eqn of tangent at Q is $x+t_2^2y=2ct_2$ —①

$\therefore$ eqn of tangent at P $x+t_1^2y=2ct_1$ —②

① − ②

$(t_2^2-t_1^2)y=2c(t_2-t_1)$

$\therefore y=\frac{2c}{t_1+t_2}$ and $t_1\neq t_2$ ✓

Sub into eqn ②

$x+t_1^2\cdot\frac{2c}{t_1+t_2}=2ct_1$

$x=2ct_1-\frac{t_1^2\,2c}{t_1+t_2}$ ✓

$=\frac{\cancel{2ct_1^2}+2ct_1t_2-\cancel{t_1^2 2c}}{t_1+t_2}$

use $t_1=2t_2$

$x=\frac{2c\cdot2t_2\cdot t_2}{3t_2}=\frac{4ct_2}{3}$ —③

and $y=\frac{2c}{3t_2}\Rightarrow t_2=\frac{2c}{3y}$ ✓

Sub in ③

$x=\frac{4c}{3}\left(\frac{2c}{3y}\right)$

$x=\frac{8c^2}{9y}$ ✓

eqn of locus of T is $xy=\frac{8}{9}c^2$

QUESTION 5

(a)

i) Circle through D, B and X has centre A ($AB = AD = AX$, equal radii)

$\therefore DAX$ is diameter

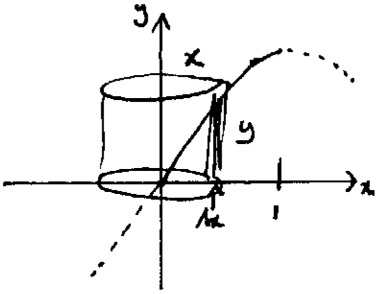
$\therefore \angle DBX = 90^\circ$ (angle in a semicircle)

ii)

P also lies on circle centre A diameter DX (given $\angle DPX = 90^\circ$ converse angle in semicircle)

$\therefore AP = AB$ (equal radii of circle)

(b) $\Delta V = 2\pi x y \, \Delta x$



$V = \lim_{\Delta x \to 0} \sum_{x=0}^{1} 2\pi x y \, \Delta x$

$= \int_0^1 (2\pi x)\, x\,(2 - x^2)\, dx$

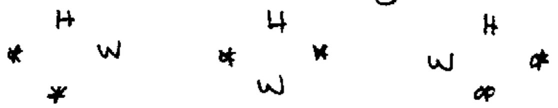
$= 2\pi \left[\frac{2x^3}{3} - \frac{x^5}{5} \right]_0^1$

$= 2\pi \left(\frac{2}{3} - \frac{1}{5} \right)$

$= \frac{14\pi}{15}$ units^3

(c)(i) let the particular couple be H and W

There are 3 possible arrangements

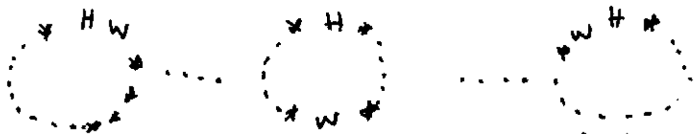


① Distance = 0 ② Distance = 1 ③ Distance = 0

$\therefore$ Average distance $= \frac{0+1+0}{3}$

$= \frac{1}{3}$

(ii) 'n' married couples ⇒ $2n$ people present. Chosen couple is H, W



Distance = 0 .... distance $= n-1$ ... distance = 0

⇒ $2n-1$ positions

distance varies from 0 to... $(n-1)$ and back to 0.

Average distance

$= \frac{0+1+2+\dots+(n-1)+\dots+2+1+0}{2n-1}$

$= \frac{2(0+1+2+\dots+n-2)+(n-1)}{2n-1}$

$= \frac{\cancel{2}\left(\frac{1}{\cancel{2}}(n-2)(n-1)\right)+(n-1)}{2n-1}$

$= \frac{(n-1)(n-2+1)}{2n-1}$

$= \frac{(n-1)^2}{2n-1}$

QUESTION 6

(a) (i) Let $P(x) = (x-\alpha)^n Q(x)$

where $Q(x)$ is a polynomial ✓

by product rule

$P'(x) = n(x-\alpha)^{n-1} \cdot Q(x) + (x-\alpha)^n \cdot Q'(x)$

$= (x-\alpha)^{n-1}\left[n\,Q(x) + (x-\alpha)Q'(x)\right]$ ✓

$\therefore \alpha$ is a root of $P'(x)$ of multiplicity $(n-1)$.

(ii) $P(x) = 2x^4+9x^3+6x^2-20x-24$ —— ①

has a triple root

$\therefore P'(x) = 8x^3+27x^2+12x-20$

has a double root

and $P''(x) = 24x^2+54x+12$

has a 1 fold root ✓

$24x^2+54x+12=0$

$4x^2+9x+2=0$

$(4x+1)(x+2)=0$

$\therefore x = -\frac{1}{4}$ or $x=-2$ ✓

Sub $x=-2$ into eqn ①

$P(-2) = 2(-2)^4 + 9(-2)^3 + 6(-2)^2 - 20(-2) - 24$

$= 32 - 72 + 24 + 40 - 24$

$= 0$

$\therefore x=-2$ is the triple root

and by inspection

$P(x) = (x+2)^3(2x-3)$ ✓

(b)(i) $f(a).f(b) < 0$ in fig. 1 ✓

$f(a).f(b) = 0$ in fig 2

(ii) let $f(x) = 2x^3-3x^2-36x+3k$

$f'(x) = 6x^2-6x-36$

$f'(x)=0$ for stat. pts

$6x^2-6x-36=0$

$x^2-x-6=0$

$(x-3)(x+2)=0$

$x=3$ or $x=-2$ ✓

$f(3) = 2\times3^3 - 3\times3^2 - 36\times3 + 3k$

$= -81 + 3k$

$f(-2) = 2(-2)^3 - 3(-2)^2 - 36(-2) + 3k$

$= 44 + 3k$

from (i) $\Rightarrow$ 3 real roots if $f(a).f(b) \le 0$ ✓

So $f(3)\times f(-2) \le 0$

$(3k-81)(3k+44) \le 0$

$-\frac{44}{3} \le k \le 27$ ✓

(iii) $P(x) = x^3 - 3m^2x + n$

$P'(x) = 3x^2 - 3m^2$

$3(x+m)(x-m) = 0$ for stat. pts

$x = \pm m$ ✓

$P(m) = m^3 - 3m^2m + n$

$= -2m^3 + n$

$P(-m) = -m^3 - 3m^2(-m) + n$

$= 2m^3 + n$ ✓

for 3 real and distinct roots

$P(m)\times P(-m) < 0$

$(-2m^3+n)(2m^3+n) < 0$

$-4m^6 + n^2 < 0$ ✓

$n^2 < 4m^6$

$n < 2m^3$ as required

(iv) Let the roots be $2\alpha, -3\alpha, 5\alpha$

sum of roots 2 at a time

$= -6\alpha^2 - 15\alpha^2 + 10\alpha^2$

$\therefore \frac{c}{a} = -3m^2 = -11\alpha^2$

$\alpha^2 = \frac{3m^2}{11}$ —— (1) ✓

$\Rightarrow \alpha =$

product of roots

$= -\frac{d}{a} = -n = 2\alpha\times -3\alpha\times5\alpha$

$-n = -30\alpha^3$

$\alpha^3 = \frac{n}{30}$ —— (2) ✓

from (1)

$\alpha^3 = \frac{3\sqrt{3}\,m^3}{11\sqrt{11}}$ —— (3)

put (2) = (3)

so $\frac{n}{30} = \frac{3\sqrt{3}\,m^3}{11\sqrt{11}}$

$\therefore 90\sqrt{3}\,m^3 = 11\sqrt{11}\,n$ ✓

QUESTION 7

(a)(i) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$y=\pm b\sqrt{1-\frac{x^2}{a^2}}$

$=\pm\frac{b}{a}\sqrt{a^2-x^2}$ ✓

Area in quadrant ① $=\int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$ ✓

$\therefore$ Area of ellipse $=\frac{4b}{a}\int_0^a\sqrt{a^2-x^2}\,dx$

(ii) $\int_0^a\sqrt{a^2-x^2}\,dx$ gives area of circle quadrant ① (radius a units)

Area $=\frac{1}{4}\times\pi r^2$ ✓

$\therefore \int_0^a\sqrt{a^2-x^2}\,dx=\frac{\pi a^2}{4}$ units²

$\therefore$ Area Ellipse $=\frac{4b}{a}\times\frac{\pi a^2}{4}$ ✓

$=\pi ab$ units²

(iii)(α) at height h cm, ellipse has $b=4\sqrt{h}$ and $a=5\sqrt{h}$ ✓

Area $=\pi ab$

$=\pi.5\sqrt{h}.4\sqrt{h}$

$=20\pi h$ cm² ✓

(β) Volume $=\int_0^{10}20\pi h\,dh$ ✓

$=\left[10\pi h^2\right]_0^{10}$

$=1000\pi$ cm³ ✓

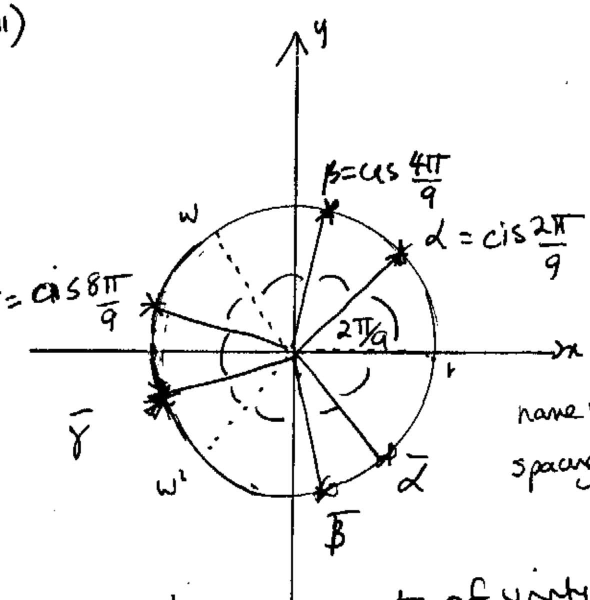
(b) $P(x)=x^6+x^3+1$

(i) $x^9-1=(x^3-1)(x^6+x^3+1)$

$\therefore$ The roots of x^9-1 are the cube roots of unity $1,\omega,\omega^2$ and the roots of $P(x)=0$ ✓

ie the roots of $P(x)=0$ are the ninth roots of unity other than $1,\omega,\omega^2$

(b)(ii)



name ✓
spacing ✓

$\therefore$ The '9' ninth roots of unity are equally spaced around the unit circle. These include $1,\omega,\omega^2$ and the other $\underline{6}$ occur in conjugate pairs and are the roots of $P(x)=0$

(iii) $(x-\alpha)(x-\bar\alpha)=\{x^2-2\,\text{Re}(\alpha).x+|\alpha$

is a factor of $P(x)$ ✓

$\therefore\left(x^2-2\cos\frac{2\pi}{9}x+1\right)$ is a factor of $P(x)$

$\therefore P(x)=k(x-\alpha)(x-\bar\alpha)(x-\beta)(x-\bar\beta).(x-\gamma)(x-\bar\gamma)$ ✓

$P(x)$ is a monic polynomial $\therefore k=1$

$P(x)=\left(x^2-2\cos\frac{2\pi}{9}x+1\right).\left(x^2-2\cos\frac{4\pi}{9}x+1\right)\left(x^2-2\cos\frac{8\pi}{9}x+1\right)$

(iv) Equating coefficients of x^2

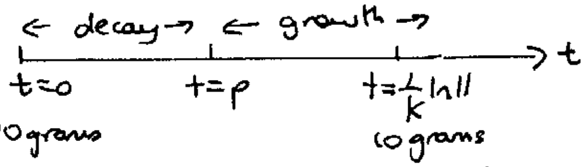
$0=3+4\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}+4\cos\frac{4\pi}{9}.\cos\frac{8\pi}{9}+4\cos\frac{8\pi}{9}.\cos\frac{2\pi}{9}$ ✓

$\therefore\cos\frac{2\pi}{9}.\cos\frac{4\pi}{9}+\cos\frac{4\pi}{9}.\cos\frac{8\pi}{9}+\cos\frac{8\pi}{9}.\cos\frac{2\pi}{9}=\frac{-3}{4}$ ✓

QUESTION 8

i) $\frac{dm_1}{dt} = -km_1 \Rightarrow$ decay $0 \le t < p$

$\frac{dm_2}{dt} = k(m_2+1) \Rightarrow$ growth $t \ge p$



since changes from decay phase to growth phase at $t=p$, quantity would not have had the chance to return to its original mass of 10g (which occurs when $t=\frac{1}{k}\ln 11$)

$\therefore p < \frac{1}{k}\ln 11$.

ii) $\frac{dm_1}{dt} = -km_1$

$m_1 = Ae^{-kt}$ where A is a constant

when $t=0, m_1=10$

$\therefore 10 = Ae^0$

$A = 10$

$M_1(t) = 10e^{-kt}$

iii) $\frac{dm_2}{dt} = k(m_2+1)$

$\therefore m_2 = -1 + Ae^{kt}$ where A is a constant

$m_2 = 10$ when $t = \frac{1}{k}\ln 11$

$\therefore 10 = -1 + Ae^{k \times \frac{1}{k}\ln 11}$

$10 = -1 + 11A$

$A = 1$

$\therefore m_2(t) = e^{kt} - 1, \ t \ge p$

iv) Since component transferred at $t=p$ hours then $m_1 = m_2$ at $t=p$

$\therefore 10e^{-kp} = e^{kp} - 1$

$\frac{10}{e^{kp}} = e^{kp} - 1$

$10 = e^{2kp} - e^{kp}$

let $e^{kp} = X$

$10 = X^2 - X$

$X^2 - X - 10 = 0$

$X = \frac{1 \pm \sqrt{(-1)^2 - 4 \times 1 \times -10}}{2}$

$= \frac{1 \pm \sqrt{41}}{2}$

$= 3.7015\ldots$ or $\cancel{-2.70\ldots}$

so $e^{kp} \doteq 3.7$

$p = \frac{1}{k}\log_e 3.7$

(v) Original mass of component

$M_1(0) = 10$

Component is in solution 1 & 2 for p hours so at $t = 2p$

$M_2(2p) = e^{2kp} - 1$

from (iv) $p = \frac{1}{k}\log_e 3.7$

$m_2(2p) = e^{2\log_e 3.7} - 1$

$\doteq 12.69\ldots$

$\therefore$ component reaches and in fact exceeds its original mass by 2.69g

PTO

Q8b)

(i) $F(x) = 1 + 2\binom{n}{1}x + 3\binom{n}{2}x^2 + \dots + (n+1)\binom{n}{n}x^n$

$\int F(x)\,dx = x + 2\binom{n}{1}\frac{x^2}{2} + 3\binom{n}{2}\frac{x^3}{3} + \dots + \frac{n+1}{n+1}\binom{n}{n}x^{n+1} + c$

$= x + \binom{n}{1}x^2 + \binom{n}{2}x^3 + \dots + \binom{n}{n}x^{n+1} + c$ — (1)

$x(1+x)^n = x\left[\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots \binom{n}{n}x^n\right]$

$= \binom{n}{0}x + \binom{n}{1}x^2 + \binom{n}{2}x^3 + \dots \binom{n}{n}x^{n+1}$ — (2)

using eqn (1) & (2)

substitute RHS of eqn (2) into eqn (1)

$\int F(x)\,dx = x(1+x)^n + c'$

Hence $F(x) = \frac{d}{dx}\left[x(1+x)^n\right]$

(ii) $xF(x) = x + 2\binom{n}{1}x^2 + 3\binom{n}{2}x^3 + \dots + (n+1)\binom{n}{n}x^{n+1}$

$\frac{d}{dx}\left(xF(x)\right) = 1^2 + 2^2\binom{n}{1}x + 3^2\binom{n}{2}x^2 + \dots + (n+1)^2\binom{n}{n}x^n$

using the chain rule $\frac{d}{dx}\left(xF(x)\right) = F(x) + xF'(x)$

$\therefore 1^2 + 2^2\binom{n}{1}x + 3^2\binom{n}{2}x^2 + \dots + (n+1)^2\binom{n}{n}x^n = F(x) + xF'(x)$