Question 1 **(Begin a new page)** **Marks**

(a) Evaluate $\int_0^1 \frac{e^x}{1+e^x}\,dx$ **2**

(b) Find $\int \frac{1-\sin x}{\cos^2 x}\,dx$ **2**

(c) (i) use the substitution $u = 1 - x^2$ to evaluate $\int_0^{\frac{\sqrt{3}}{2}} \frac{x^3}{\sqrt{1-x^2}}\,dx$ **2**

(ii) Hence use integrating by parts to find $\int_0^{\frac{\sqrt{3}}{2}} 3x^2 \cos^{-1} x\,dx$. **2**

(d) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\pi/3} \frac{1}{5-4\cos x}\,dx$, expressing the answer in simplest exact form. **3**

(e) (i) Find the real numbers A and B such that **2**

$$\frac{x^2+3x}{(x^2+1)(x+1)} = \frac{Ax+1}{x^2+1} + \frac{B}{x+1}$$

(ii) Hence find $\int \frac{x^2+3x}{(x^2+1)(x+1)}\,dx$ **2**

Question 2 **(Begin a new page)** **Marks**

(a) Show that $(1 + i)^3 = 2(i - 1)$ **1**

(b) Show that the complex number $z = \dfrac{6 - 2i}{3 + 4i} - \dfrac{6}{5i}$ is real. **2**

(c) Sketch the region in the Argand diagram where the inequalities **3**

$$|z - 2 + 2i| \leq 2 \text{ and } -\frac{\pi}{3} < \arg z \leq -\frac{\pi}{6}$$

holds simultaneously.

(d) $z = x + iy$ is a complex number such that $|z - 3 - 4i| = 2$.

(i) Find the maximum value of $|z|$. **2**

(ii) Find the minimum value of $\arg z$. **2**

(e) For any non-zero complex number z,

(i) show that $\arg\left(\dfrac{z}{\bar{z}}\right) = 2\arg z$. **1**

(ii) Let z be a complex number for which $z \neq 0$, $z \neq 1$ and **2**

$$\frac{z}{\bar{z}} = -\frac{z-1}{\overline{z-1}}$$

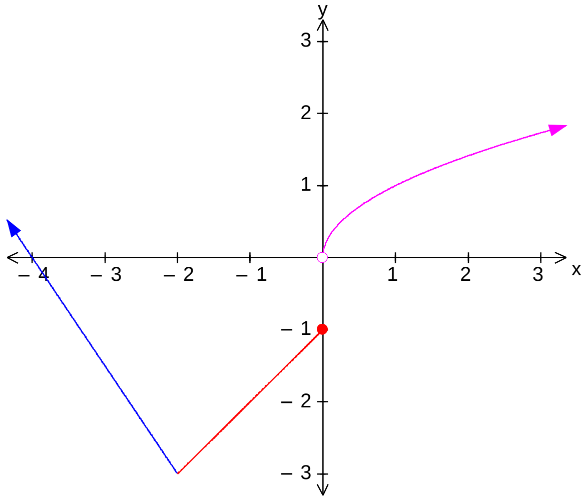
Show that $\arg z = \arg(z - 1) + \dfrac{\pi}{2}$ or $\arg z = \arg(z - 1) - \dfrac{\pi}{2}$

(iii) Hence sketch the locus of all points z that satisfy **2**

$$\frac{z}{\bar{z}} = -\frac{z-1}{\overline{z-1}}$$

Question 3 **(Begin a new page)** **Marks**

(a) The diagram below show the discontinuous function $y = f(x)$.



Draw large (half page), separate sketches of each of the following:

(i) $y = |f(x-1)|$ **2**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = \sqrt{-f(x)}$ **2**

(iv) $y = \ln(f(x))$ **2**

(v) $y^2 = f(x)$ **2**

(b) It is given that $3-i$ is a root of $P(z) = z^3 + kz + 60$,where k is a real number.

(i) State why $3+i$ is also a root of $P(z)$. **1**

(ii) Solve the equation $P(z) = 0$ **1**

(iii) Hence determine the value of k. **1**

(c) For $z = r(\cos\theta + i\sin\theta)$, find r and the smallest value of θ which satisfies the equation $2z^3 = 9 + 3\sqrt{3}\,i$ **2**

Question 4 **(Begin a new page)** **Marks**

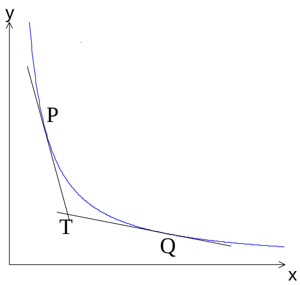
(a) The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $\quad a > b > 0$ has eccentricity e.

(i) Show that the line through the focus F $(ae, 0)$ that is perpendicular to the asymptote $y = \frac{bx}{a}$ has equation $ax + by - a^2e = 0$ **1**

(ii) Show that this line meets the asymptote at a point on the corresponding directrix . **2**

(b) Find the equation of the normal to the curve $x^4 + 3xy - 2y^2 + 13 = 0$ at the point (-1, 2). **3**

(c) The point $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ $p \neq q$ lies on the same branch of the hyperbola $xy = c^2$. The tangents at P and Q meet at point T.

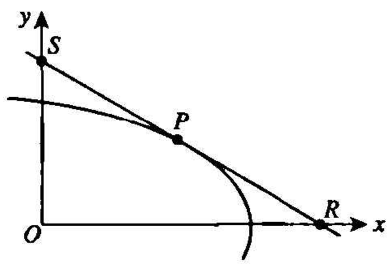


(i) Show that the equation of the tangent to the hyperbola at Q is $x + q^2y = 2cq$. **2**

(ii) Show that T has coordinate $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$ **2**

(iii) If P and Q move so that $pq = \text{k}$ where k is a constant, show that the locus of T is a straight line and give its equation in terms of k. **2**

(d) In the diagram, the tangent to the ellipse at $P(\sqrt{2}\cos\theta, 2\sqrt{2}\sin\theta)$ intersect the x-axis at R and the y-axis at S.

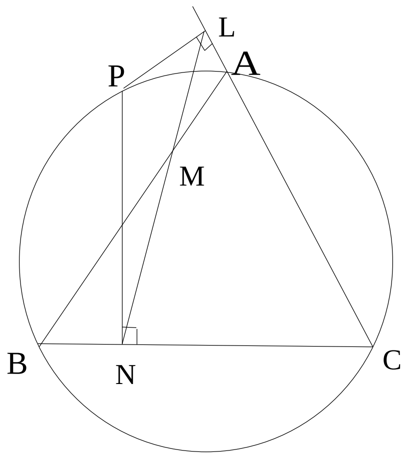


(i) Show that the area of ΔORS is $\frac{4}{\sin 2\theta}$, where O is the origin. [You may assume the equation of the tangent at P to be $\frac{x\cos\theta}{\sqrt{2}} + \frac{y\sin\theta}{2\sqrt{2}} = 1$]. **1**

(ii) Find the coordinate of P where the area is a minimum. **2**

Question 5 **(Begin a new page)** **Marks**

(a) ABC is an acute-angled triangle inscribed in a circle. P is a point on the minor arc AB of the circle. PL and PN are the perpendiculars from P to CA (produced) and CB respectively. LN cuts AB at M.



(i) Copy the diagram.

(ii) Explain why PNCL is a cyclic quadrilateral. **1**

(iii) Show that $\angle PBM = \angle PNM$. **2**

(iv) Hence deduce that

(α) PBNM is a cyclic quadrilateral **1**

(β) PM is perpendicular to AB. **2**

(b) The roots of $x^3 + px^2 + qx + r = 0$ form an arithmetic progression, prove that $2p^3 + 27r - 9pq = 0$. **3**

(c) Use the following identity to answer the following question.

$$\tan 4\theta = \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$

(i) Solve the equation $x^4 + 4\sqrt{3}\,x^3 - 6x^2 - 4\sqrt{3}\,x + 1 = 0$. **3**

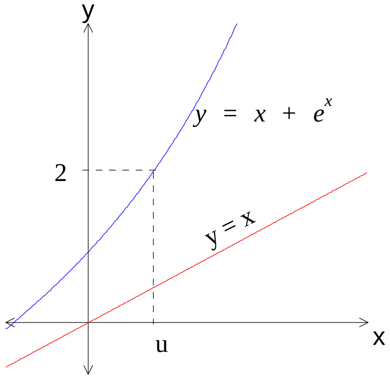
(ii) Hence show that

(α) $\tan\frac{\pi}{24} + \tan\frac{7\pi}{24} + 4\sqrt{3} = \tan\frac{5\pi}{24} + \tan\frac{11\pi}{24}$ **2**

(β) $\tan\frac{5\pi}{24}\tan\frac{7\pi}{24} = \cot\frac{\pi}{24}\cot\frac{11\pi}{24}$ **1**

Question 6 **(Begin a new page)** **Marks**

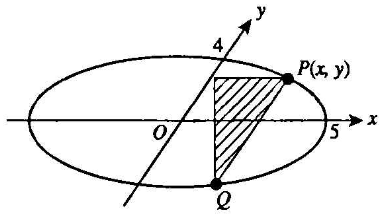
(a) The diagram shows the graph of the curve $y = x + e^x,\ \ x \geq 0$. The region bounded by the curve and the line $y = x$ between $x = 0$ and $x = u$ is rotated through one complete revolution about the y- axis.



(i) use the method of cylindrical shells to show that the volume of the solid of revolution is given by $$V = 2\pi \int_0^u xe^x\,dx$$ **3**

(ii) Hence show that $V = 2\pi\,(3u - u^2 - 1)$ **3**

(b) The base of a certain solid is the ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1.$ Every cross-section perpendicular to the x-axis is an equilateral triangle. The shaded cross-section is an equilateral triangle with base PQ.



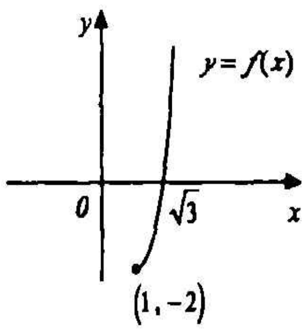
(i) Show that the shaded cross-sectional area is given by **1**

$$A = \sqrt{3}\,y^2$$

(ii) Hence express the cross-sectional area as a function of x. **1**

(iii) Find the volume of the solid. **2**

(c) The diagram shows the graph of $y = f(x)$ where $f(x) = x^3 - 3x;\ \ x \geq 1$



(i) Copy the diagram. On your diagram sketch the graph of the inverse function $y = f^{-1}(x)$, showing any intercepts on the coordinate axes and the coordinates of any end points. Draw in the line of $y = x$. **2**

(ii) Find the coordinate of any point of intersection of the curve $y = f(x)$ and $y = f^{-1}(x)$. **1**

(iii) Hence find the area of the region in the first quadrant bounded by the curve $y = f(x)$, $y = f^{-1}(x)$ and the coordinate axes. **2**

Question 7 **(Begin a new page)** **Marks**

(a) Solve for x, $\tan^{-1}(4x) - \tan^{-1}(3x) = \tan^{-1}\left(\frac{1}{7}\right)$ **3**

(b) A sequence of number u_n is defined by

$$u_n = 8u_{n-1} - 15u_{n-2} \qquad n \geq 3$$

and $u_1 = 2$, $u_2 = 16$

(i) Prove that $u_n = 5^n - 3^n$ for $n \geq 1$ by the method of mathematical induction. **4**

(ii) Hence show that **2**

$$u_1 + u_2 + \ldots\ldots + u_n = \frac{5^{n+1} - 2 \times 3^{n+1} + 1}{4}$$

(c) If $I_n = \int_0^1 x(1-x)^n \, dx \quad n = 0, 1, 2....$

(i) Show that $I_n = \frac{n}{n+2} I_{n-1} \quad n = 1, 2\ldots\ldots$ **2**

(ii) Hence deduce that $I_n = \frac{1}{2\ {}^{n+2}C_2} \quad n = 1, 2, 3\ldots..$ **2**

(iii) Find n for which $I_n \leq \frac{1}{132}$ **2**

Question 8 **(Begin a new page)** **Marks**

(a) Consider the polynomial $P(x) = \frac{x^4}{4} - \frac{x^3}{3} + x^2 - 2x + p$, p is a real number.

(i) Show that $P(x)$ has exactly one turning point for all real values of p. **2**

(ii) For what value of p does the equation $P(x) = 0$ have no real roots? **2**

(b) A particle moves in a straight line with velocity v and its acceleration given by $a = -v\sqrt{1-v^2}$. The displacement x, of the particle from a fixed origin O is initially zero and its velocity at that time is V.

(i) Show that **4**

$$x = \sin^{-1}(V\sqrt{1-v^2} - v\sqrt{1-V^2})$$

(ii) The time that has passed since the particle began its movement is given by t. **3**
By considering $a = \frac{dv}{dt}$, show that

$$t = \ln\left[\frac{V(1+\sqrt{1-v^2})}{v(1+\sqrt{1-V^2})}\right]$$

[You may assume the following result

$$\int \frac{dv}{v\sqrt{1-v^2}} = \log_e\left(\frac{v}{1+\sqrt{1-v^2}}\right)$$, DON'T PROVE IT.]

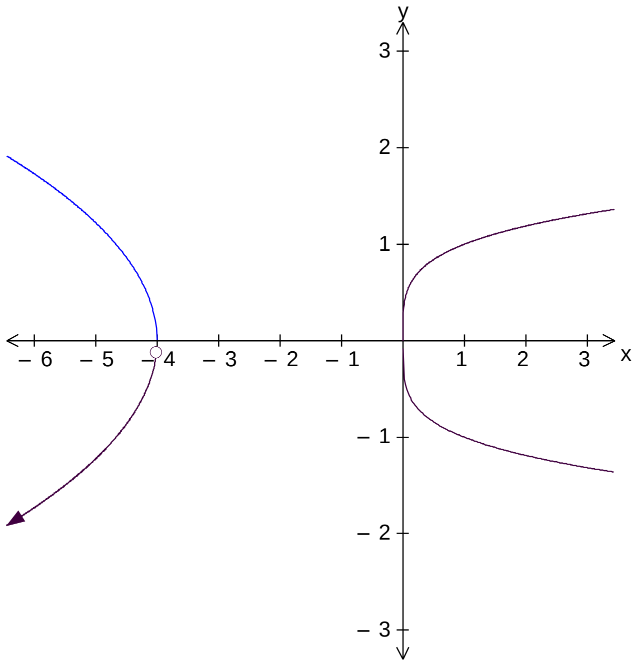
(c) Consider $f(x) = \ln(1+x) - \frac{x}{1+x}$ and $g(x) = \ln(1+x) - x$, $x > 0$

(i) Show that $g(x)$ is a decreasing function for all positive values of x. **1**

(ii) Deduce that $x > \ln(1+x)$ **1**

(iii) Hence show that $\frac{x}{1+x} < \ln(1+x) < x$ **2**

End of Paper


y
3
2
1
− 1
− 2
− 3
− 6
− 5
− 4
− 3
− 2
− 1
1
2
3
x

<u>Question 1</u>

a) $\displaystyle\int_0^1 \frac{e^x}{1+e^x}\,dx = \Big[\log_e(1+e^x)\Big]_0^1$ ✓

$\displaystyle = \log_e \frac{1+e}{2}$ ✓

b) $\displaystyle\int \frac{1-\sin x}{\cos^2 x}\,dx = \int(\sec^2 x - \tan x \sec x)\,dx$ ✓

$= \tan x - \sec x + C$ ✓

c) (i) $u = 1-x^2 \qquad \therefore x^2 = 1-u$

$du = -2x\,dx \qquad x\,dx = -\frac{du}{2}$

when $x = \frac{\sqrt3}{2}, \qquad u = \frac14$

when $x = 0 \qquad u = 1$

$$\int_0^{\frac{\sqrt3}{2}} \frac{x^3}{\sqrt{1-x^2}}\,dx = \int_0^{\frac{\sqrt3}{2}} \frac{x^2}{\sqrt{1-x^2}}\,x\,dx$$

$$= -\int_1^{\frac14} \frac{1-u}{2\sqrt u}\,du$$ ✓

$$= \frac12\int_{\frac14}^1 \left(\frac{1}{\sqrt u} - \sqrt u\right)du$$

$$= \frac12\left[2\sqrt u - \frac23 u^{\frac32}\right]_{\frac14}^1$$

$$= \frac12\left(2-\frac23\right) - \frac12\left(2\cdot\frac12 - \frac23\cdot\frac18\right)$$

$$= \frac{5}{24}$$ ✓

(ii) $$\int_0^{\frac{\sqrt3}{2}} 3x^2\cos^{-1}x\,dx = \int_0^{\frac{\sqrt3}{2}} \cos^{-1}x\,d(x^3)$$

$$= \Big[x^3\cos^{-1}x\Big]_0^{\sqrt3/2} - \int_0^{\frac{\sqrt3}{2}} x^3\,d\cos^{-1}x$$ ✓

$$= \frac{3\sqrt3}{8}\cdot\frac{\pi}{6} + \int_0^{\frac{\sqrt3}{2}} \frac{x^3}{\sqrt{1-x^2}}\,dx$$

$$= \frac{\sqrt3\pi}{16} + \frac{5}{24}$$ ✓

(d) Put $t = \tan\frac{x}{2}$

$dx = \frac{2\,dt}{1+t^2}$

$x = \frac{\pi}{3}, \qquad t = \frac{1}{\sqrt3}$ ✓

$x = 0 \qquad t = 0$

$$\int_0^{\frac{\pi}{3}} \frac{1}{5-4\cos x}\,dx = \int_0^{\frac{1}{\sqrt3}} \frac{2\,dt}{(1+t^2)\left(5-\frac{4(1-t^2)}{1+t^2}\right)}$$

$$= \int_0^{\frac{1}{\sqrt3}} \frac{2\,dt}{5+5t^2-4+4t^2}$$

$$= \int_0^{\frac{1}{\sqrt3}} \frac{2\,dt}{1+9t^2}$$ ✓

$$= \frac19\int_0^{\frac{1}{\sqrt3}} \frac{2\,dt}{\frac19+t^2}$$

$$= \frac29\cdot 3\Big[\tan^{-1}3t\Big]_0^{\frac{1}{\sqrt3}}$$

$$= \frac{2\pi}{9}$$ ✓

Q1 (cont'd)

1(e) (i) $(Ax+1)(x+1)+B(x^2+1) \equiv x^2+3x$

Put $x=0$ $\quad B+1 = 0$

$\therefore B = -1$ ✓

Equating coeff of x^2 $\quad A+B = 1$

$\therefore A = 1-B$

$= 2$ ✓

(ii) $\displaystyle\int \frac{x^2+3x}{(x^2+1)(x+1)}\,dx = \int \frac{2x+1}{x^2+1} - \frac{1}{x+1}\,dx$ ✓

$= \log_e(x^2+1) - \log_e(x+1) + \tan^{-1}x + C$

$= \log_e \dfrac{x^2+1}{x+1} + \tan^{-1}x + C$ ✓

Question 2

(a) $(1+i)^3 = 1+3i-3-i$

$= 2i-2$

$= 2(i-1)$ ✓

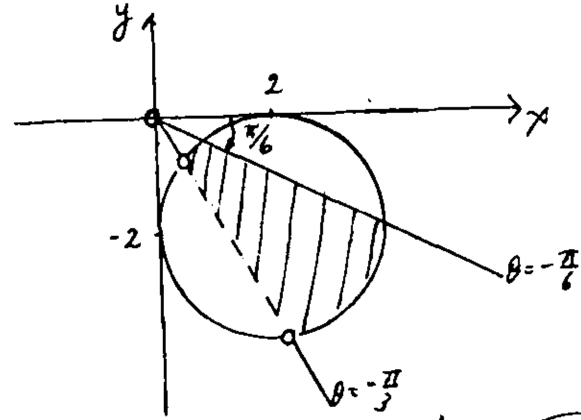
(b) $z = \dfrac{6-2i}{3+4i} - \dfrac{6}{5i}$

$= \dfrac{(6-2i)(3-4i)}{(3+4i)(3-4i)} + \dfrac{6i}{5}$ ✓

$= \dfrac{18-8-30i}{9+16} + \dfrac{6i}{5}$

$= \dfrac{2}{5}$ which is real ✓

(c)

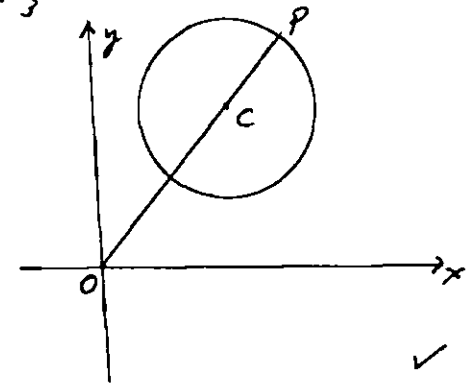


✓ for circle
✓ for $\theta = -\frac{\pi}{6}$
✓ for $\theta = -\frac{\pi}{3}$ (dotted)

d) (i) Let C be the centre of the circle

$|z-3-4i| = 2$

$\therefore$ C is $(3,4)$.



Let P be the point such that O, C, P are collinear. ✓

$OP = OC + CP$

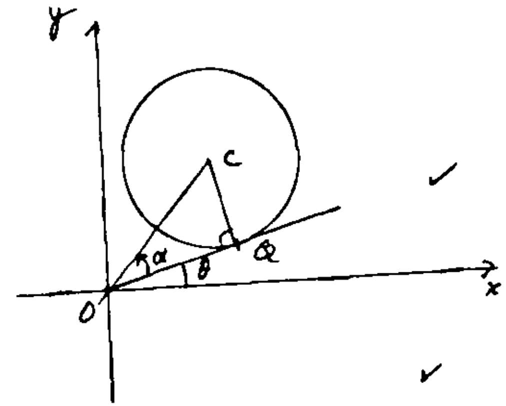
$= \sqrt{3^2+4^2} + 2$

$= 7$ ✓

$\therefore$ max value of $|z|$ is 7

(ii) $OC = 5$, $CQ = 2$

$\therefore \alpha = \sin^{-1}\frac{2}{5}$



but $\alpha+\theta = \arg(3+4i)$

$= \tan^{-1}\frac{4}{3}$ ✓

$\therefore \theta = \tan^{-1}\frac{4}{3} - \sin^{-1}\frac{2}{5}$

or $= 0.5157$ radians ✓

$\therefore$ min $\arg z = 0.5157$ radians.
or $\tan^{-1}\frac{4}{3} - \sin^{-1}\frac{2}{5}$

Q2 (cont'd)

2(e)

(i) Let $z = r\,\text{cis}\,\theta$,

then $\bar{z} = r\,\text{cis}(-\theta)$

$$\therefore \quad \frac{z}{\bar{z}} = \frac{r\,\text{cis}\,\theta}{r\,\text{cis}(-\theta)}$$

$$= \text{cis}\,2\theta$$

$$\therefore \arg\left(\frac{z}{\bar{z}}\right) = 2\arg z.$$

✓

(ii) from (i) $\arg\left(\frac{z}{\bar{z}}\right) = 2\arg z$

$$\arg\left(\frac{z-1}{\overline{z-1}}\right) = 2\arg(z-1)$$

✓

Since given $\frac{z}{\bar{z}} = -\frac{z-1}{\overline{z-1}}$

$$\therefore \arg\left(\frac{z}{\bar{z}}\right) = \arg\left(-\frac{z-1}{\overline{z-1}}\right)$$

$$= \arg(-1) + \arg\left(\frac{z-1}{\overline{z-1}}\right)$$

✓

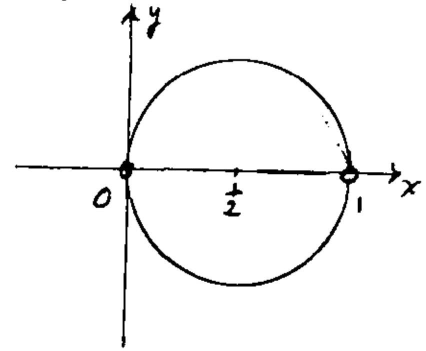
$$2\arg z = \pi + 2\arg(z-1) \text{ or } -\pi + 2\arg(z-1)$$

$$\therefore \arg z = \arg(z-1) + \frac{\pi}{2} \quad \text{or} \quad \arg z = \arg(z-1) - \frac{\pi}{2}$$

(iii) From (ii) $\arg z - \arg(z-1) = \pm\frac{\pi}{2}$

$$\arg\left(\frac{z}{z-1}\right) = \pm\frac{\pi}{2}$$

$\therefore$ Locus of $P(z)$ is

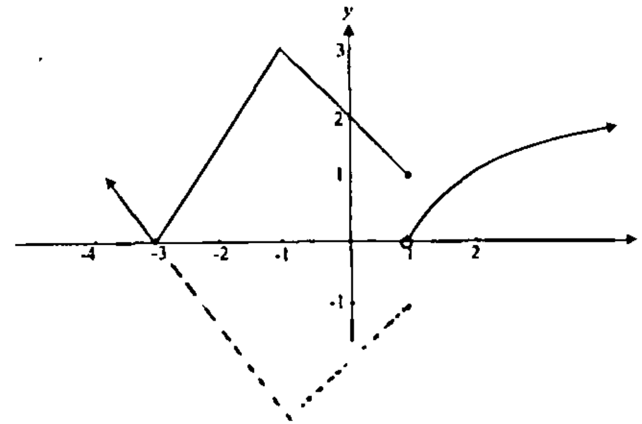


✓ 2 semi-circles

✓ excluding $z = 0$ and $z = 1$.

Question 3

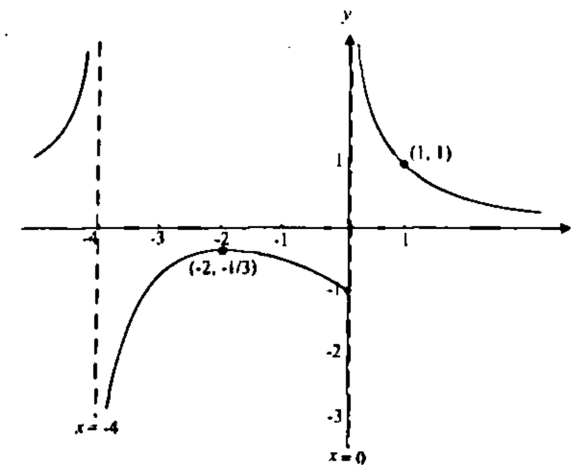
The graph of $y = |f(x-1)|$ is shown below.



✓ for $y = f(x-1)$

✓ for $|f(x-1)|$

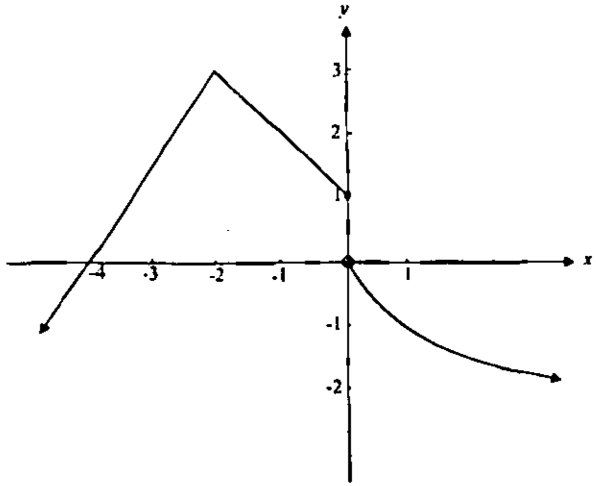
(ii) The graph of $y = \frac{1}{f(x)}$ is shown below.



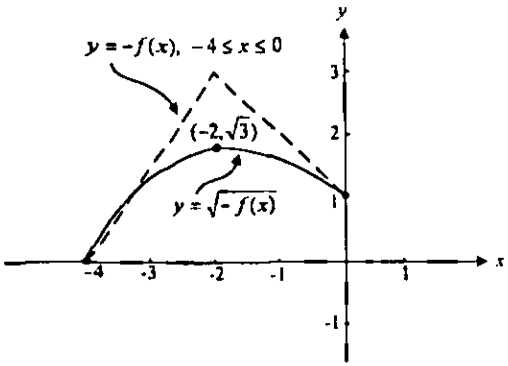
✓ for asymptotes

✓ for correct end points.

(iii) The graph of $y = -f(x)$ is shown below.

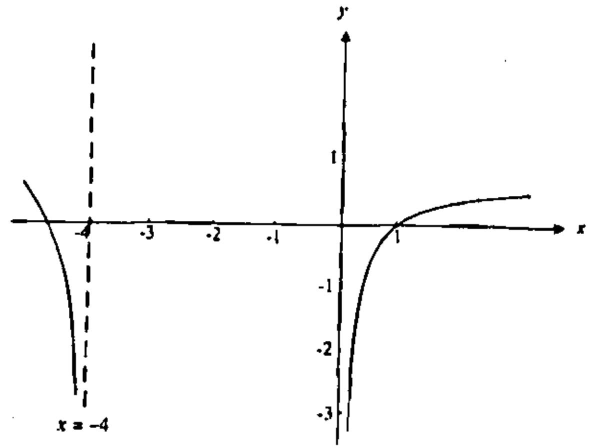


Only those values in the domain of $y = -f(x)$ for which $y \geq 0$ are included in the graph of $y = \sqrt{-f(x)}$. That is, only values of x where $-4 \leq x \leq 0$ are included.
For values of $y = -f(x)$ that are less than 1, $\sqrt{-f(x)} > -f(x)$.
For values of $y = -f(x)$ that are greater than 1, $\sqrt{-f(x)} < -f(x)$.
Note that $\sqrt{-f(0)} = 1$ and $\sqrt{-f(-4)} = 0$.
The graph of $y = -f(x)$ for $-4 \leq x \leq 0$ together with the graph of $y = \sqrt{-f(x)}$ are shown below.



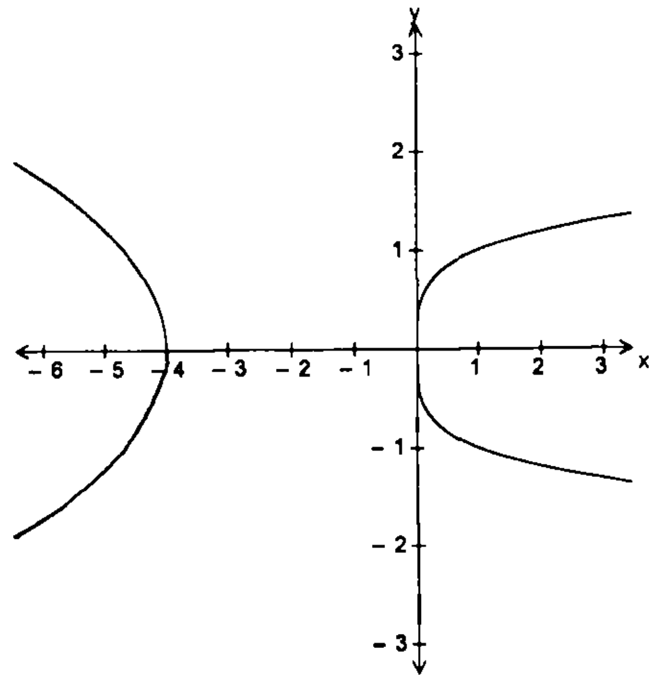
✓ for correct shape
✓ for correct curvature above & below $y=1$

(iv) The graph of $y = \ln(f(x))$ only exists for $f(x) > 0$, that is, for $x < -4$ and $x > 0$.
The graph of $y = \ln(f(x))$ is shown below.



✓ for asymptote
✓ for correct shape

(v) $y^2 = f(x)$



✓ for $y = \sqrt{f(x)}$
✓ for the other half

Q3 (b)

(i) Since complex roots of any polynomial equation with real coefficients occur in conjugate pairs.

$\therefore$ if $z = 3-i$ is a root, then $\bar{z} = 3+i$ must also be a root.

(ii) Let the other root be β.

$\therefore$ Roots are $3-i, 3+i, \beta$.

Sum of roots $(3-i)+(3+i)+\beta = 0$

$$6+\beta = 0$$

$$\therefore \beta = -6$$

$\therefore$ Roots are $3-i,\ 3+i,\ -6$.

(iii) Since $P(-6) = 0$

$$\therefore (-6)^3 + k(-6) + 60 = 0$$

$$-216 - 6k + 60 = 0$$

$$\therefore k = -26$$

(c)

$$2z^3 = 9+3\sqrt{3}\,i$$

$$= 6\sqrt{3}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

$$= 6\sqrt{3}\ \text{cis}\,\frac{\pi}{6}$$

$$z^3 = 3\sqrt{3}\ \text{cis}\,\frac{\pi}{6}$$

$$\therefore z = \sqrt{3}\ \text{cis}\left(\frac{\frac{\pi}{6}+2k\pi}{3}\right) \qquad k = 0, 1, 2$$

$$\therefore r = \sqrt{3}$$

smallest value of $\theta = \frac{\pi}{18}$

Question 4

(a) (i) gradient of the line perpendicular to the asymptote is $-\frac{a}{b}$

$\therefore$ Equation of the line is

$$y - 0 = -\frac{a}{b}(x - ae)$$

$$y = -\frac{ax}{b} + \frac{a^2e}{b}$$

$$by = -ax + a^2e$$

ie $ax + by - a^2e = 0 \qquad (1)$

(ii) when (1) meets $y = \frac{bx}{a}$

$$ax + b\left(\frac{bx}{a}\right) - a^2e = 0$$

$$(a^2+b^2)x - a^3e = 0$$

$$x = \frac{a^3e}{a^2+b^2}$$

$$= \frac{a^3e}{a^2+a^2e^2-a^2} \qquad \because b^2 = a^2(e^2-1)$$

$$= \frac{a}{e}$$

$\therefore$ They meet at the point where $x = \frac{a}{e}$ which lies on the corresponding directrix.

Q4 (b) $x^4+3xy-2y^2+13=0$

$4x^3+3y+3xy'-4yy'=0$

$\therefore (3x-4y)y' = -(4x^3+3y)$

$y' = -\dfrac{4x^3+3y}{3x-4y}$ ✓

$\therefore$ gradient at $(-1,2) = -\dfrac{4(-1)+3(2)}{3(-1)-4(2)}$

$= \dfrac{2}{11}$

$\therefore$ gradient of normal $= -\dfrac{11}{2}$ ✓

$\therefore$ Equation of normal is

$y-2 = -\frac{11}{2}(x+1)$

$2y-4 = -11x-11$

$11x+2y+7=0$ ✓

(c) (i) $xy=c^2$

$\therefore y = \dfrac{c^2}{x}$

$y' = -\dfrac{c^2}{x^2}$

$\therefore$ at Q $y' = -\dfrac{c^2}{c^2q^2}$

$= -\dfrac{1}{q^2}$ ✓

$\therefore$ Equation of the tangent at Q is

$y-\dfrac{c}{q} = -\dfrac{1}{q^2}(x-cq)$

$q^2y-cq = -x+cq$

ie $x+q^2y=2cq$ (1) ✓

(ii) Similarly, tangent at P is

$x+p^2y=2cp$ (2)

(1)-(2) $(q^2-p^2)y = 2c(q-p)$

$\therefore y = \dfrac{2c(q-p)}{q^2-p^2}$

$= \dfrac{2c}{p+q}$ ✓

Put into (1) $x+\dfrac{2cq^2}{p+q} = 2cq$

$\therefore x = \dfrac{2cpq}{p+q}$ ✓

$\therefore T$ is $\left(\dfrac{2cpq}{p+q}, \dfrac{2c}{p+q}\right)$

(iii) At T $x = \dfrac{2cpq}{p+q}$ (3)

$y = \dfrac{2c}{p+q}$ (4)

4(c) (cont'd)

(3)/(4) $\frac{x}{y} = pq$ ✓

$= k$ (given)

$\therefore \quad x = ky$

$\therefore$ Locus of T is the straight line $x = ky$

or $y = \frac{x}{k}$ ✓

(d) Equation of RS :

$$\frac{x\cos\theta}{\sqrt{2}} + \frac{y\sin\theta}{2\sqrt{2}} = 1$$

$\therefore$ At R $\quad y = 0$.

$\therefore x = \frac{\sqrt{2}}{\cos\theta}$ $\qquad R\left(\frac{\sqrt{2}}{\cos\theta}, 0\right)$

At S, $\quad x = 0$

$y = \frac{2\sqrt{2}}{\sin\theta}$ $\qquad S\left(0, \frac{2\sqrt{2}}{\sin\theta}\right)$

$\therefore$ Area of $\Delta ORS = \frac{1}{2}\, OR \times OS$

$= \frac{1}{2} \cdot \frac{\sqrt{2}}{\cos\theta} \cdot \frac{2\sqrt{2}}{\sin\theta}$

$= \frac{4}{2\sin\theta\cos\theta}$

$= \frac{4}{\sin 2\theta}$ ✓

(ii) ΔORS is minimum when $\sin 2\theta = 1 \quad \therefore \theta = \frac{\pi}{4}$ ✓

ie $\quad \sin\theta = \cos\theta = \frac{1}{\sqrt{2}}$

$\therefore P$ is $\left(\sqrt{2} \cdot \frac{1}{\sqrt{2}},\ 2\sqrt{2} \cdot \frac{1}{\sqrt{2}}\right)$ ✓

$= (1, 2)$.

<u>Question 5</u>

(ii) $\angle PNC + \angle PLC = 90° + 90°$ (given)

$= 2$ rt $\angle$s ✓

$\therefore PNCL$ is a cyclic quad (opp $\angle$'s of cyclic quad are supplementary)

(iii) $\angle PBM = \angle PBA$

$= \angle PCA$ ($\angle$s in same segment)

$= \angle PCL$

$= \angle PNL$ $\angle$'s in same segment (PNCL is a cyclic quad)

$= \angle PNM$

(iv) (α) $\angle PBM = \angle PNM$ (proved) ✓

$\therefore$ PBNM is a cyclic quad

($\angle$'s subtended by same arc in same segment are equal) ✓

(β) $\angle PMB = \angle PNB$ ($\angle$'s in same segment are equal) (PBNM is a cyclic quad)

$= 90°$ ($\angle PNB = 90°$, given)

$\therefore PM \perp AB$ ✓

Q 5 (b)

Let the roots be $\alpha-d,\ \alpha,\ \alpha+d$.

$\therefore$ sum of roots: $\alpha-d+\alpha+\alpha+d=-p$

$$\therefore\quad p=-3\alpha \qquad (1)$$

Sum of root taken 2 at a time:

$$\alpha(\alpha-d)+\alpha(\alpha+d)+(\alpha-d)(\alpha+d)=q$$
$$3\alpha^2-d^2=q$$
$$q=3\alpha^2-d^2 \qquad (2)$$

Product of roots

$$\alpha(\alpha-d)(\alpha+d)=-r$$
$$\alpha^3-\alpha d^2=-r$$
$$r=\alpha d^2-\alpha^3 \qquad (3)$$

$$\therefore\ 2p^3+27r-9pq$$
$$=2(-3\alpha)^3+27(\alpha d^2-\alpha^3)-9(-3\alpha)(3\alpha^2-d^2)$$
$$=-54\alpha^3+27\alpha d^2-27\alpha^3+81\alpha^3-27\alpha d^2$$
$$=0$$

(c) (i) $x^4+4\sqrt{3}x^3-6x^2-4\sqrt{3}x+1=0$

$$\therefore\ 1-6x^2+x^4=4\sqrt{3}x-4\sqrt{3}x^3$$
$$\therefore\ \frac{(4x-4x^3)\sqrt{3}}{1-6x^2+x^4}=1$$

$$\frac{4x-4x^3}{1-6x^2+x^4}=\frac{1}{\sqrt{3}}$$

Let $x=\tan\theta$

$$\therefore\ \frac{4\tan\theta-4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}=\frac{1}{\sqrt{3}}$$
$$\tan 4\theta=\frac{1}{\sqrt{3}}$$

$$\therefore\ 4\theta=\frac{\pi}{6},\ \frac{7\pi}{6},\ \frac{13\pi}{6},\ \frac{19\pi}{6},\ \frac{25\pi}{6}\ \ldots$$
$$\theta=\frac{\pi}{24},\ \frac{7\pi}{24},\ \frac{13\pi}{24},\ \frac{19\pi}{24},\ \frac{25\pi}{24},\ \ldots$$

$\therefore$ solutions are $x=\tan\frac{\pi}{24},\ \tan\frac{7\pi}{24},\ \tan\frac{13\pi}{14},\ \tan\frac{19\pi}{24}$

[since $\tan\frac{25\pi}{24}=\tan\frac{\pi}{24}$ etc...]

(ii) (α) Sum of roots:

$$\tan\frac{\pi}{24}+\tan\frac{7\pi}{24}+\tan\frac{13\pi}{24}+\tan\frac{19\pi}{24}=-4\sqrt{3}$$
$$\tan\frac{\pi}{24}+\tan\frac{7\pi}{24}+\tan\left(\pi-\frac{11\pi}{24}\right)+\tan\left(\pi-\frac{5\pi}{24}\right)=-4\sqrt{3}$$
$$\tan\frac{\pi}{24}+\tan\frac{7\pi}{24}-\tan\frac{11\pi}{24}-\tan\frac{5\pi}{24}=-4\sqrt{3}$$
$$\therefore\ \tan\frac{\pi}{24}+\tan\frac{7\pi}{24}+4\sqrt{3}=\tan\frac{5\pi}{24}+\tan\frac{11\pi}{24}$$

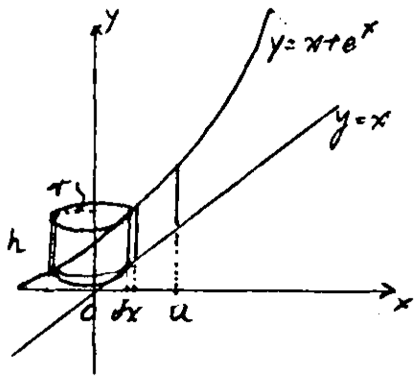
(β) product of roots

$$\tan\frac{\pi}{24}\cdot\tan\frac{7\pi}{24}\left(-\tan\frac{5\pi}{24}\right)\left(-\tan\frac{11\pi}{24}\right)=1$$

ie $\tan\frac{5\pi}{24}\tan\frac{7\pi}{24}=\dfrac{1}{\tan\frac{\pi}{24}\tan\frac{11\pi}{24}}$

$$=\cot\frac{\pi}{24}\cot\frac{11\pi}{24}$$

Question 6



(i) $r = x$

$h = (x+e^x) - x$

$= e^x$ ✓

$\therefore$ Volume of the cylindrical shell

$\delta V = 2\pi r h\, \delta x$

$= 2\pi x e^x \delta x$ ✓

$\therefore$ Volume $= \int dV$

$= \int_0^u 2\pi x e^x dx$

$= 2\pi \int_0^u x e^x dx$ ✓

(ii) $V = 2\pi \int_0^u x e^x dx$

$= 2\pi \left[x e^x\right]_0^u - 2\pi \int_0^u e^x dx$

$= 2\pi u e^u - 2\pi \left[e^x\right]_0^u$

$= 2\pi u e^u - 2\pi e^u + 2\pi$ ✓

$= 2\pi(u e^u - e^u + 1)$ (1)

since $y = 2$ when $x = u$ on $y = x + e^x$

$\therefore \quad u + e^u = 2$

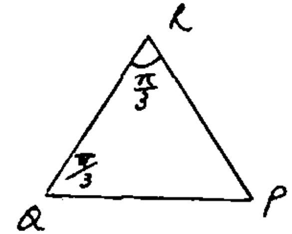
$e^u = 2 - u$ (2) ✓

$\therefore$ put (2) into (1)

$V = 2\pi\left[u(2-u) - (2-u) + 1\right]$

$= 2\pi\left(2u - u^2 - 2 + u + 1\right)$

$= 2\pi\left(3u - u^2 - 1\right)$ ✓



(b) (i) $PQ = 2y$

$\therefore QR = PQ = PR = 2y$

$\therefore \triangle PQR = \frac{1}{2} PQ \cdot QR \sin\frac{\pi}{3}$

$= \frac{1}{2} \cdot 2y \cdot 2y \cdot \frac{\sqrt{3}}{2}$

$= \sqrt{3}\, y^2$ ✓

(ii) $\triangle PQR = \sqrt{3} \cdot 16\left(1 - \frac{x^2}{25}\right)$

$= \frac{16\sqrt{3}(25 - x^2)}{25}$ ✓

(iii) Volume of the slice

$\delta V = \frac{16\sqrt{3}(25 - x^2)}{25} \delta x$

$\therefore V = \frac{16\sqrt{3}}{25} \int_{-5}^{5} 25 - x^2\, dx$ ✓

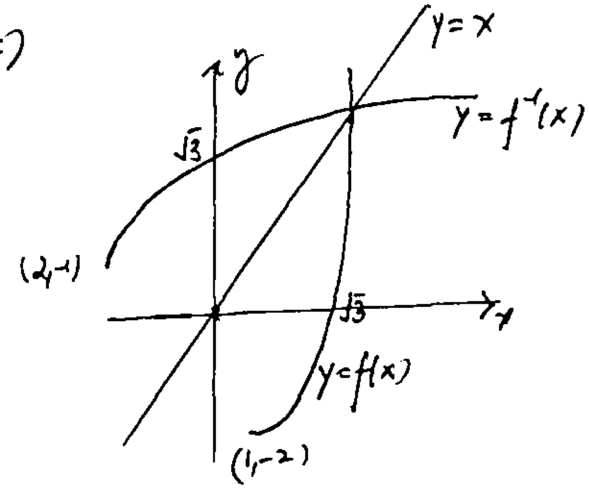
$= \frac{32\sqrt{3}}{25} \int_0^5 25 - x^2\, dx$

$= \frac{32\sqrt{3}}{25} \left[25x - \frac{x^3}{3}\right]_0^5$

$= \frac{320\sqrt{3}}{3}$ unit^3 ✓

Q6 (c)

(i)



✓ for end-pt & intercept

✓ for correct shape

(ii) At pt of intersection,

$$x = f(x)$$

$$x = x^3 - 3x$$

$$\therefore x^3 - 4x = 0$$

$$x(x^2-4) = 0$$

$$x(x-2)(x+2)$$

$$\therefore x = 0,\ x = 2,\ x = -2$$

but domain of $y = f(x)$ is $x \geq 1$

$$\therefore x = 2$$

$\therefore$ pt of intersection is $(2,2)$ ✓

(iii) Area $= 2\left[\frac{1}{2}\times 2\times 2 - \int_{\sqrt{3}}^{2} x^3 - 3x\,dx\right]$

$$= 2\left[2 - \left[\frac{x^4}{4} - \frac{3x^2}{2}\right]_{\sqrt{3}}^{2}\right] \quad ✓$$

$$= \frac{7}{2} \text{ unit}^2 \quad ✓$$

Question 7

(a) $\tan^{-1}(4x) - \tan^{-1}(3x) = \tan^{-1}\left(\frac{1}{7}\right)$

$$\tan\left(\tan^{-1}(4x) - \tan^{-1}(3x)\right) = \frac{1}{7} \quad ✓$$

$$\frac{4x - 3x}{1 + (4x)(3x)} = \frac{1}{7} \quad ✓$$

$$7x = 1 + 12x^2$$

$$\therefore 12x^2 - 7x + 1 = 0$$

$$(3x-1)(4x-1) = 0$$

$$\therefore x = \frac{1}{3} \text{ or } x = \frac{1}{4} \quad ✓$$

(b) (i) when $n=1$ $\quad 5^1 - 3^1 = 2$

when $n=2$ $\quad 5^2 - 3^2 = 16$

$\therefore u_n = 5^n - 3^n$ is true for $n = 1, 2$. ✓

Assume it is true for $n = k-1, k-2$.

ie $u_{k-1} = 5^{k-1} - 3^{k-1}$ ✓

$u_{k-2} = 5^{k-2} - 3^{k-2}$

$$\therefore u_k = 8u_{k-1} - 15u_{k-2}$$

$$= 8(5^{k-1} - 3^{k-1}) - 15(5^{k-2} - 3^{k-2})$$

$$= 40\cdot 5^{k-2} - 24\cdot 3^{k-2} - 15\cdot 5^{k-2} + 15\cdot 3^{k-2}$$

$$= 25\cdot 5^{k-2} - 9\cdot 3^{k-2} \quad ✓$$

$$= 5^k - 3^k$$

$\therefore$ It will be true for $n=k$ if it is true for $n=k-1$, $n=k-2$ ✓

Since it is proven true for $n=1, 2$, $\therefore$ it will be true for $n=3, 4, 5, \ldots$ ie true for all positive integer n.

(ii) $u_1+u_2+u_3+\cdots+u_n$

$$= (5-3)+(5^2-3^2)+(5^3-3^3)+\cdots+(5^n-3^n)$$

$$= (5+5^2+5^3+\cdots+5^n)-(3+3^2+3^3+\cdots+3^n) \quad ✓$$

$$= \frac{5(5^n-1)}{5-1} - \frac{3(3^n-1)}{3-1}$$

$$= \frac{5^{n+1}-5}{4} - \frac{3^{n+1}-3}{2}$$

$$= \frac{5^{n+1}-5-2\times 3^{n+1}+6}{4}$$

$$= \frac{5^{n+1}-2\times 3^{n+1}+1}{4} \quad ✓$$

(c)(i) $I_n = \int_0^1 x(1-x)^n\,dx$

$$= \left[\frac{x^2}{2}(1-x)^n\right]_0^1 - \int_0^1 \frac{x^2}{2}\,d(1-x)^n$$

$$= \frac{n}{2}\int_0^1 x^2(1-x)^{n-1}\,dx$$

$$2I_n = n\int_0^1 x^2(1-x)^{n-1}\,dx$$

$$= n\int_0^1 x\left[1-(1-x)\right](1-x)^{n-1}\,dx \quad ✓$$

$$= n\int_0^1 x(1-x)^{n-1}\,dx - n\int_0^1 x(1-x)^n\,dx$$

$$= n\,I_{n-1} - n\,I_n$$

$$(n+2)I_n = n\,I_{n-1}$$

$$\therefore I_n = \frac{n}{n+2}\,I_{n-1} \quad ✓$$

(ii) $I_0 = \int_0^1 x\,dx = \frac{1}{2}$

$$I_1 = \frac{1}{3}I_0$$

$$I_2 = \frac{2}{4}I_1$$

$$I_3 = \frac{3}{5}I_2$$

$$\vdots \qquad \vdots$$

$$I_{n-1} = \frac{n-1}{n+1}I_{n-2}$$

$$I_n = \frac{n}{n+2}I_{n-1}$$

multiplying

$$I_n = \frac{n(n-1)(n-2)\cdots 1}{(n+2)(n+1)\cdots 5\cdot 4\cdot 3}\cdot\frac{1}{2} \quad ✓$$

ie $I_n = \dfrac{n!\,2!}{(n+2)!\quad 2}$

$= \dfrac{1}{2\ {}^{n+2}C_2}$ ✓

(iii) $I_n \leq \dfrac{1}{132}$

$\dfrac{1}{2\ {}^{n+2}C_2} \leq \dfrac{1}{132}$

$66 \leq {}^{n+2}C_2$ ✓

ie $\dfrac{(n+2)(n+1)}{2} \geq 66$

$n^2+3n+2 \geq 132$

$n^2+3n-130 \geq 0$

$(n-10)(n+13) \geq 0$

$\therefore\ n \leq -13$ or $n \geq 10$

$\therefore\ n \geq 10$ ✓

Question 8

(a) $P(x) = \dfrac{x^4}{4} - \dfrac{x^3}{3} + x^2 - 2x + p$

(i) $P'(x) = x^3 - x^2 + 2x - 2$

$= x^2(x-1) + 2(x-1)$

$= (x^2+2)(x-1)$

when $P'(x) = 0$

$(x^2+2)(x-1) = 0$

$x = 1 \qquad x^2+2 \neq 0.$ ✓

$\therefore$ There is only one turning point. ✓

(ii) Since $P(x)$ is a quartic polynomial,

& $P''(x) = 3x^2 - 2x + 2$

$P''(1) = 3 > 0$

$\therefore$ The turning pt is a minimum turning pt. ✓

The equation $P(x) = 0$ will have no real roots if the graph of $y = P(x)$ does not touch or cut the x-axis, ie when $P(1) > 0$

ie $\frac{1}{4} - \frac{1}{3} + 1 - 2 + p > 0$

$p > \dfrac{13}{12}$ ✓

Q 8 (b)

(i) $a = -v\sqrt{1-v^2}$

$\therefore v\dfrac{dv}{dx} = -v\sqrt{1-v^2}$ ✓

$$\int_V^v \frac{dv}{\sqrt{1-v^2}} = -\int_0^x dx$$

$$\left[\sin^{-1} v\right]_V^v = -x$$ ✓

$\therefore x = \sin^{-1} V - \sin^{-1} v$

Let $\alpha = \sin^{-1} V$, $\beta = \sin^{-1} v$ $\quad$ $\sin\alpha = V$, $\sin\beta = v$; $\cos\alpha = \sqrt{1-V^2}$, $\cos\beta = \sqrt{1-v^2}$ ✓

$x = \alpha - \beta$

$\sin x = \sin(\alpha - \beta)$

$= \sin\alpha\cos\beta - \sin\beta\cos\alpha$

$= V\sqrt{1-v^2} - v\sqrt{1-V^2}$

$\therefore x = \sin^{-1}\left[V\sqrt{1-v^2} - v\sqrt{1-V^2}\right]$ ✓

(ii) $a = \dfrac{dv}{dt} = -v\sqrt{1-v^2}$

$$\int_V^v \frac{dv}{v\sqrt{1-v^2}} = -\int_0^t dt$$ ✓

$$\therefore \left[\ln v - \ln\left(1+\sqrt{1-v^2}\right)\right]_V^v = -t$$

$\therefore t = \left[\ln V - \ln\left(1+\sqrt{1-V^2}\right)\right] - \left[\ln v - \ln\left(1+\sqrt{1-v^2}\right)\right]$ ✓

$= \ln\dfrac{V}{v} + \ln\dfrac{1+\sqrt{1-v^2}}{1+\sqrt{1-V^2}}$

$= \ln\dfrac{V\left(1+\sqrt{1-v^2}\right)}{v\left(1+\sqrt{1-V^2}\right)}$ ✓

(c) (i) $g(x) = \ln(1+x) - x$

$g'(x) = \dfrac{1}{1+x} - 1$

$= -\dfrac{x}{1+x} < 0$ $\quad$ since $x > 0$ ✓

$\therefore g(x)$ is a decreasing function for all $x > 0$

(ii) $\therefore g(x) > g(0)$ $\quad$ for all $x > 0$ ✓

$\therefore \ln(1+x) - x > 0$

$\ln(1+x) > x$ $\quad$ $\forall x > 0$

(iii) $f(x) = \ln(1+x) - \dfrac{x}{1+x}$

$= \ln(1+x) - \left[1 - \dfrac{1}{1+x}\right]$

$f'(x) = \dfrac{1}{1+x} - \dfrac{1}{(1+x)^2}$

8(c) (cont'd)

$$= \frac{1+x-1}{(1+x)^2}$$

$$= \frac{x}{(1+x)^2} > 0 \qquad \forall x > 0$$

$\therefore f(x)$ is an increasing function of x for all $x>0$

$\therefore f(x) > f(0)$ for all $x>0$ ✓

$$\ln(1+x) - \frac{x}{1+x} > 0$$

$$\ln(1+x) > \frac{x}{1+x}$$

but from (c)(ii)

$$x > \ln(1+x)$$

$$\therefore \frac{x}{1+x} < \ln(1+x) < x$$

✓