GF

Name:

Class: 12MTZ1

Teacher: MR FARDOULY

CHERRYBROOK TECHNOLOGY HIGH SCHOOL

2006 AP4

YEAR 12 TRIAL HSC EXAMINATION

MATHEMATICS EXTENSION 2

Time allowed - 3 HOURS
(Plus 5 minutes' reading time)

DIRECTIONS TO CANDIDATES:

- Attempt all questions.
- All questions are of equal value.
- Each question is to be commenced on a new page clearly marked Question 1, Question 2, etc on the top of the page. **
- Each question is to be returned in a separate bundle.
- All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used.
- Your solutions will be collected in one bundle stapled in the top left corner.

Please arrange them in order, Q1 to 8.

Each page must show your name and your class. *

Question 1 **Marks**

(a) Find $\int x\sin(x^2+3)dx$ **2**

(b) Find $\int \frac{dt}{\sqrt{7+6t-t^2}}$. **2**

(c) Using the substitution $t=\tan\frac{\theta}{2}$, find $\int \frac{2}{4+3\sin\theta}d\theta$. **3**

(d) (i) Show that $\frac{1}{(x^2+3)(x^2+1)}=\frac{1}{2}\left[\frac{1}{x^2+1}-\frac{1}{x^2+3}\right]$ **2**

(ii) Hence evaluate $\int_0^1 \frac{dx}{(x^2+3)(x^2+1)}$ **2**

(e) If $I_m=\int_0^k (k^2-x^2)^m dx$, for $m\geq 1$, show that $I_m=\frac{2k^2m}{2m+1}.I_{m-1}$. **4**

(Hint: $\frac{x^2}{k^2-x^2}=\frac{k^2}{k^2-x^2}-1$)

Question 2 (Begin a new page)

(a) Simplify $\frac{(\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12})(\cos\frac{3\pi}{12}+i\sin\frac{3\pi}{12})}{\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}}$ **2**

(b) If $z=\sqrt{3}+i$, find z^4, writing the answer in modulus-argument form. **2**

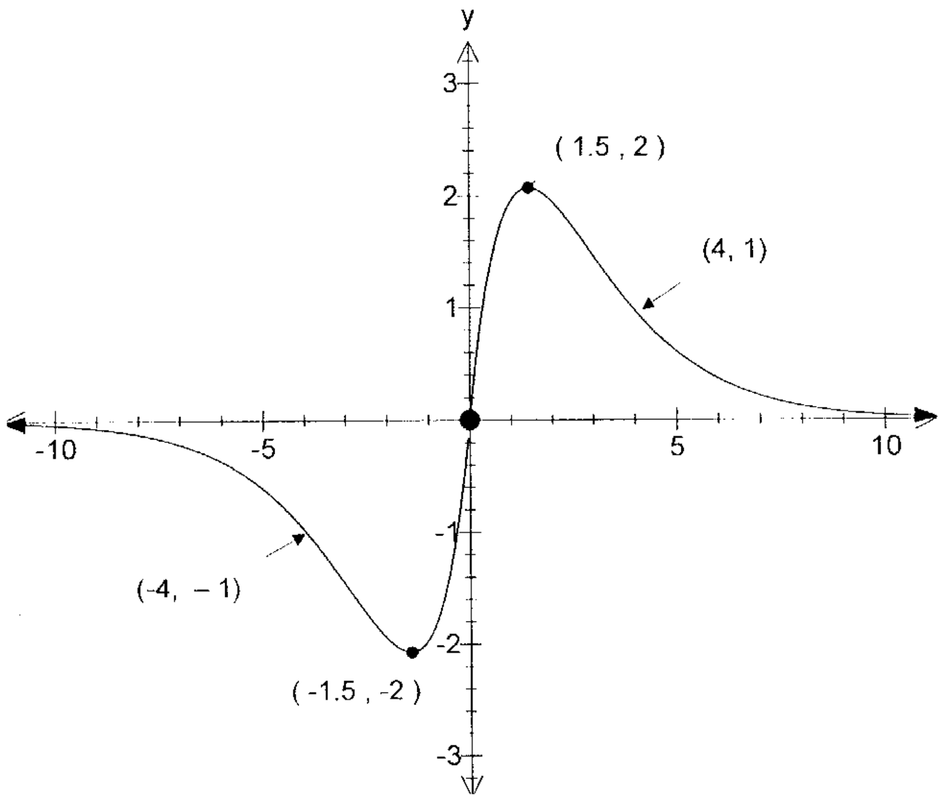
(c) The equation $z^2-(a+bi)z-6i=0$, where a and b are real, has roots α and β such that $\alpha^2+\beta^2=5$.

(i) Show that $\alpha^2-\beta^2=5$ and $\alpha\beta=-6$. **2**

(ii) Hence find the values of a and b. **2**

Question 3 (Begin a new page) **Marks**

(a) The diagram shows the graph of $y = f(x)$



Draw separate sketches of the following:

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = [f(x)]^2$ **2**

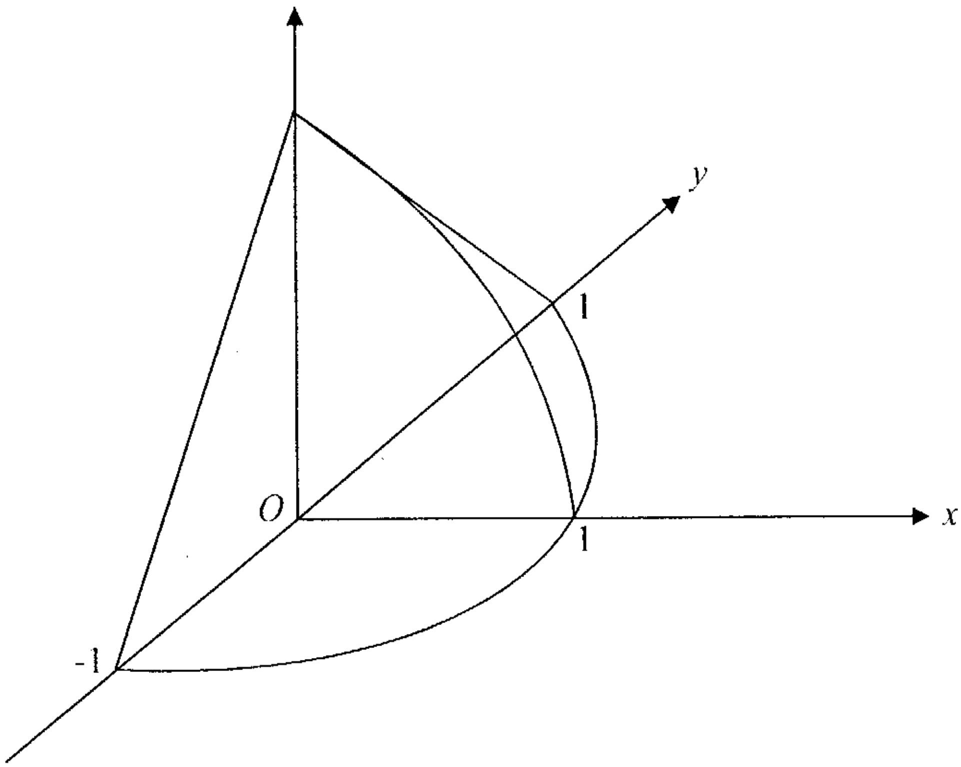
(iii) $y = f'(x)$ **2**

(iv) $y = \int f(x)dx$, if $x = 0$ when $y = 0$ **2**

(v) $y = x + f(x)$ **2**

(Question 3 continued)

(b) **Marks**



The base of a solid is the semi-circular region in the x - y plane with the straight edge running from the point $(0,-1)$ to the point $(0,1)$ and the point $(1,0)$ on the curved edge of the semicircle. Each cross-section perpendicular to the x-axis is an isosceles triangle with each of the two equal sidelengths three quarters the length of the third side.

(i) Show that the area of the triangular cross-section at $x = a$ is $\frac{\sqrt{5}}{2}\left(1 - a^2\right)$. **3**

(ii) Find the volume of the solid. **2**

<u>Question 4</u> (Begin a new page)

(a) If p, q and r are the roots of the equation $x^3 + 4x^2 - 3x + 1 = 0$, find the equation whose roots are $\frac{1}{p}, \frac{1}{q}$ and $\frac{1}{r}$. **2**

 Marks

(b) (i) Let k be a zero of the polynomial $F(x)$ and also of its derivative $F'(x)$.
Prove that k is a zero of $F(x)$ of multiplicity at least 2. **3**

(ii) Show that $y = 1$ is a double root of the equation $y^{2t} - ty^{t+1} = 1 - ty^{t-1}$, where t is a positive integer. **2**

(c) (i) Determine the complex roots of $z^5 = 1$. **2**

(ii) Hence factorise $z^5 - 1$ over the

(α) Complex field **2**

(β) Real field. **2**

(d) Show that for the complex number $z = \dfrac{1 - t^2 + 2it}{1 + t^2}$, $|z| = 1$ for all values of t. **2**

Question 5 (Begin a new page)

(a) (i) The hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, where $0 < a < b$, has eccentricity e. S is the focus of the hyperbola on the positive x-axis and the line through S perpendicular to the x-axis intersects the hyperbola at P and Q.

(i) Show that $PQ = \dfrac{2b^2}{a}$. **2**

(ii) If P and Q have coordinates (9, 24) and (9, -24) respectively, show that $a = 3$ and $b = 6\sqrt{2}$. **3**

(iii) For these values of a and b, sketch the graph of the hyperbola showing clearly the x-intercepts, the coordinates of the foci, and the equations of the directrices and asymptotes. **4**

(b) An ellipse can be described as the locus of a point moving so that the sum of its distances from two fixed points (foci) is constant.

(i) If the two fixed points are A(-4, 0) and B(4, 0) and the sum of the distances of $P(x, y)$ from these points is 10 units, show that the equation of the ellipse is $\frac{x^2}{25}+\frac{y^2}{9}=1$. **2**

(ii) Show that the ellipse can be represented parametrically by the equations $x = 5\cos\theta$ and $y = 3\sin\theta$, and find the equation of the tangent to the ellipse at the point where $\theta = \frac{\pi}{6}$. **4**

Question 6 (Begin a new page)

(a) A body is projected vertically upwards from the ground with initial velocity v_0 in a medium that produces a resistance force per unit mass of kv^2, where v is the velocity and k is a positive constant. Taking acceleration due to gravity as $g\ ms^{-2}$,

(i) Prove that the maximum height H of the body above the ground is $H = \frac{1}{2k}\log_e\left(1+\frac{kv_0^2}{g}\right)$. **4**

(ii) Show that in order to double the maximum height reached, the initial velocity must be increased by a factor of $\left(e^{2kH}+1\right)^{\frac{1}{2}}$. **4**

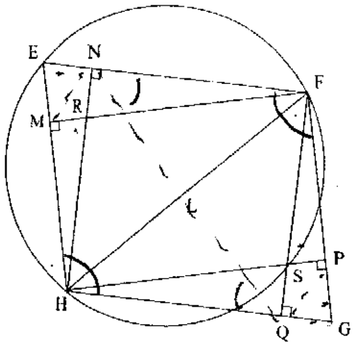
(b) A body of mass m kg is moving in a horizontal straight line. At time t seconds it has displacement x metres from a fixed point O on the line and velocity v ms^{-1} and acceleration a ms^{-2}. If the body is initially at O with velocity V ms^{-1}, and $a = -\frac{1}{10}\sqrt{v}\left(1+\sqrt{v}\right)$ms^{-2},

(i) Show that $t = 20\log_e\left(\frac{1+\sqrt{V}}{1+\sqrt{v}}\right)$ **3**

(ii) Find the distance travelled before the body comes to rest. **4**

Question 7 (Begin a new page) **Marks**

(a) The vertices E, F and H of the parallelogram EFGH lie on a circle. L is the midpoint of the diagonal FH. R is the point of intersection of the altitudes HN and FM in the triangle EFH. S is the point of intersection of the altitudes HP and FQ in the triangle FGH. If S lies on the circle,



(i) Prove that the points R, L and S are collinear. **4**

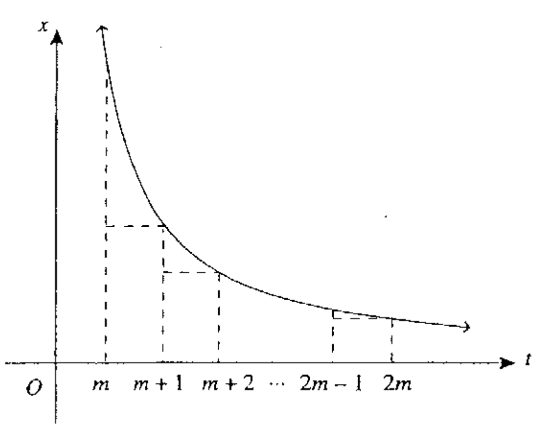
(ii) Show that the hexagon MNFPQH is cyclic. **1**

(b) (i) Prove that $\dfrac{1}{2p+1}+\dfrac{1}{2p+2}>\dfrac{1}{p+1}$, for all $p>0$ **2**

(ii) Prove the following statement by mathematical induction **4**

$$\frac{1}{m+1}+\frac{1}{m+2}+\dots\dots+\frac{1}{2m}\geq\frac{37}{60},\text{ for } m\geq 3.$$

(iii) The diagram below shows the graph of $x=\dfrac{1}{t}$, for $t>0$.



(α) By comparing areas, show that $\displaystyle\int_m^{m+1}\frac{1}{t}\,dt>\frac{1}{m+1}$. **1**

(β) Hence, without using a calculator, show that $\log_e 2>\dfrac{37}{60}$. **3**

Question 8 (Begin a new page) **Marks**

(a) Given a, b and c are three non negative numbers, show that the arithmetic mean is greater than or equal to the geometric mean. **3**

(b) Polynomial $P(x)$ gives remainders -2 and 1 when divided by $2x-1$ and $x-2$ respectively. What is the remainder when $P(x)$ is divided by $2x^2-5x+2$? **3**

(c) The equation $ax^4+bx^3+cx^2+dx+e=0$ has a quadruple root α.

(i) Find α in terms of a and b. **2**

(ii) Hence, show that $\left(1+\dfrac{b}{4a}\right)^4=\dfrac{a+b+c+d+e}{a}$. **2**

(d) (i) If $n=-1$, show that $\displaystyle\int_1^e x^n \log x\,dx=\frac{1}{2}$ **2**

(ii) If $n\neq -1$, show that $\displaystyle\int_1^e x^n \log x\,dx=\frac{ne^{n+1}+1}{(n+1)^2}$ **3**

END OF TEST

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

APY EXTENSION 2 SOLUTIONS 2006

Question 1

(a) Let $u = x^2 + 3$

$du = 2x\,dx$

$\int x \sin(x^2+3)\,dx =$

$\frac{1}{2}\int \sin u\,du$ ✓

$= -\frac{1}{2}\cos u + c$

$= -\frac{1}{2}\cos(x^2+3) + c$ ✓

(b) $\int \frac{dt}{\sqrt{16-(t-3)^2}}$ ✓

$= \sin^{-1}\left(\frac{t-3}{4}\right) + c$ ✓

(c) $\frac{dt}{d\theta} = \frac{1}{2}\sec^2\frac{\theta}{2}$

$d\theta = \frac{2\,dt}{1+t^2}$ ✓

$\therefore \int \frac{2}{4+3\sin\theta}\,d\theta =$

$\int \frac{2}{4+3\left(\frac{2t}{1+t^2}\right)} \cdot \frac{2\,dt}{1+t^2}$

$\int \frac{2}{2t^2+3t+2}\,dt$ ✓

$= \int \frac{1}{\left(t+\frac{3}{4}\right)^2 + \frac{7}{16}}\,dt$

$= \int \frac{1}{\left(\frac{\sqrt{7}}{4}\right)^2 + \left(t+\frac{3}{4}\right)^2}\,dt$

$= \frac{4}{\sqrt{7}}\tan^{-1}\left(\frac{4t+3}{\sqrt{7}}\right) + c$ ✓

(d) (i) Let $\frac{1}{(x^2+3)(x^2+1)} = \frac{A}{x^2+3} + \frac{B}{x^2+1}$

$\therefore 1 = A(x^2+1) + B(x^2+3)$

let $x^2 = -1$, $\therefore 1 = 2B$

$B = \frac{1}{2}$ ✓

let $x^2 = -3$ $\therefore 1 = A \cdot -2$

$\therefore A = \frac{-1}{2}$

$\therefore \frac{1}{(x^2+3)(x^2+1)} = \frac{1}{2}\left[\frac{1}{x^2+1} - \frac{1}{x^2+3}\right]$ ✓

OR

{Show RHS expanded + simplified equals LHS}

(ii) $\int_0^1 \frac{dx}{(x^2+3)(x^2+1)} =$

$\frac{1}{2}\int_0^1 \frac{1}{x^2+1} - \frac{1}{x^2+3}\,dx$

$= \frac{1}{2}\left[\tan^{-1}x - \frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}}\right]_0^1$ ✓

$= \frac{1}{2}\left[\frac{\pi}{4} - \frac{1}{\sqrt{3}}\cdot\frac{\pi}{6}\right]$

$= \frac{\pi}{4}\left[\frac{1}{2} - \frac{1}{3\sqrt{3}}\right]$ ✓

(e) $I_m = \int_0^k (k^2-x^2)^m \cdot 1\,dx$

let $u = (k^2-x^2)^m$ $\qquad v' = 1$

$du = m(k^2-x^2)^{m-1}\cdot -2x$ $\qquad v = x$

$\therefore I_m = \left[x(k^2-x^2)^m\right]_0^k + \int_0^k 2mx^2(k^2-x^2)^{m-1}\,dx$ ✓

$= \left[k(k^2-k^2)^m - 0\right] + 2m\int_0^k x^2(k^2-x^2)^{m-1}\,dx$

$= 2m\int_0^k \frac{x^2(k^2-x^2)^m}{k^2-x^2}\,dx$

$= 2m\int_0^k (k^2-x^2)^m\left(\frac{k^2}{k^2-x^2} - 1\right)dx$ ✓

$= 2mk^2\int_0^k (k^2-x^2)^{m-1}\,dx - 2m\int (k^2-x^2)^m\,dx$

$\therefore I_m = 2k^2m\,I_{m-1} - 2m\,I_m$ ✓

$I_m(1+2m) = 2k^2m\,I_{m-1}$

$I_m = \frac{2k^2m}{2m+1}\cdot I_{m-1}$ ✓

Question 2

(a) $\cos\left(\frac{5\pi}{12} + \frac{3\pi}{12} - \frac{2\pi}{3}\right) + i\sin\left(\frac{5\pi}{12} + \frac{3\pi}{12} - \frac{2\pi}{3}\right)$ ✓

$= \cos 0 + i\sin 0$

$= 1$ ✓

(b) $z = \sqrt{3} + i$

$= 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

$= 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ ✓

$= 2\,\text{cis}\,\frac{\pi}{6}$

$\therefore z^4 = 2^4\,\text{cis}\,4\frac{\pi}{6}$

$= 16\,\text{cis}\,\frac{2\pi}{3}$ ✓

(c) (i) $\alpha + \beta = a + bi$

$\alpha\beta = -bi$

$(\alpha+\beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$

$\therefore (a+bi)^2 = 5 - 12i$

$a^2 - b^2 + 2abi = 5 - 12i$

equating real + imaginary parts

$a^2 - b^2 = 5$
$ab = -6$ ✓

WRONG QUESTION - should have been "show that $a^2 - b^2 = 5$ and $ab = -6$".

+2 marks to all students.

(ii) $a^2 - b^2 = 5$

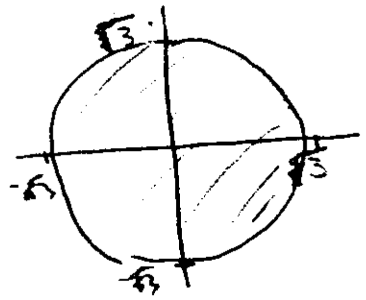
$\therefore a^4 - a^2b^2 = 5a^2$ ✓

$a^4 - 5a^2 - 36 = 0$

$(a^2 - 9)(a^2 + 4) = 0$

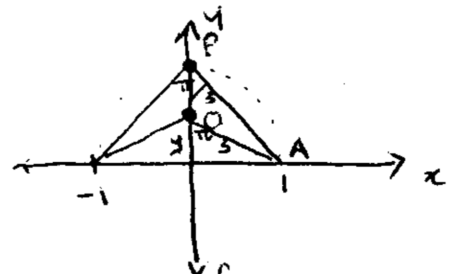
$\therefore a = 3, b = -2$
or
$a = -3, b = 2$ ✓

(d)



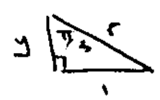
✓

(e)



P lies on circumference

$\therefore$ Centre subtends $\frac{\pi}{3}^c$ with origin and $(1,0)$ ✓



$\tan\frac{\pi}{3} = \frac{1}{y}$

$y = \frac{1}{\sqrt{3}}$

Centre $\left(0, \frac{1}{\sqrt{3}}\right)$

$\sin\frac{\pi}{3} = \frac{1}{r}$

$r = \frac{1}{\sqrt{3}/2}$ radius

✓ 1 only for 2nd mk.

$x^2 + \left(y - \frac{1}{\sqrt{3}}\right)^2 = \frac{4}{3}$ ✓

(f) $z_2 = iz_1$ ✓

$z_1 + z_2 = z_1 + iz_1$

$= z_1(1 + i)$

$(z_1 + z_2)^2 = z_1^2(1+i)^2$ ✓

$= z_1^2(1 + 2i - 1)$

$= 2iz_1^2$

$= 2z_1(iz_1)$

$= 2z_1z_2$ ✓

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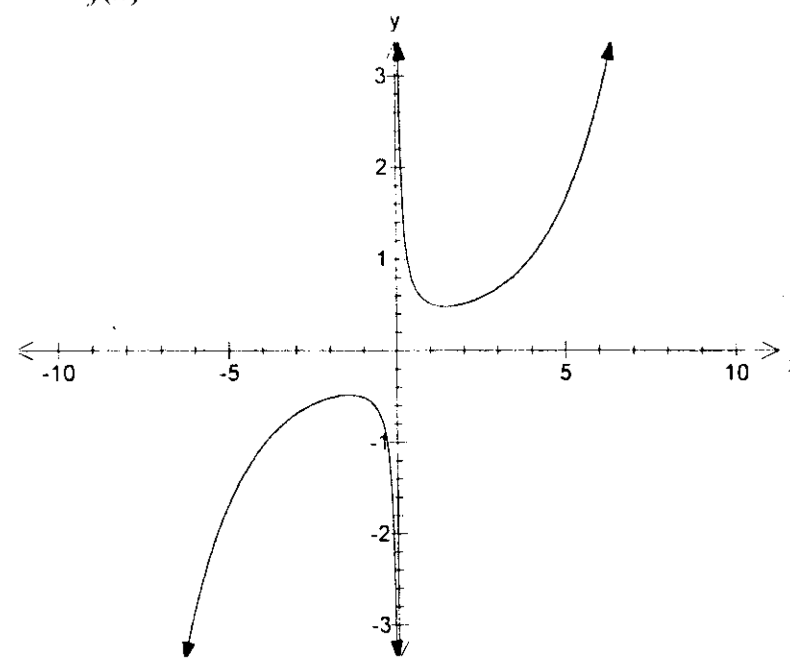
Question 3

(a) (i) $y = \dfrac{1}{f(x)}$

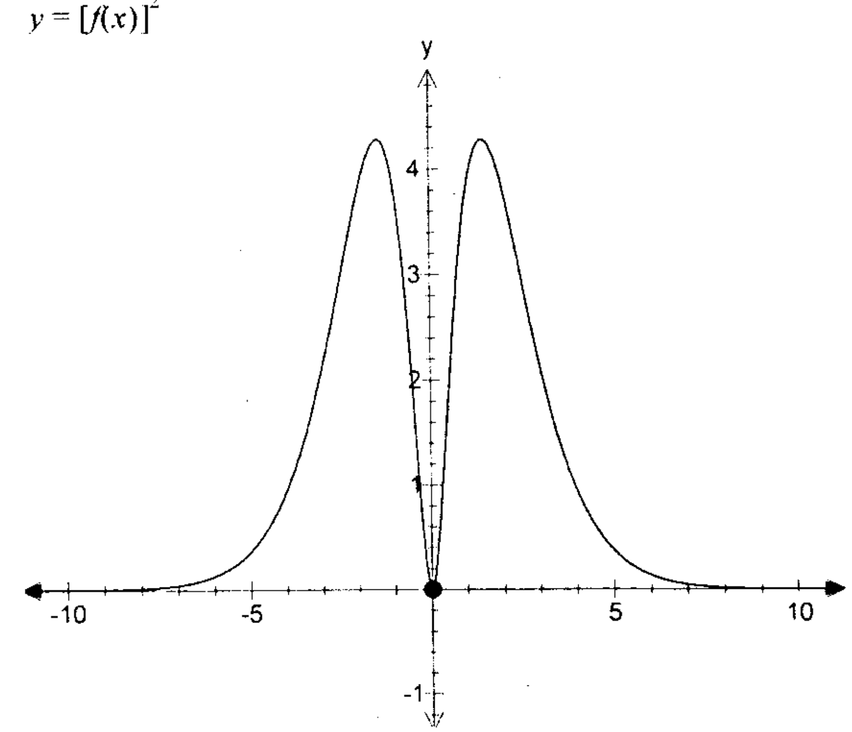


✓ asymptote at $x = 0$

① $(4,1)$ $(-4,-1)$

or $(1.5, \frac{1}{2})$ $(-1.5, \frac{1}{2})$

(ii) $y = [f(x)]^2$



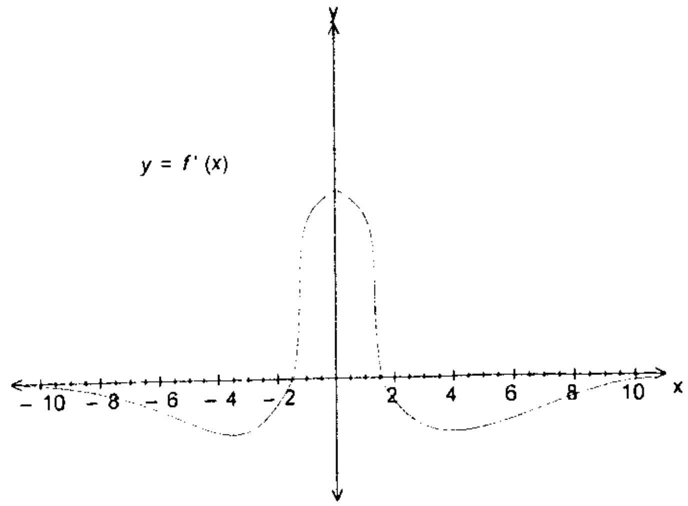
✓ corner at origin

✓ shape. (max tp at $x = \pm 1.5, 1.5$)

(iii)

✓ max tp at $x=0$

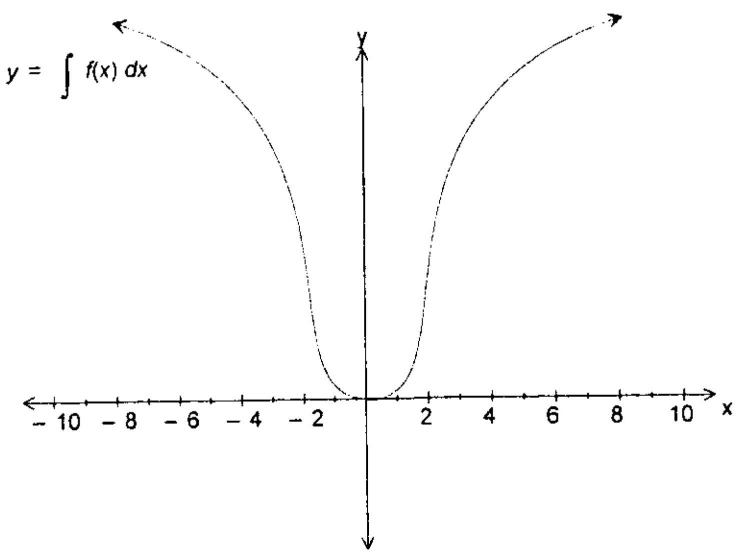
✓ x intercepts at $x=-1.5, 1.5$



(iv)

✓ min t.p at (0,0)

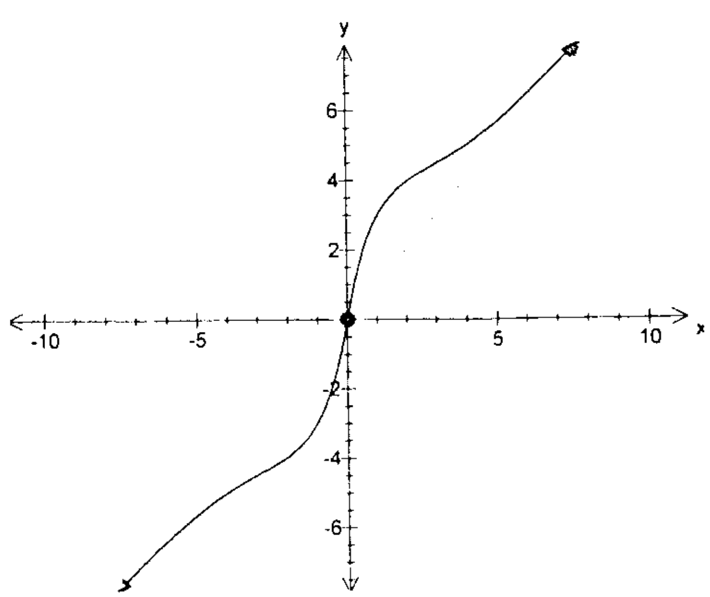
✓ dec. at dec. rate $x<0$ increasing at dec. rate $x>0$



(v) $y = x + f(x)$

✓ thru (0,0)

✓ symmetric about origin

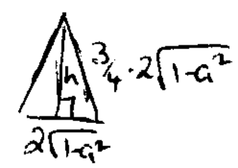


Qn. 3. (cont.)

(b) (i) when $x=a$,

$y = \pm\sqrt{1-a^2}$

$\therefore$ length of base $= 2\sqrt{1-a^2}$ ✓



$h^2 = \left(\frac{3}{2}\sqrt{1-a^2}\right)^2 - \left(\sqrt{1-a^2}\right)^2$

$= \frac{9}{4}(1-a^2) - (1-a^2)$

$= \frac{5}{4}(1-a^2)$

$\therefore h = \frac{\sqrt{5}\sqrt{(1-a^2)}}{2}$ ✓

$\therefore$ Area $= \frac{1}{2} \cdot 2\sqrt{1-a^2} \cdot \frac{\sqrt{5}}{2}\sqrt{1-a^2}$

$= \frac{\sqrt{5}}{2}(1-a^2)$ units^2 ✓

(ii) $\delta V = A \cdot \delta x$ where A is area of isosceles Δ

$= \frac{\sqrt{5}}{2}(1-x^2)\,\delta x$

$\therefore V = \frac{\sqrt{5}}{2}\int_0^1 (1-x^2)\,dx$ ✓

$= \frac{\sqrt{5}}{2}\left[x - \frac{x^3}{3}\right]_0^1$

$= \frac{\sqrt{5}}{2}\left[\left(1-\frac{1}{3}\right) - 0\right]$

$= \frac{\sqrt{5}}{3}$ units^3 ✓

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QUESTION 4

(a) Let $m = \frac{1}{x}$

Since $x = p, q, r$ then

$m = \frac{1}{p}, \frac{1}{q}, \frac{1}{r}$

Subst. $x = \frac{1}{m}$ into eqn.

$\frac{1}{m^3} + \frac{4}{m^2} - \frac{3}{m} + 1 = 0$ ✓

$1 + 4m - 3m^2 + m^3 = 0$

∴ the equation is

$x^3 - 3x^2 + 4x + 1 = 0$ ✓

(b) (i) $F(x) = (x-k)Q(x)$ ✓

$F'(x) = Q(x) + (x-k)Q'(x)$

$\therefore F'(k) = Q(k)$ ✓

since k is a zero of $F'(x)$,

$Q(k) = 0$

So, by the factor thm

$(x-k)$ is a factor of $Q(x)$

So, $F(x) = (x-k)(x-k)Q^*(x)$

$\therefore (x-k)^2$ is a factor of $F(x)$ ✓

(ii) $y^{2t} - ty^{t+1} = 1 - ty^{t-1}$

$y^{2t} - ty^{t+1} + ty^{t-1} - 1 = 0$

Let $P(y) = y^{2t} - ty^{t+1} + ty^{t-1} - 1$

$P(1) = 0$ ✓

$P'(y) = 2ty^{2t-1} - t(t+1)y^{t} + t(t-1)y^{t-2}$

$P'(1) = 0$

So, $y = 1$ is a zero of $P(y)$ and $P'(y)$

∴ it must be a double root

(root of multiplicity 2) of $P(y) = 0$ ✓

(d) (i) $(\cos\theta + i\sin\theta)^5 = 1$

$\cos 5\theta + i\sin 5\theta = 1$

$\therefore \cos 5\theta = 1$ ✓

$5\theta = 0, 2\pi, 4\pi, 6\pi, 8\pi$

$\therefore \theta = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$

$\therefore z = 1, \text{cis}\frac{2\pi}{5}, \text{cis}\frac{4\pi}{5},$

$\text{cis}\frac{6\pi}{5}, \text{cis}\frac{8\pi}{5}$ ✓

(ii) (α)

$z^5 - 1 = (z-1)(z - \text{cis}\frac{2\pi}{5})(z - \text{cis}\frac{4\pi}{5})$

$(z - \text{cis}\frac{6\pi}{5})(z - \text{cis}\frac{8\pi}{5})$ ✓

(β)

$z^5 - 1 = (z-1)(z - \text{cis}\frac{2\pi}{5})(z - \text{cis}\frac{4\pi}{5})$

$(z - \text{cis}(-\frac{4\pi}{5}))(z - \text{cis}(-\frac{2\pi}{5}))$

$\text{cis}(-\frac{4\pi}{5}) = \cos(-\frac{4\pi}{5}) + i\sin(-\frac{4\pi}{5})$

$= \cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5}$

$(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5})(\cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5})$

$= \cos^2\frac{4\pi}{5} + \sin^2\frac{4\pi}{5}$

$= 1$ ✓

$\therefore (z - \text{cis}\frac{4\pi}{5})(z - \text{cis}(-\frac{4\pi}{5}))$

$= z^2 - 2z\cos\frac{4\pi}{5} + 1$

$\therefore z^5 - 1 = (z-1)(z^2 - 2z\cos\frac{2\pi}{5} + 1)$

$(z^2 - 2z\cos\frac{4\pi}{5} + 1)$ ✓

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Qn. 4 (e).

$z = \frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2} \cdot i$

$|z| = \sqrt{\left(\frac{1-t^2}{1+t^2}\right)^2 + \left(\frac{2t}{1+t^2}\right)^2}$ ✓

as $\{ z = x + iy, \; |z| = \sqrt{x^2+y^2} \}$

$= \sqrt{\frac{1 - 2t^2 + t^4 + 4t^2}{(1+t^2)^2}}$

$= \sqrt{\frac{t^4 + 2t^2 + 1}{(1+t^2)^2}}$

$= \sqrt{\frac{(t^2+1)^2}{(1+t^2)^2}}$ ✓

$= 1$

Question 5

(a) (i) $S(ae, 0)$

∴ line thru S has equation $x = ae$

∴ At P and Q,

$$\frac{(ae)^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$y^2 = b^2(e^2 - 1)$$ ✓

but $b^2 = a^2(e^2-1)$

$$\therefore e^2 - 1 = \frac{b^2}{a^2}$$

$$\therefore y^2 = b^2 \cdot \frac{b^2}{a^2}$$

$$y^2 = \frac{b^4}{a^2}$$

$$y = \pm \frac{b^2}{a}$$ ✓

∴ P, Q have coords.

$$\left(ae, \pm\frac{b^2}{a}\right)$$

$$\therefore PQ = \frac{2b^2}{a}$$

(ii) $\frac{2b^2}{a} = 2.24 = 48$

$\therefore b^2 = 24a$ ✓

subst. into $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\frac{9^2}{a^2} - \frac{24^2}{b^2} = 1$$ ✓

$$a^2 + 24a - 81 = 0$$

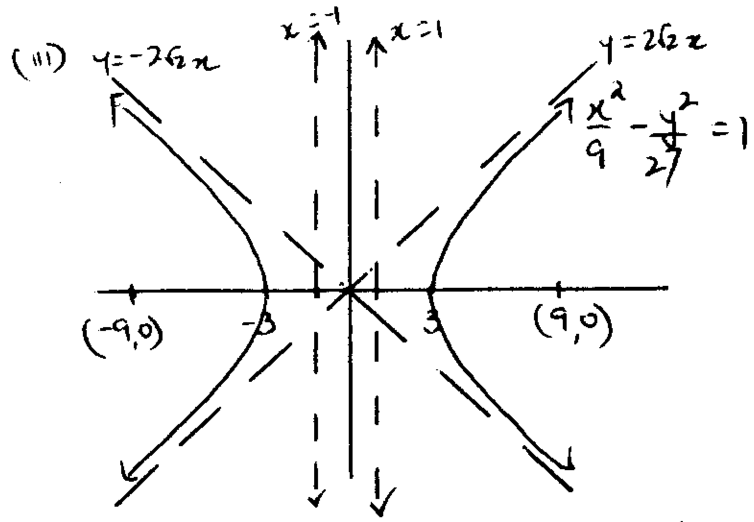
$(a+27)(a-3) = 0$

∴ $a = 3$ (not $a = -27$ as $0 < a < b$)

and $b^2 = 24.3 = 72$

$b = \sqrt{72} = 6\sqrt{2}$ ✓

(iii)



x-intercepts $x = \pm 3$ ✓

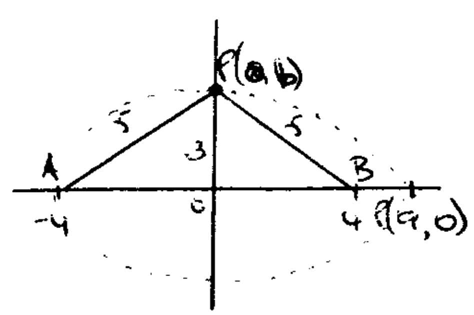
Foci $x = \pm ae = \pm 9$ ✓

Directrices $x = \pm \frac{a}{e} = \pm 1$ ✓

Asymptotes $y = \pm \frac{b}{a} x$

$= \pm 2\sqrt{2}x$ ✓

(b). (i)



$OP = 3$ (by Pythagoras)

∴ $b = 3$ ✓

when P is at $(a, 0)$, $a = 5$

subst. into $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

∴ eqn. is $\frac{x^2}{25} + \frac{y^2}{9} = 1$ ✓

Qn. 5 b)

(ii) $$\frac{x^2}{25} + \frac{y^2}{9} = \frac{5^2\cos^2\theta}{25} + \frac{3^2\sin^2\theta}{9}$$

$= \cos^2\theta + \sin^2\theta$

$= 1.$

$= RHS$ ✓

$x = 5\cos\theta$

$\frac{dx}{d\theta} = -5\sin\theta$

$y = 3\sin\theta$

$\frac{dy}{d\theta} = 3\cos\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$= 3\cos\theta \cdot \frac{-1}{5\sin\theta}.$$ ✓

when $\theta = \frac{\pi}{6}$,

$$\frac{dy}{dx} = \frac{-3 \cdot \sqrt{3}/2}{5 \cdot 1/2}$$

$$= \frac{-3\sqrt{3}}{5}$$ ✓

$x = 5 \cdot \frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{2}$

$y = 3 \cdot \frac{1}{2} = \frac{3}{2}$

$$\therefore y - \frac{3}{2} = -\frac{3\sqrt{3}}{5}\left(x - \frac{5\sqrt{3}}{2}\right)$$

or

$3\sqrt{3}x + 5y - 30 = 0$ ✓

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Question 8

(a).

Show that $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$ ✓

$a^2+b^2 \geq 2ab$ ①
$b^2+c^2 \geq 2bc$ ②
$c^2+a^2 \geq 2ca$ ③

① + ② + ③
$2(a^2+b^2+c^2) \geq 2(ab+bc+ca)$
$\therefore a^2+b^2+c^2 \geq ab+bc+ca$
$a^2+b^2+c^2-ab-bc-ca \geq 0$ ✓

Multiplying b.s. by $(a+b+c)$

$(a+b+c)(a^2+b^2+c^2-ab-bc-ca) \geq 0$
$a^3+ab^2+ac^2-a^2b-abc-a^2c$
$+ba^2+b^3+bc^2-ab^2-b^2c-abc$
$+a^2c+b^2c+c^3-abc-bc^2-ac^2 \geq 0$

$a^3+b^3+c^3-3abc \geq 0$
$a^3+b^3+c^3 \geq 3abc$
let $a^3=a,\ b^3=b,\ c^3=c$ ✓
$\therefore a+b+c \geq 3\sqrt[3]{abc}$
$\therefore \frac{a+b+c}{3} \geq \sqrt[3]{abc}$

(b) $P(x) = (2x^2-5x+2)Q(x)+R(x)$
Since deg $D(x) >$ deg $R(x)$
deg $R(x) < 2$
Let $R(x) = ax+b$

$\therefore P(x) = (2x-1)(x-2)Q(x)+ax+b$ ✓

$P(\frac{1}{2}) = \frac{a}{2}+b = -2$ ... ①
$P(2) = 2a+b = 1$ ... ② ✓

② − ①

$\frac{3a}{2} = 3$

$\therefore a=2,\ b=-3$

$\therefore R(x) = 2x-3$ ✓

(c)(i).

$P(x) = ax^4+bx^3+cx^2+dx+e = 0$
$P'(x) = 4ax^3+3bx^2+2cx+d$
$P''(x) = 12ax^2+6bx+2c$
$P'''(x) = 24ax+6b = 0$ ✓
as α is a quadruple root

$\therefore x = -\frac{6b}{24a} = -\frac{b}{4a}$ ✓

(ii) Since $\alpha = -\frac{b}{4a}$ is a quadruple root,
$P(x) = a(x-\alpha)^4 = ax^4+bx^3+cx^2+dx+e = 0$

$\therefore (x-\alpha)^4 = \frac{ax^4+bx^3+cx^2+dx+e}{a}$ ✓

Subst. $x=1$ and $\alpha = -\frac{b}{4a}$

$\left(1+\frac{b}{4a}\right)^4 = \frac{a+b+c+d+e}{a}$ ✓

Qn 8(d)

(i) $\int_1^e x^{-1}\log x\,dx$

$= \frac{1}{2}\left[\log x\right]_1^e$ ✓

$= \frac{1}{2}$ ✓

(ii) Let $I = \int_1^e x^n \log x\,dx$

let $u = \log x$ $v' = x^n$
$\frac{du}{dx} = \frac{1}{x}$ $v = \frac{x^{n+1}}{n+1}$

$\therefore I = \frac{1}{n+1}\left[x^{n+1}\log x\right]_1^e - \frac{1}{n+1}\int_1^e \frac{x^{n+1}}{x}\,dx$ ✓

$= \frac{1}{n+1}(e^{n+1}) - \frac{1}{n+1}\int_1^e x^n\,dx$

$= \frac{1}{n+1}e^{n+1} - \frac{1}{(n+1)^2}\left[x^{n+1}\right]_1^e$ ✓

$= \frac{1}{(n+1)^2}\left[(n+1)e^{n+1} - (e^{n+1}-1)\right]$

$= \frac{ne^{n+1}+1}{(n+1)^2}$ ✓

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