Question 1 **(Begin a new page)** **Marks**

(a) (i) Show that $\dfrac{1}{e^x - 1} = \dfrac{e^{-x}}{1 - e^{-x}}$. **1**

(ii) Hence find $\displaystyle\int \frac{1}{e^x - 1}\,dx$. **1**

(b) Find $\displaystyle\int \frac{x}{\sqrt{1 + x}}\,dx$ using the substitution $u = 1 + x$. **2**

(c) Use partial fractions to find $\displaystyle\int \frac{x^2 + x - 4}{(2x + 1)(x^2 + 4)}\,dx$ **3**

(d) (i) Use integration by parts to find $\displaystyle\int \tan^{-1}x\,\,dx$. **2**

(ii) Hence find $\displaystyle\int x\tan^{-1}(x^2)\,\,dx$. **2**

(e) (i) Find $\displaystyle\int \frac{dx}{x^2 + 2x + 10}$ **2**

(ii) Use the result $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(a + b - x)\,dx$ to evaluate the integral **2**

$\displaystyle\int_0^\pi x\cos 2x\,dx$.

Question 2 **(Begin a new page)**

(a) Given $w = 2 + 5\mathrm{i}$, $z = 4 - 3\mathrm{i}$ and $u = a + b\mathrm{i}$, evaluate:

(i) $w + \bar{z}$ **1**

(ii) $|\,w + \bar{z}\,|$ **1**

(iii) $(w + \bar{z})(\overline{w} + z)$ **2**

(iv) If $\dfrac{w(9 - 8\mathrm{i})}{u}$ is a real number,

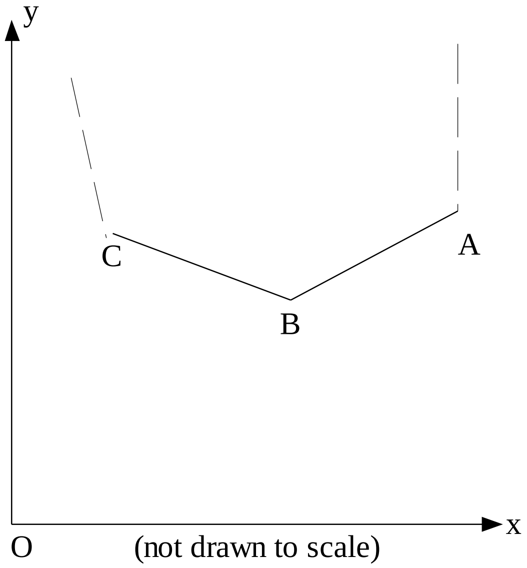
(α) show that $a = 2b$, **2**

(β) hence find u if $|\,u\,| = 5$. **2**

Question 2 (continued)

Marks

(b) A, B are the points representing the complex numbers $z_1 = 13 + 11i$ and $z_2 = 10 + 8i$ respectively. Find the complex number that is represented by the point C where A, B and C are adjacent vertices of a regular hexagon. **3**

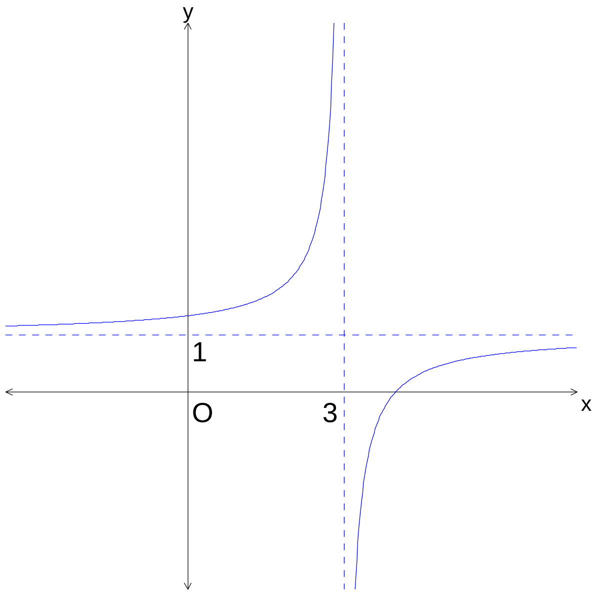


(c) (i) Express $z = \dfrac{1 + i}{\sqrt{3} + i}$ in the form of $r \operatorname{cis} \theta$. **3**

(ii) Find the smallest positive integer n such that z^n is a purely imaginary number. **1**

Question 3 (Begin a new page)

(a) The diagram shows the graph of the function $f(x) = \dfrac{x - 4}{x - 3}$. On separate diagrams, sketch the following graphs, showing clearly any intercepts on the axes and the equations of the asymptotes.



(i) $y = f(-x)$ **1**

(ii) $y = |f(x)|$ **1**

(iii) $y = f(|x|)$ **2**

(iv) $y = e^{f(x)}$ and **2**

(v) $y = \sqrt{f(x)}$ **1**

(b) The line $y = mx$ through the origin O (0, 0) touches the curve $y = \dfrac{x - 4}{x - 3}$ at point $P(x_1, y_1)$.

(i) By considering the gradient of the line OP, show that **2**

$$x_1^{\,2} - 8x_1 + 12 = 0$$

Question 3b continued on p.3…

Marks

Question 3b (continued)

(ii) Hence find the equation of the two tangents to $y = \dfrac{x-4}{x-3}$ from the origin. **2**

(c) (i) Sketch the region in the complex plane for which $|z - 5 - 12i| \le 3$. **2**

(ii) Find the maximum modulus of the complex number z that satisfies $|z - 5 - 12i| \le 3$. **2**

Question 4 **(Begin a new page)**

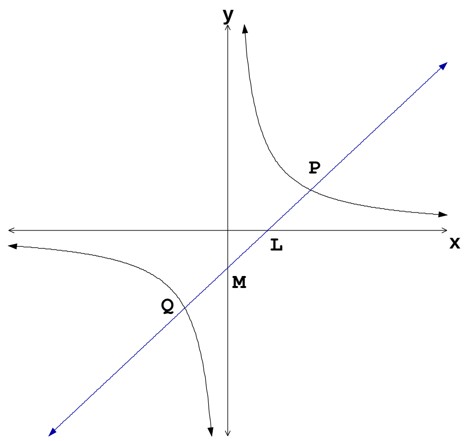
(a) Given the equation of an ellipse is $E : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

(i) Show that if the line $y = mx + c$ is a tangent to E, then $c^2 = a^2m^2 + b^2$. **2**

(ii) Show that if the line $y = mx + c$ is a tangent to the circle $x^2 + y^2 = r^2$, then $c^2 = r^2m^2 + r^2$. **1**

(iii) Hence or otherwise, find the equations of the common tangents to the ellipse $\dfrac{x^2}{6} + \dfrac{y^2}{3} = 1$ and the circle $2x^2 + 2y^2 = 9$. **3**

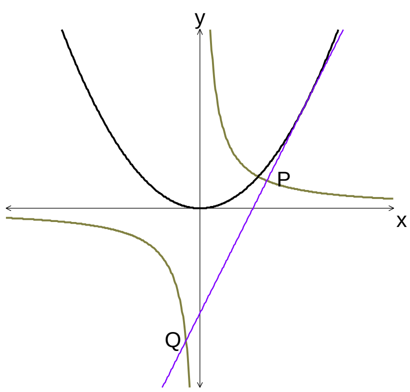
(b) $P\left(cp, \dfrac{c}{p}\right)$ and $Q\left(cq, \dfrac{c}{q}\right)$ are two points on the rectangular hyperbola $xy = c^2$. The chord PQ meets the asymptotes at L and M.



(i) Show that the equation of the chord PQ is $x + pqy = c(p + q)$. **1**

(ii) Prove that $PL = QM$. **3**

(iii) A tangent to the parabola $x^2 = 4y$ meets the hyperbola $xy = 9$ at two points P and Q.



(α) show that $1 + 3pq(p + q) = 0$. **2**

(β) Hence show that the equation of the locus of N, the mid-point of PQ, is $2x^2 + y = 0$. **3**

Question 5 **(Begin a new page)** **Marks**

(a) The equation $x^3 - x^2 + 3x + 4 = 0$ has roots α, β and γ.

(i) Find the values of $\alpha^2 + \beta^2 + \gamma^2$ and hence deduce that two of the roots must be complex. **3**

(ii) Find the monic polynomial equation with roots α^2, β^2, and γ^2. **2**

(b) (i) Show that if $x = \alpha$ is a repeated root of the polynomial equation $P(x) = 0$, then it must also be a root of $P'(x) = 0$. **1**

(ii) The hyperbola $xy = c^2$ touches the ellipse $\dfrac{(x-1)^2}{6} + y^2 = 1$ at point Q which lies in the 3rd quadrant. Deduce that the equation $x^4 - 2x^3 - 5x^2 + 6c^4 = 0$ has a negative repeated root α. **2**

(iii) Hence determine the values of α and c^4. **2**

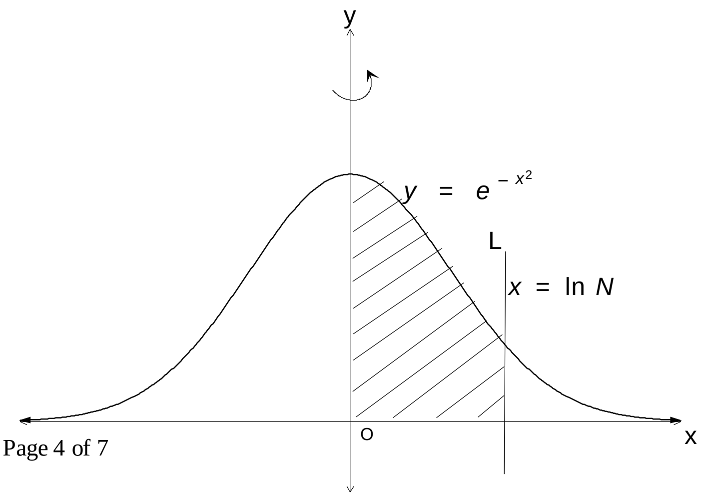
(c) (i) By using the identity $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$, show that the roots of the equation $16x^5 - 20x^3 + 5x - 1 = 0$ are $x = \cos\dfrac{2r\pi}{5}$, $r = 0, 1, 2, 3, 4$. **2**

(ii) Hence show that $\cos\dfrac{\pi}{5}\cos\dfrac{2\pi}{5} = \dfrac{1}{4}$. **3**

Question 6 **(Begin a new page)**

(a) (i) Find $\int x e^{-x^2}\,dx$ using the substitution $u = x^2$. **2**

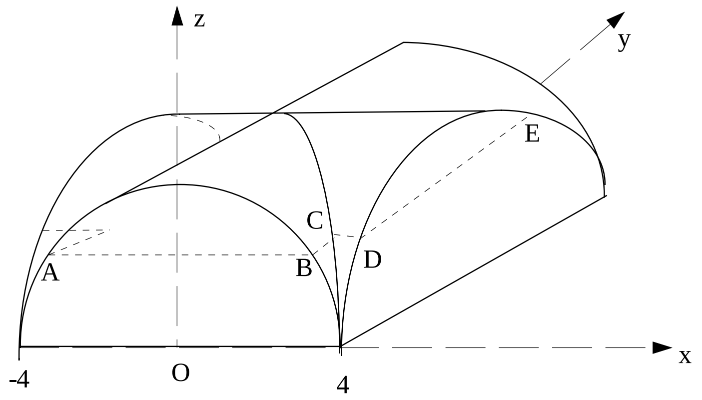
(ii) The region bounded between the curve $y = e^{-x^2}$, the x-axis, the y-axis and the line L: $x = \log_e N$ is rotated about the y-axis. Show that the volume of the solid formed is $\pi\left[1 - \left(\dfrac{1}{N}\right)^{\ln N}\right]$ by using the cylindrical shell method. **4**

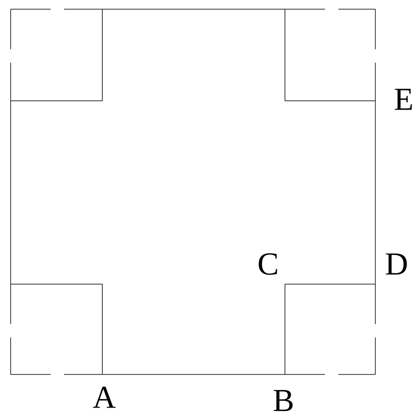


Question 6 (continued) **Marks**

(iii) Hence find the volume of the largest solid possible by sliding the line L indefinitely towards the right. **2**

(b)





The solid shown above is formed in the shape of 2 identical intersecting semi-circular cylinders, each of radius 4. The base of the solid is a square of sides 8 cm. On the front face, the equation of the semi-circle is $x^2 + z^2 = 16,\ \ z \geq 0$, where x is the horizontal distance measured from the mid-point of the base of the front face and z is the height.

The shape of a horizontal slice of thickness δz taken at a height z is also shown above. It is a square with 4 smaller squares removed, one from each corner.

(i) Find x in terms of z. **1**

(ii) Show that the volume of the solid is given by the expression **3**

$$V = \int_0^4 \{8^2 - 4[4 - \sqrt{16 - z^2}\ ]^2\}\, dz$$

(iii) Hence find the volume of the solid. **3**

Question 7 **(Begin a new page)**

(a) The air resistance encountered by a body of mass m moving at a speed v is given by $kmgv$ where m is the mass of the body, g is the acceleration due to gravity and k is a constant.

(i) Find the terminal velocity V_T. **2**

(ii) The body is projected vertically upward with an initial speed of U, show that the maximum height reached will be given by $H = \frac{U}{gk} - \frac{1}{gk^2}\ln(1 + kU)$. **3**

Question 7a continued on p.6…

Question 7a (continued) **Marks**

(iii) Show that, if the initial speed of projection is V_T , then the time taken to reach the maximum height is $T = \dfrac{V_T}{g} \ln 2$. **3**

(b) A body of mass *m* is projected vertically downward with a speed $\frac{1}{3} V_T$, where V_T is the terminal velocity, in a resistive medium. The resistance is mkv^2 , where *k* is a constant and *v* is the speed of the body *t* seconds after being projected. Show that it takes $\dfrac{V_T}{2g} \ln \dfrac{3}{2}$ seconds to reach a speed of $\frac{1}{2} V_T$. **4**

(c) The gravitational force exerted by the earth on a body of mass *m* at a distance *x* $(x \geq R)$ from its centre is given by $\dfrac{mk}{x^2}$, where *R* is the radius of the earth and *k* is a constant.

(i) Show that $k = gR^2$. **1**

(ii) A body is projected vertically upward with an initial speed *U*. Show that the speed of the body when it is at a distance *x* from the centre of the earth is given by **2**

$$v^2 = U^2 + 2gR^2 \left(\frac{1}{x} - \frac{1}{R} \right) .$$

Question 8 **(Begin a new page)**

(a) A sequence of number $\{ u_n \}$ is defined by $u_1 = 1$, $u_2 = 5$ and **3**

$u_{n+2} = 5 u_{n+1} - 6 u_n \quad n \geq 1$

Use mathematical induction to show that $u_n = 3^n - 2^n \qquad n \geq 1$

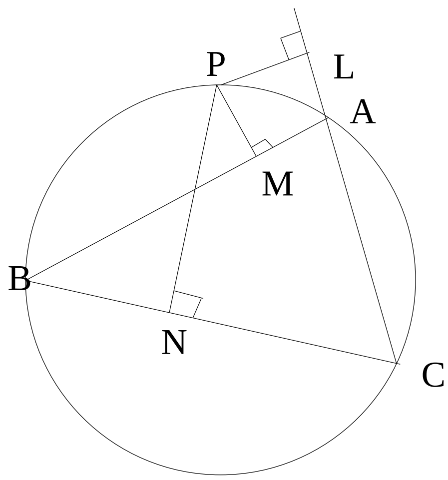
(b) Show that **3**

$$\sin \theta + \sin 2\theta + ... + \sin n\theta = \frac{1}{2 \sin \frac{\theta}{2}} \left[\cos \frac{\theta}{2} - \cos \frac{(2n+1)\theta}{2} \right] .$$

[You may assume the identity $2 \sin A \sin B = \cos (A - B) - \cos (A + B)$]

Question 8 (continued)

(c) P is a point on the minor arc AB. L, M and N are the feet of perpendiculars from P to CA, AB and BC respectively.



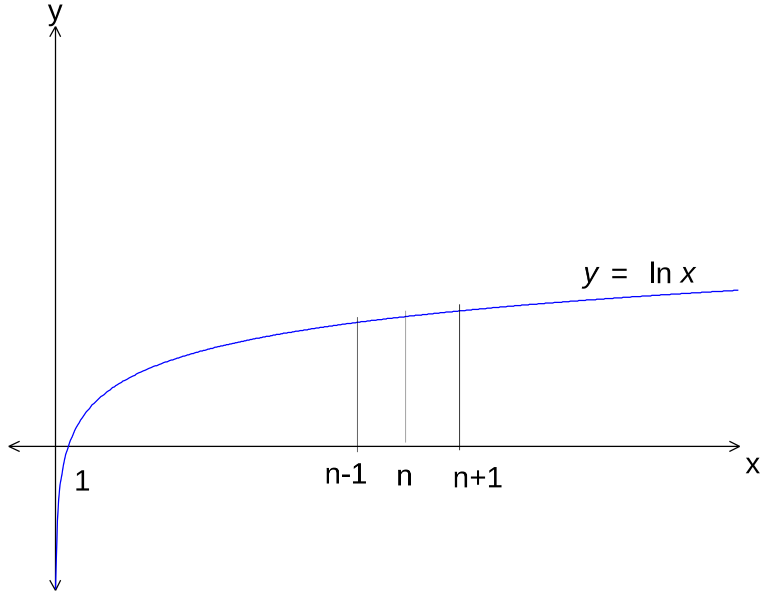
(i) Copy the diagram. **1**

(ii) State the reason why P, M, A and L are concyclic. **1**

(iii) State the reason why P, B, N and M are concyclic. **1**

(iv) Show that L, M and N are collinear. **2**

(d)



(i) By considering the appropriate area under the graph of $y = \log_e x$, show that **2**

$$\log_e n < \int_n^{n+1} \log_e x \, dx < \log_e (n+1) \qquad n \geq 1$$

(ii) Hence, show that $1 + \dfrac{1}{n} < e^{\frac{1}{n}} < \left(1 + \dfrac{1}{n}\right)^{\frac{n+1}{n}}$ **3**

End of Paper

Solution to CTHS AP4 Ext 2 Maths

Question 1

(a) (i)
$$\frac{1}{e^x-1} = \frac{1}{e^x-1} \times \frac{e^{-x}}{e^{-x}}$$
$$= \frac{e^{-x}}{1-e^{-x}}$$

(ii)
$$\int \frac{1}{e^x-1}\,dx = \int \frac{e^{-x}}{1-e^{-x}}\,dx$$
$$= \int \frac{d(1-e^{-x})}{1-e^{-x}}$$
$$= \ln(1-e^{-x})$$

(b) $u = 1+x$

$\therefore x = u-1$

$dx = du$

$$\int \frac{x}{\sqrt{1-x}}\,dx = \int \frac{u-1}{\sqrt{u}}\,du$$
$$= \int \sqrt{u} - \frac{1}{\sqrt{u}}\,du$$
$$= \frac{2}{3}u^{3/2} - 2\sqrt{u} + C$$
$$= \frac{2}{3}(1+x)^{\frac{3}{2}} - 2\sqrt{1+x} + C$$

(c) Let
$$\frac{x^2+x-4}{(2x+1)(x^2+4)} = \frac{A}{2x+1} + \frac{Bx+C}{x^2+4}$$

$$\therefore A(x^2+4) + (Bx+C)(2x+1) \equiv x^2+x-4 \qquad (1)$$

Put $x = -\frac{1}{2}$
$$A\left(\tfrac{1}{4}+4\right) = \tfrac{1}{4} - \tfrac{1}{2} - 4$$
$$\tfrac{17}{4}A = -\tfrac{17}{4}$$
$$A = -1$$

Equating coeff. of x^2 in (1)
$$A + 2B = 1$$
$$2B = 1 - A$$
$$2B = 2$$
$$B = 1$$

Put $x = 0$ in (1)
$$4A + C = -4$$
$$C = -4 - 4A$$
$$= -4 + 4$$
$$= 0$$

$$\therefore \frac{x^2+x-4}{(2x+1)(x^2+4)} = -\frac{1}{2x+1} + \frac{x}{x^2+4}$$

$$\therefore \int \frac{x^2+x-4}{(2x+1)(x^2+4)}\,dx = -\int \frac{1}{2x+1}\,dx + \int \frac{x}{x^2+4}\,dx$$
$$= -\tfrac{1}{2}\ln(2x+1) + \tfrac{1}{2}\ln(x^2+4) + C$$
$$= \tfrac{1}{2}\ln\frac{x^2+4}{2x+1} + C$$

(d) (i)
$$\int \tan^{-1}x\,dx = x\tan^{-1}x - \int x\,d(\tan^{-1}x)$$
$$= x\tan^{-1}x - \int \frac{x}{1+x^2}\,dx$$
$$= x\tan^{-1}x - \frac{1}{2}\int \frac{d(1+x^2)}{1+x^2}$$
$$= x\tan^{-1}x - \tfrac{1}{2}\ln(1+x^2) + C$$

(ii) Put $u = x^2$

$du = 2x\,dx$

$$\therefore \int x\tan^{-1}(x^2)\,dx = \tfrac{1}{2}\int \tan^{-1}u\,du$$
$$= \tfrac{1}{2}u\tan^{-1}u - \tfrac{1}{4}\ln(1+u^2) + C$$
$$= \frac{x^2}{2}\tan^{-1}(x^2) - \tfrac{1}{4}\ln(1+x^4) + C$$

(e) (i)
$$\int \frac{dx}{x^2+2x+10} = \int \frac{dx}{(x+1)^2+9}$$
$$= \tfrac{1}{3}\tan^{-1}\frac{x+1}{3} + C$$

(ii)
$$\int_0^\pi x\cos 2x\,dx = \int_0^\pi (\pi-x)\cos 2(\pi-x)\,dx$$
$$= \int_0^\pi (\pi-x)\cos 2x\,dx$$
$$= \pi\int_0^\pi \cos 2x\,dx - \int_0^\pi x\cos 2x\,dx$$
$$2\int_0^\pi x\cos 2x\,dx = \pi\int_0^\pi \cos 2x\,dx$$

$$\therefore \int_0^\pi x\cos 2x\,dx = \tfrac{1}{2}\pi\int_0^\pi \cos 2x\,dx$$
$$= \tfrac{1}{4}\pi\left[\sin 2x\right]_0^\pi$$
$$= 0$$

Question 2

(a) (i)
$$w + \bar{z} = (2+5i) + \overline{(4-3i)}$$
$$= 2+5i+4+3i$$
$$= 6+8i$$

(ii)
$$|w+\bar{z}| = \sqrt{6^2+8^2}$$
$$= 10$$

(iii)
$$(w+\bar{z})(\bar{w}+z) = (w+\bar{z})\overline{(w+\bar{z})}$$
$$= |w+\bar{z}|^2$$
$$= 10^2$$
$$= 100$$

(iv)(α)
$$\frac{w(9-8i)}{u} = \frac{(2+5i)(9-8i)}{a+bi}$$
$$= \frac{(18+40)+(45-16)i}{a+bi}$$
$$= \frac{58+29i}{a+bi} \times \frac{a-bi}{a-bi}$$
$$= \frac{(58a+29b)+(29a-58b)i}{a^2-b^2}$$

$\frac{w(9-8i)}{u}$ is real only if $Im\left(\frac{w(9-8i)}{u}\right)=0$

ie $\frac{29a-58b}{a^2-b^2}=0$ ✓

ie $a=2b$

(β) $u=a+bi$

$\therefore |u|=\sqrt{a^2+b^2}$

$|u|=5$

$a^2+b^2=25$

$(2b)^2+b^2=25$ ✓

$5b^2=25$

$b^2=5$

$b=\pm\sqrt{5}$

$\therefore u=\pm\sqrt{5}(2+i)$ ✓

(b) $\overrightarrow{BA}=z_1-z_2$

$=(13+11i)-(10+8i)$

$=3+3i$

$\overrightarrow{BC}=\overrightarrow{BA}\times cis\frac{2\pi}{3}$ ✓

$=(3+3i)(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3})$

$=(3+3i)(-\frac{1}{2}+\frac{\sqrt{3}}{2}i)$

$=-\frac{3}{2}-\frac{3\sqrt{3}}{2}+\left(\frac{3\sqrt{3}}{2}-\frac{3}{2}\right)i$ ✓

but $\overrightarrow{BC}=z_3-z_2$ where z_3 is the complex no. that is represented by C.

$\therefore z_3=z_2+\overrightarrow{BC}$

$=10+8i-\frac{3}{2}-\frac{3\sqrt{3}}{2}+\left(\frac{3\sqrt{3}}{2}-\frac{3}{2}\right)i$

$=\left(\frac{17}{2}-\frac{3\sqrt{3}}{2}\right)+\left(\frac{13}{2}+\frac{3\sqrt{3}}{2}\right)i$

$=\frac{17-3\sqrt{3}}{2}+\frac{13+3\sqrt{3}}{2}i$ ✓

(c) (i) $z=\frac{1+i}{\sqrt{3}+i}$

$=\frac{\sqrt{2}(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i)}{2(\frac{\sqrt{3}}{2}+\frac{1}{2}i)}$

$=\frac{1}{\sqrt{2}}\frac{cis\frac{\pi}{4}}{cis\frac{\pi}{6}}$

$=\frac{1}{\sqrt{2}}cis(\frac{\pi}{4}-\frac{\pi}{6})$

$=\frac{1}{\sqrt{2}}cis\frac{\pi}{12}$

(ii) $z^n=\left(\frac{1}{\sqrt{2}}\right)^n cis\frac{n\pi}{12}$

$=\left(\frac{1}{\sqrt{3}}\right)^n\left(\cos\frac{n\pi}{12}+i\sin\frac{n\pi}{12}\right)$

if z^n is a purely imaginary number, then $\cos\frac{n\pi}{12}=0$

ie $\frac{n\pi}{12}=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\frac{7\pi}{2},\ldots$

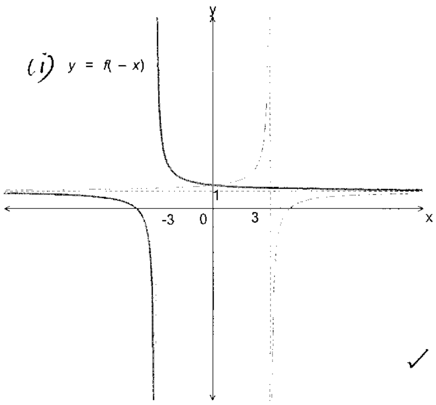
$n=6,18,30,42,\ldots$

$\therefore$ The smallest positive integral value of $n=6$.

Question 3

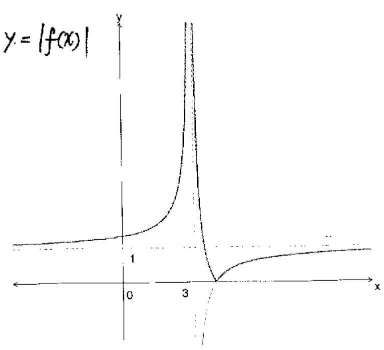
(a)

(i)



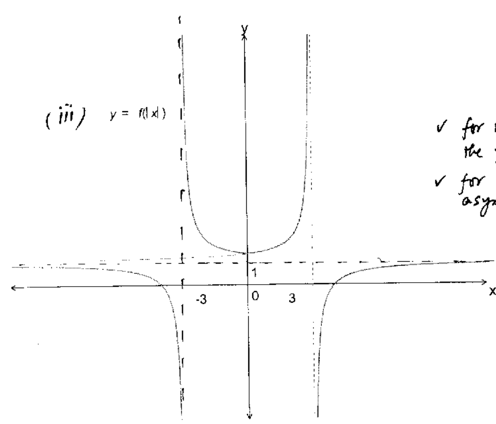
✓

(ii) $y=|f(x)|$



✓

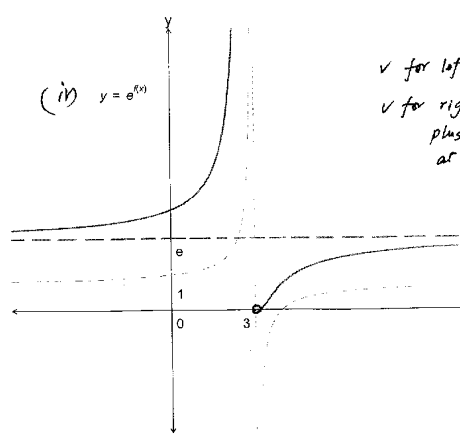
(iii)



✓ for reflexion about the y-axis

✓ for all correct asymptotes

(iv)

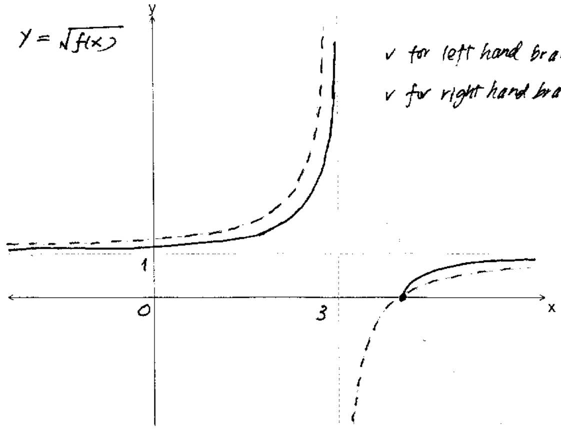


✓ for left hand branch

✓ for right hand branch plus open end-pt at $x=3$.

3(a)

(v) $y=\sqrt{f(x)}$



✓ for left hand branch

✓ for right hand branch

(b) (i) $m_{op} = \frac{y_1}{x_1}$ (1)

$y = \frac{x-4}{x-3}$

$y' = \frac{(x-3)-(x-4)}{(x-3)^2}$

$= \frac{1}{(x-3)^2}$

$\therefore$ gradient at $P(x_1,y_1)$ is $\frac{1}{(x_1-3)^2}$ (2)

From (1) & (2) $\frac{y_1}{x_1} = \frac{1}{(x_1-3)^2}$ ✓

but $y_1 = \frac{x_1-4}{x_1-3}$

$\therefore \frac{x_1-4}{x(x_1-3)} = \frac{1}{(x_1-3)^2}$ ✓

$(x_1-3)(x_1-4) = x_1$

$x_1^2-7x_1+12 = x_1$

$x_1^2-8x_1+12=0$

(ii) $x_1^2-8x_1+12=0$

$(x_1-2)(x_1-6)=0$

$\therefore x_1 = 2$ or $x_1=6$

when $x_1=2$, $y_1 = \frac{2-4}{2-3}$

$=2$

$\therefore$ Equation of tangent is $y=x$ ✓

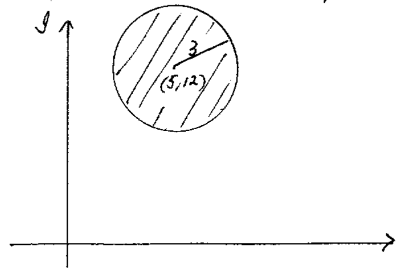
when $x_1=6$, $y_1=\frac{6-4}{6-3}$

$=\frac{2}{3}$

$\therefore$ Equation of tangent is $y=\frac{x}{9}$

or $x-9y=0$ ✓

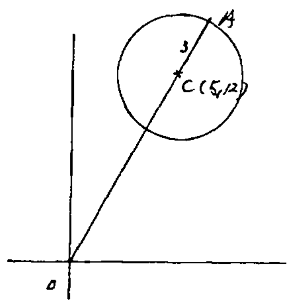
(c) (i)



1 for correct region

1 for correct centre & radius.

(ii) OCA is a straight line where C is the centre of the circle



$OC = \sqrt{5^2+12^2}$

$=13$ ✓

$\therefore OA = 13+3$

$=16$ ✓

Question 4

(a) (i) $y=mx+c$ (1)

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (2)

Put (1) into (2)

$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$

$b^2x^2+a^2(mx+c)^2=a^2b^2$

$b^2x^2+a^2(m^2x^2+2mcx+c^2)=a^2b^2$

$b^2x^2+a^2m^2x^2+2a^2mcx+a^2c^2-a^2b^2=0$

$(b^2+a^2m^2)x^2+2a^2mcx+(a^2c^2-a^2b^2)=0$ (3) ✓

If (1) is a tangent to E, then $\Delta=0$

$(2a^2mc)^2-4(b^2+a^2m^2)(a^2c^2-a^2b^2)=0$

$4a^4m^2c^2-4a^2(b^2+a^2m^2)(c^2-b^2)=0$

$a^2m^2c^2-b^2c^2+b^4-a^2m^2c^2+a^2b^2m^2=0$

$b^4+a^2b^2m^2-b^2c^2=0$ ✓

$b^2+a^2m^2-c^2=0$ $\because b\neq 0$

$\therefore c^2=a^2m^2+b^2$

(ii) Substitute $a^2=b^2=r^2$ into $c^2=a^2m^2+b^2$ ✓

$\therefore c^2=r^2m^2+r^2$

(iii) If $y=mx+c$ is a common tangent,

then $c^2=a^2m^2+b^2=r^2m^2+r^2$

ie $6m^2+3=\frac{9}{2}m^2+\frac{9}{2}$ ✓

$12m^2+6=9m^2+9$

$3m^2=3$

$m=\pm 1$ ✓

when $m=1$, $c^2=6+3$

$=9$

$c=\pm 3$

$\therefore$ The equations of the common tangent are

$y=x\pm 3$

when $m=-1$, $c^2=6+3$

$c=\pm 3$ ✓

$\therefore$ The equations are $y=-x\pm 3$.

(b)
(i) $P(cp, \frac{c}{p})$, $Q(cq, \frac{c}{q})$

$\therefore$ Equation of chord PQ is

$$\frac{y-\frac{c}{q}}{x-cq} = \frac{\frac{c}{p}-\frac{c}{q}}{cp-cq}$$

$$y-\frac{c}{q} = \frac{\frac{1}{p}-\frac{1}{q}}{p-q}(x-cq)$$

$$pq(y-\frac{c}{q}) = pq\frac{\frac{1}{p}-\frac{1}{q}}{p-q}(x-cq)$$

$$pqy - cp = \frac{q-p}{p-q}(x-cq)$$

$$pqy - cp = cq - x$$

ie $x + pqy = c(p+q)$ (1) ✓

(ii) At L, $y=0$

$\therefore$ $x = c(p+q)$

$\therefore$ L is $(c(p+q), 0)$

At M, $x=0$,

from (1) $pqy = c(p+q)$

$\therefore$ $y = \frac{c(p+q)}{pq}$

$\therefore$ M is $\left(0, \frac{c(p+q)}{pq}\right)$ ✓

mid-pt of PQ is

$$N\left(\frac{c(p+q)}{2}, \frac{c}{2}\left(\frac{1}{p}+\frac{1}{q}\right)\right)$$

$$= \left(\frac{c(p+q)}{2}, \frac{c}{2}\cdot\frac{p+q}{pq}\right)$$

$$= \left(\frac{c(p+q)}{2}, \frac{c(p+q)}{2pq}\right)$$

mid-pt of LM is

$$N' = \left(\frac{c(p+q)}{2}, \frac{c(p+q)}{2pq}\right)$$

$\therefore$ $N = N'$ ✓

Hence $PN = QN$

$LN = MN$

$PN - LN = QN - MN$

$\therefore$ $PL = QM$. ✓

(iii)(α) $xy = 9$ $c=3$

$\therefore$ Let P & Q be the pts $(3p, \frac{3}{p})$ & $(3q, \frac{3}{q})$ respectively.

By result of (i) eq^n of PQ is

$$x + pqy = 3(p+q) \quad (2)$$

when this line meets the parabola $x^2 = 4y$,

Put $y = \frac{x^2}{4}$ into (2)

P.7

$$x + \frac{pqx^2}{4} = 3(p+q)$$

$$4x + pqx^2 = 12(p+q)$$

$$pqx^2 + 4x - 12(p+q) = 0$$ ✓

Since PQ is a tangent to $x^2 = 4y$

$\therefore$ $\Delta = 0$

ie $4^2 + 4pq(12)(p+q) = 0$

$16 + 48pq(p+q) = 0$

$1 + 3pq(p+q) = 0$ (3) ✓

(β) At N. $x = \frac{3}{2}(p+q)$ (4)

$y = \frac{3}{2}(\frac{1}{p}+\frac{1}{q})$

$= \frac{3(p+q)}{2pq}$ (5)

$\frac{(4)}{(5)}$ $pq = \frac{x}{y}$ (6) ✓

From (3) $p+q = \frac{2x}{3}$ (7) ✓

Put (6) & (7) into (3)

$$1 + 3\left(\frac{x}{y}\right)\left(\frac{2x}{3}\right) = 0$$ ✓

$$1 + \frac{2x^2}{y} = 0$$

ie $2x^2 + y = 0$

$\therefore$ Equation of the locus of N is $2x^2 + y = 0$

Question 5

(a) (i) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= 1 - 2(3)$

$= -5$ ✓

Since sum of squares of real numbers must be non-negative, hence not all α, β & γ are real, ✓ but complex roots of a polynomial equation occur in conjugate pairs, $\therefore$ exactly 2 of α, β & γ are complex ✓

(ii) Let $y = x^2$

$\therefore$ $x^3 - x^2 + 3x + 4 = 0$ is transformed into

$$y\sqrt{y} - y + 3\sqrt{y} + 4 = 0$$

$$\sqrt{y}(y+3) = y - 4$$ ✓

$$y(y+3)^2 = (y-4)^2$$

$$y(y^2+6y+9) = y^2 - 8y + 16$$

$$y^3 + 6y^2 + 9y = y^2 - 8y + 16$$

$$y^3 + 5y^2 + 17y - 16 = 0$$

$\therefore$ Equation whose roots are $\alpha^2, \beta^2, \gamma^2$ is

$$x^3 + 5x^2 + 17x - 16 = 0$$ ✓

P.8

(b) (i) Let $P(x) = (x-\alpha)^2 Q(x)$

$$P'(x) = 2(x-\alpha)Q(x) + (x-\alpha)^2 Q'(x)$$
$$= (x-\alpha)[2Q(x) + (x-\alpha)Q'(x)]$$

$\therefore P'(\alpha) = 0$

hence $x = \alpha$ must also be a root of $P'(x) = 0$

(ii) When $xy = c^2$ meets $\frac{(x-1)^2}{6} + y^2 = 1$

$$\frac{(x-1)^2}{6} + \left(\frac{c^2}{x}\right)^2 = 1$$

$$x^2(x-1)^2 + 6c^4 = 6x^2$$
$$x^2(x^2-2x+1) + 6c^4 = 6x^2$$
$$x^4 - 2x^3 + x^2 + 6c^4 - 6x^2 = 0$$
$$x^4 - 2x^3 - 5x^2 + 6c^4 = 0 \qquad (1)$$

Since $xy = c^2$ touches $\frac{(x-1)^2}{6} + y^2 = 1$ at Q

$\therefore$ equation (1) must have a repeated root.

Q lies in the 3rd quadrant, $\therefore$ x-coordinate of Q must be negative

$\therefore \quad \alpha < 0$

$\therefore$ (1) must have a negative repeated root α

(iii) Let $P(x) = x^4 - 2x^3 - 5x^2 + 6c^4$

$$P'(x) = 4x^3 - 6x^2 - 10x$$
$$P'(x) = 0$$
$$2x(2x^2 - 3x - 5) = 0$$
$$2x(2x-5)(x+1) = 0$$

$\therefore x = 0, \frac{5}{2}$ or -1.

Since $\alpha < 0$

$\therefore \quad \alpha = -1$

$$P(-1) = 0$$
$$(-1)^4 - 2(-1)^3 - 5(-1)^2 + 6c^4 = 0$$
$$1 + 2 - 5 + 6c^4 = 0$$
$$6c^4 = 2$$
$$\therefore c^4 = \frac{1}{3}$$

(c) (i) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

Put $x = \cos\theta$

Roots of $\cos 5\theta = 1 = \text{cis}\, 0$

are $\cos\theta = \text{cis}\,\frac{2r\pi}{5} \qquad r = 0, 1, 2, 3, 4$

$\therefore$ Roots of $16x^5 - 20x^3 + 5x = 1$

ie $16x^5 - 20x^3 + 5x - 1 = 0$

are $x = \text{cis}\,\frac{2r\pi}{5} \qquad r = 0, 1, 2, 3, 4.$

(ii) Product of roots of $16x^5 - 20x^3 + 5x - 1 = 0$

is $\cos 0 \cos\frac{2\pi}{5}\cos\frac{4\pi}{5}\cos\frac{6\pi}{5}\cos\frac{8\pi}{5} = \frac{1}{16}$

$$\cos\frac{2\pi}{5}\cos\left(\pi - \frac{\pi}{5}\right)\cos\left(\pi + \frac{\pi}{5}\right)\cos\left(2\pi - \frac{2\pi}{5}\right) = \frac{1}{16}$$
$$\cos\frac{2\pi}{5}\left(-\cos\frac{\pi}{5}\right)\left(-\cos\frac{\pi}{5}\right)\cos\frac{2\pi}{5} = \frac{1}{16}$$
$$\left(\cos\frac{\pi}{5}\cos\frac{2\pi}{5}\right)^2 = \frac{1}{16}$$
$$\cos\frac{\pi}{5}\cos\frac{2\pi}{5} = \frac{1}{4}$$

since $\cos\frac{\pi}{5} > 0, \quad \cos\frac{2\pi}{5} > 0$

Question 6

(a) (i) $u = x^2$

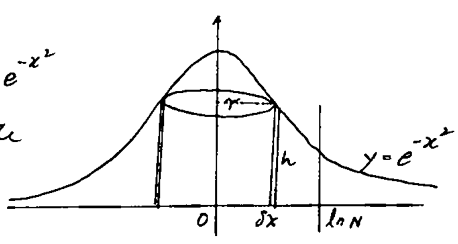
$du = 2x\,dx$

$$\therefore \int xe^{-x^2}dx = \frac{1}{2}\int e^{-u}\,du$$
$$= -\frac{1}{2}e^{-u} + C$$
$$= -\frac{1}{2}e^{-x^2} + C$$

(ii) $r = x$,

$h = y = e^{-x^2}$

Volume of the cylindrical shell is



$$\delta V = 2\pi r h \delta x$$
$$= 2\pi x e^{-x^2}\delta x$$

$$\therefore \text{Volume} = \lim_{\delta x \to 0}\sum \delta V$$
$$= \lim_{\delta x \to 0}\sum 2\pi x e^{-x^2}\delta x$$
$$= 2\pi\int_0^{\ln N} xe^{-x^2}dx$$
$$= 2\pi\left[-\frac{1}{2}e^{-x^2}\right]_0^{\ln N} \qquad \text{(from (i))}$$
$$= \pi\left(1 - e^{-(\ln N)^2}\right)$$
$$= \pi\left[1 - e^{(-\ln N)\ln N}\right]$$
$$= \pi\left[1 - \left(\frac{1}{N}\right)^{\ln N}\right]$$

(iii) Let $u = \left(\frac{1}{N}\right)^{\ln N}$

$$\therefore \ln u = \ln N \ln\left(\frac{1}{N}\right)$$
$$= -(\ln N)^2$$

$\therefore$ as $N \to \infty$ $\ln u$ will decrease without bound,

ie $u \to 0$

$$\therefore \lim_{N\to\infty}\pi\left[1 - \left(\frac{1}{N}\right)^{\ln N}\right] = \pi(1 - 0)$$
$$= \pi$$

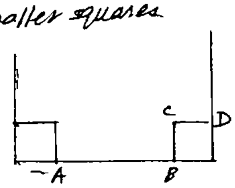
(b) (i)

$$x^2+z^2=16$$
$$x^2=16-z^2$$
$$x=\sqrt{16-z^2}$$ ✓

(ii) Each side of the square = 8

$$AB = 2x = 2\sqrt{16-z^2}$$

∴ Length of each side of the smaller squares at the corners



$$= \tfrac{1}{2}\left[8 - 2\sqrt{16-z^2}\right] = 4-\sqrt{16-z^2}$$ ✓

∴ Sum of areas of the squares at the corners

$$= 4\left[4-\sqrt{16-z^2}\right]^2$$

∴ Area of the slice $= 8^2 - 4\left[4-\sqrt{16-z^2}\right]^2$ ✓

∴ Volume of the slice $\delta V = \left\{8^2 - 4\left[4-\sqrt{16-z^2}\right]^2\right\}\delta z$ ✓

∴ Volume of the solid

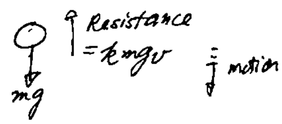
$$V = \lim_{\delta z \to 0} \sum \delta V = \int_0^4 \left\{8^2 - 4\left[4-\sqrt{16-z^2}\right]\right\} dz$$

(iii)

$$V = \int_0^4 \left(64 - 16 + 4\sqrt{16-z^2}\right) dz$$
$$= \int_0^4 \left(48 + 4\sqrt{16-z^2}\right) dz$$ ✓
$$= \left[48z\right]_0^4 + 2 \times \text{area of semi-circle of radius } 4$$
$$= 192 + 2 \times \tfrac{1}{2}\pi(4^2)$$ ✓
$$= 192 + 16\pi$$ ✓

Question 7

(a) (i) When the body is moving downward, take downward as positive direction.

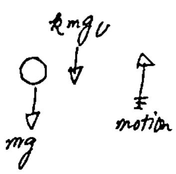


$$m\ddot{x} = mg - kmgv = mg(1-kv)$$ ✓

at terminal velocity, $\ddot{x}=0$

$$\therefore 1 - kV_T = 0$$
$$V_T = \frac{1}{k}$$ ✓

(ii) Take upward as positive direction.



$$m\ddot{x} = -mg + kmgv = -mg(1+kv)$$
$$\ddot{x} = -g(1+kv)$$ ✓

$$v\frac{dv}{dx} = -g(1+kv)$$
$$\int_u^0 \frac{v\,dv}{1+kv} = -\int_0^H g\,dx$$ ✓
$$\frac{1}{k}\int_u^0 \left(1 - \frac{1}{1+kv}\right) dv = -g\left[x\right]_0^H$$
$$\frac{1}{k}\left[v - \frac{1}{k}\ln(1+kv)\right]_u^0 = -gH$$ ✓
$$\frac{1}{k}\left[\left(0-\frac{1}{k}\ln 1\right) - u + \frac{1}{k}\ln(1+ku)\right] = -gH$$
$$\therefore H = \frac{1}{gk}\left[u - \frac{1}{k}\ln(1+ku)\right] = \frac{u}{gk} - \frac{1}{gk^2}\ln(1+ku)$$

(iii) For upward motion:

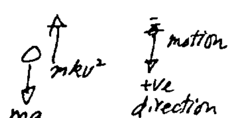
$$\ddot{x} = -g(1+kv)$$
$$\frac{dv}{dt} = -g(1+kv)$$
$$\int_{V_T}^0 \frac{dv}{1+kv} = -\int_0^T g\,dt$$ ✓
$$\frac{1}{k}\left[\ln(1+kv)\right]_{V_T}^0 = -\left[gt\right]_0^T$$

$$\frac{1}{k}\ln(1+kV_T) = gT$$
$$\therefore T = \frac{1}{gk}\ln(1+kV_T)$$ ✓

but $V_T = \frac{1}{k}$

$$\therefore T = \frac{V_T}{g}\ln\left(1 + k\cdot\frac{1}{k}\right)$$
$$\therefore T = \frac{V_T}{g}\ln 2$$ ✓

(b)



$$m\ddot{x} = mg - mkv^2$$
$$\ddot{x} = g - kv^2$$
$$\frac{dv}{dt} = g - kv^2$$
$$\int_{\frac{V_T}{3}}^{\frac{V_T}{2}} \frac{dv}{g-kv^2} = \int_0^T dt$$ ✓
$$\frac{1}{k}\int_{V_T/3}^{\frac{V_T}{2}} \frac{dv}{\frac{g}{k}-v^2} = T$$
$$T = \frac{1}{k}\int_{\frac{V_T}{3}}^{\frac{V_T}{2}} \frac{dv}{\left(\sqrt{\frac{g}{k}}-v\right)\left(\sqrt{\frac{g}{k}}+v\right)}$$

when terminal velocity is attained

$$\ddot{x}=0$$

$g - kV_T^2 = 0$

$\therefore V_T = \sqrt{\frac{g}{k}}$ or $k = \frac{g}{V_T^2}$

$\therefore T = \frac{1}{k}\int_{\frac{V_T}{3}}^{\frac{V_T}{2}} \frac{dv}{(V_T-v)(V_T+v)}$

$= \frac{1}{2kV_T}\int_{\frac{V_T}{3}}^{\frac{V_T}{2}}\left(\frac{1}{V_T-v}+\frac{1}{V_T+v}\right)dv$

$= \frac{1}{2kV_T}\left[\ln(V_T+v) - \ln(V_T-v)\right]_{\frac{V_T}{3}}^{V_T/2}$

$= \frac{1}{2kV_T}\left[\ln\left(\frac{V_T+v}{V_T-v}\right)\right]_{\frac{V_T}{3}}^{\frac{V_T}{2}}$

$= \frac{1}{2kV_T}\left[\ln\left(\frac{V_T+\frac{V_T}{2}}{V_T-\frac{V_T}{2}}\right) - \ln\left(\frac{V_T+\frac{V_T}{3}}{V_T-\frac{V_T}{3}}\right)\right]$

$= \frac{1}{2kV_T}\left[\ln 3 - \ln 2\right]$

$= \frac{1}{2V_T}\cdot\frac{V_T^2}{g}\ln\frac{3}{2}$ $\quad \because k = \frac{g}{V_T^2}$

$= \frac{V_T}{2g}\ln\frac{3}{2}$

(c) (i) $m\ddot{x} = -\frac{mk}{x^2}$

at surface of earth, $x = R$ & $m\ddot{x} = -mg$

$\therefore -mg = -\frac{mk}{R^2}$

$\therefore k = gR^2$

(ii) $m\ddot{x} = -\frac{mk}{x^2}$ $\qquad \frac{mk}{x^2}$ $\qquad$ +ve motion

$\ddot{x} = -\frac{k}{x^2}$

$= -\frac{gR^2}{x^2}$

$\frac{d}{dx}\left(\frac{v^2}{2}\right) = -\frac{gR^2}{x^2}$

$\therefore \int_U^V d\left(\frac{v^2}{2}\right) = -\int_R^x \frac{gR^2}{x^2}\,dx$

$\left[\frac{v^2}{2}\right]_U^V = \left[\frac{gR^2}{x}\right]_R^x$

$\frac{V^2}{2} - \frac{U^2}{2} = gR^2\left(\frac{1}{x} - \frac{1}{R}\right)$

$\therefore V^2 = U^2 + 2gR^2\left(\frac{1}{x} + \frac{1}{R}\right)$

P.13

Question 8

(a) $U_n = 3^n - 2^n$

when $n=1$, $U_1 = 3 - 2 = 1$

when $n = 2$ $U_2 = 3^2 - 2^2 = 5$

$\therefore$ It is true for $n=1$ & $n=2$.

Assume it is true for $n=k$ and $n=k+1$

ie $U_k = 3^k - 2^k$

$U_{k+1} = 3^{k+1} - 2^{k+1}$

then $U_{k+2} = 5U_{k+1} - 6U_k$

$= 5(3^{k+1} - 2^{k+1}) - 6(3^k - 2^k)$

$= 15(3^k) - 10(2^k) - 6(3^k) + 6(2^k)$

$= 9(3^k) - 4(2^k)$

$= 3^2\cdot 3^k - 2^2\cdot 2^k$

$= 3^{k+2} - 2^{k+2}$

Hence it will be true for $n=k+2$ if it is true for $n=k$ and $n=k+1$. Since it is proved true for $n=1$ and $n=2$,

$\therefore$ it will be true for $n = 3, 4, 5, \dots$ ie all positive integers n.

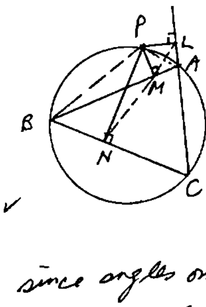
(b) Let $S = \sin\theta + \sin 2\theta + \dots + \sin(n-1)\theta + \sin n\theta$

$\therefore 2S\cdot\sin\frac{\theta}{2} = 2\sin\theta\sin\frac{\theta}{2} + 2\sin2\theta\sin\frac{\theta}{2} + \dots + 2\sin(n-1)\theta\sin\frac{\theta}{2} + 2\sin n\theta\sin\frac{\theta}{2}$

$= \left[\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}\right] + \left[\cos\frac{3\theta}{2} - \cos\frac{5\theta}{2}\right] + \dots + \left[\cos\frac{(2n-3)\theta}{2} - \cos\frac{(2n-1)\theta}{2}\right] + \left[\cos\frac{(2n-1)\theta}{2} - \cos\frac{(2n+1)\theta}{2}\right]$

$= \cos\frac{\theta}{2} - \cos\frac{(2n+1)\theta}{2}$

$\therefore S = \frac{1}{2\sin\frac{\theta}{2}}\left[\cos\frac{\theta}{2} - \cos\frac{(2n+1)\theta}{2}\right]$



(c) (i) $\angle PMA + \angle PLA = 2$ rt $\angle$s

$\therefore P, M, A, L$ are concyclic since opposite angles are supplementary

(ii) $\angle PMB = \angle PNB$

$\therefore P, B, M, N$ are concyclic since angles on the same chord in the same segment are equal

(iii) $\angle PAL = \angle PBC$ (ext $\angle$ of cyclic quad)

$\angle AML = \angle APL$ (angles in the same segment)

$= 90° - \angle PAL$ ($\angle$s sum of $\triangle$)

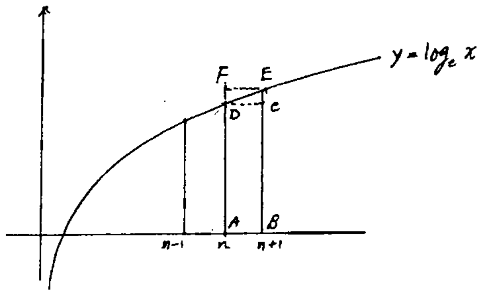
$\therefore \angle AML = 90° - \angle PBC$

$= \angle BPN$ ($\angle$ sum of ΔPBN) ✓

$= \angle BMN$ ($\angle$'s in the same segment)

$\therefore L, M, N$ must be collinear (vert. opp $\angle$'s equal)

(d) (i)



$AD = \ln n$

$AB = 1$

$\therefore$ Area of $ABCD = \log_e n$

Area of $ABCD <$ area under $y = \log_e x$ between $x = n$ and $x = n+1$

$\therefore \log_e n < \int_n^{n+1} \log_e x \, dx$

Similarly $\int_n^{n+1} \log_e x \, dx <$ area of $ABEF$

$\int_n^{n+1} \log_e x \, dx < \log_e (n+1)$

$\therefore \log_e n < \int_n^{n+1} \log_e x \, dx < \log_e (n+1)$

(ii) $\log_e n < \left[x \ln x - x\right]_n^{n+1} < \ln(n+1)$

$\ln n < (n+1)\ln(n+1) - (n+1) - n \ln n + n < \ln(n+1)$

$\ln n < (n+1)\ln(n+1) - n\ln n - 1 < \ln(n+1)$ (1) ✓

By considering the LHS & middle section

$(n+1)\ln n < (n+1)\ln(n+1) - 1$

$1 < (n+1)\ln\left(\frac{n+1}{n}\right)$

$\frac{1}{n} < \left(\frac{n+1}{n}\right)\ln\left(\frac{n+1}{n}\right)$

$e^{\frac{1}{n}} < \left(\frac{n+1}{n}\right)^{\frac{n+1}{n}}$

$e^{\frac{1}{n}} < \left(1+\frac{1}{n}\right)^{\frac{n+1}{n}}$ (2) ✓

Excluding the left-hand most side of (1)

$(n+1)\ln(n+1) - n\ln n - 1 < \ln(n+1)$

$n\ln(n+1) - n\ln n < 1$

$n\ln\left(\frac{n+1}{n}\right) < 1$

$\ln\left(\frac{n+1}{n}\right) < \frac{1}{n}$

$\therefore 1+\frac{1}{n} < e^{\frac{1}{n}}$ (3) ✓

From (2) & (3) $1+\frac{1}{n} < e^{\frac{1}{n}} < \left(1+\frac{1}{n}\right)^{\frac{n+1}{n}}$