AW

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Class: 12MTZ1
Teacher: MRS WIDMER

CHERRYBROOK TECHNOLOGY HIGH SCHOOL

2009 AP4

YEAR 12 TRIAL HSC EXAMINATION

MATHEMATICS EXTENSION 2

Time allowed - 3 HOURS
(Plus 5 minutes' reading time)

DIRECTIONS TO CANDIDATES:

- Attempt all questions.
- All questions are of equal value.
- Each question is to be commenced on a new page clearly marked Question 1, Question 2, etc on the top of the page. **
- All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used. Standard Integral Tables are provided.
- Your solutions will be collected in one bundle stapled in the top left corner.

Please arrange them in order, Q1 to 8.

*****Each page must show your name and your class. *****

QUESTION ONE. (15 MARKS)

MARKS

(a) Evaluate $\int_0^3 \frac{x}{\sqrt{16+x^2}}\,dx$. 3

(b) Find $\int \frac{x^2+x+1}{x(x^2+1)}\,dx$. 2

(c) By using the substitution $t = \tan\frac{\vartheta}{2}$, evaluate $\int_0^{\frac{\pi}{2}} \frac{d\vartheta}{2+\sin\vartheta}$. 3

(d) Find $\int xe^{-x}\,dx$. 2

(e) (i) If $I_n = \int_0^1 (1+x^2)^n\,dx,\ n = 0,1,2, \ldots..$ show that 3

$(2n+1)\,I_n = 2^n + 2n\,I_{n-1}$ for $n = 1,2, \ldots ....$

(ii) Hence find a reduction formula for $J_m = \int_0^{\frac{\pi}{4}} \sec^{2m}x\,dx$. 2

QUESTION TWO. (15 MARKS) (START A NEW PAGE)

(a) If z is such that $|z| = 3$ and $\arg z = \frac{\pi}{3}$, mark on the same Argand diagram 4

(i) z (ii) $\bar{z}$ (iii) iz (iv) z^{-1}

(b) In an Argand Diagram, the point P representing the complex number z moves so that $|z-(1+i)| = 1$.

(i) Sketch the locus of P 1

(ii) Shade the region where $|z-(1+i)| = 1$ and $0 < \arg(z-1) < \frac{\pi}{4}$. 1

Question 2 continued on next page.

QUESTION TWO CONTINUED. **MARKS**

(c) Given $z = cos\vartheta + isin\vartheta$ and for the positive integers n,

$$z^n + \frac{1}{z^n} = 2cosn\vartheta \quad \text{and} \quad z^n - \frac{1}{z^n} = 2isinn\vartheta.$$

(i) Expand $(z + \frac{1}{z})^4 + (z - \frac{1}{z})^4$ to show that $cos^4\vartheta + sin^4\vartheta = \frac{1}{4}(cos4\vartheta + 3)$ **2**

(ii) By letting $x = cos\vartheta$, show that the equation $8x^4 + 8(1 - x^2)^2 = 7$ **2**

has roots $\pm cos\frac{\pi}{12}$, $\pm cos\frac{5\pi}{12}$

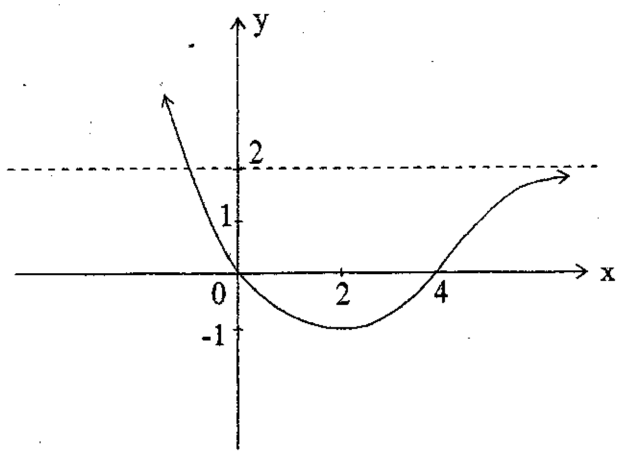
(iii) Show that $cos\frac{\pi}{12} . cos\frac{5\pi}{12} = \frac{1}{4}$ and **3**

$$cos\frac{\pi}{12} + cos\frac{5\pi}{12} = \sqrt{\frac{3}{2}}$$

(iv) Hence or otherwise, find a surd expression for $cos\frac{\pi}{12}$ **2**

QUESTION THREE. (15 MARKS) (START A NEW PAGE) **MARKS**

(a) The diagram shows the graph of $f(x)$.



Sketch on separate diagrams, the following curves, indicating clearly any turning points and asymptotes.

(i) $y = \left|\dfrac{1}{f(x)}\right|$ **2**

(ii) $y = (f(x))^2$ **2**

(iii) $y^2 = f(x)$ **2**

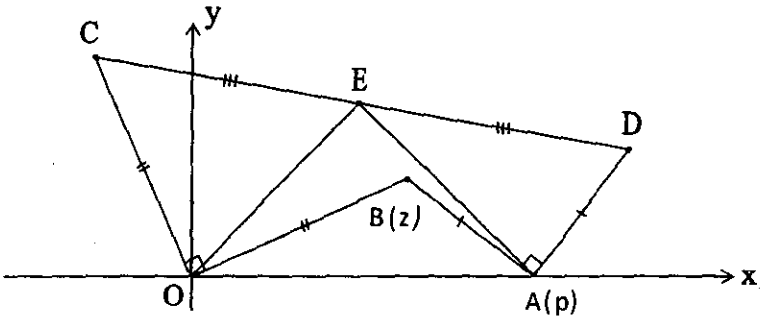
(iv) $y = x f(x)$ **2**

(b) (i) Show that the area enclosed by a parabola $x^2 = 4ay$ and its latus rectum is given by $A = \dfrac{8a^2}{3}$ units^2. **3**

(ii) A solid is formed such that its base is a semicircle of radius one metre. Vertical sections parallel to the diameter are parabolas with each latus rectum being a chord of the semicircle parallel to the diameter. By using the result from (i) and the technique of slicing, find the volume of this solid. **4**

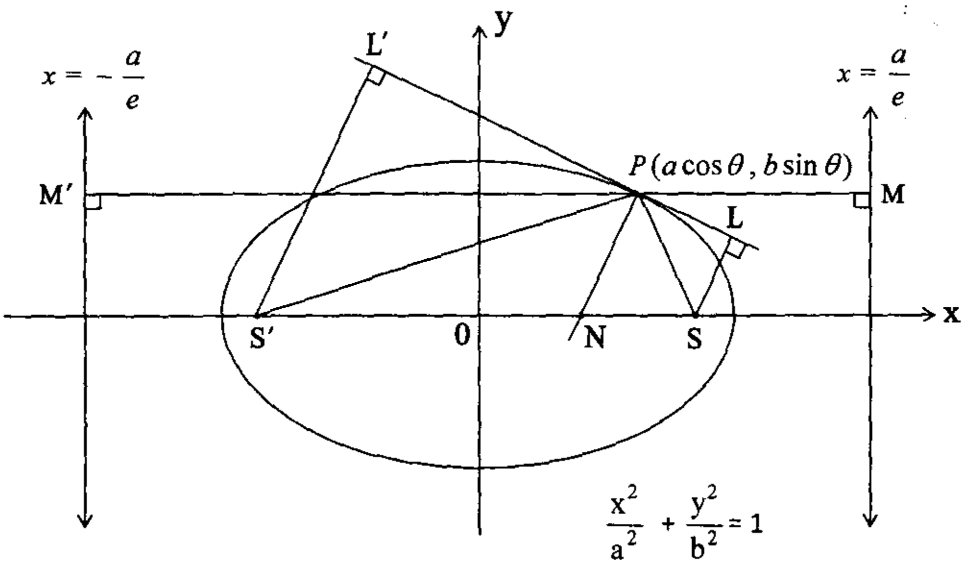
QUESTION FOUR. (15 MARKS) (START A NEW PAGE) **MARKS**

(a) In the Argand diagram O, A and B represent the origin, the number p and the complex number z respectively. By rotating B about A by 90° in a clockwise direction we get the point D and by rotating B about O in an anticlockwise direction we get the point C. Let E be the midpoint of CD. **3**



Show that ΔOEA is a right angled isosceles triangle with a right angle at E.

(b) Lines drawn from the foci S and S' of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, are perpendicular to the tangent drawn at $P(a\cos\theta, b\sin\theta)$. They meet this tangent respectively at L and L'.
The line parallel to the x-axis passing through P intersects the directrices at M and M' and the normal at P meets the x-axis at N.



Question 4 continued on next page.

QUESTION 4(b) CONTINUED. **MARKS**

(b) (i) Show that $PS = a(1 - e\cos\theta)$ **2**

(ii) Write down a similar expression for PS'. **1**

(iii) Show that the equation of the tangent at P is **2**

$bx\cos\theta + ay\sin\theta - ab = 0$

(iv) Find the distances SL and $S'L'$ from the foci S and S' to the tangent at P. **2**

(v) Hence, or otherwise, show that PN bisects $\angle SPS'$. **3**

(vi) Show that $\frac{PS}{NS} = \frac{PS'}{NS'}$ **2**

QUESTION FIVE. (15 MARKS) (START A NEW PAGE)

(a) Given that a, b, c and d represent positive integers and that $a + b + c = 3d$ show that $100a + 10b + c$ is divisible by 3. **2**

(b) The roots of $x^3 + 3px + q = 0$ are α, β and γ, (none of which are equal to 0).

(i) Find the monic equation with roots $\frac{\beta\gamma}{\alpha}$, $\frac{\alpha\gamma}{\beta}$ and $\frac{\alpha\beta}{\gamma}$, giving the coefficients in terms of p and q. **4**

(ii) Deduce that if $\gamma = \alpha\beta$ then $(3p - q)^2 + q = 0$. **2**

(c) Determine the values of a and b given that $(x + 1)^2$ is a factor of $P(x) = x^5 + 2x^2 + ax + b$. **3**

(d) Show that $\frac{\sin(2k+1)\alpha}{sin\alpha} - \frac{\sin(2k-1)\alpha}{sin\alpha} = 2\cos(2k\alpha)$ if $0 < \alpha < \frac{\pi}{2}$. **2**

(e) Given that $sin^{-1}x, cos^{-1}x$ and $sin^{-1}(1 - x)$ are acute, show that **2**

$\sin(sin^{-1}x - cos^{-1}x) = 2x^2 - 1$.

QUESTION SIX. (15 MARKS) (START A NEW PAGE) **MARKS**

(a) A toy car of mass *1kg*, initially at rest, starts to move by a propelling force of *50N*, provided by its engine along a straight road.

The car experiences a resistance force of kv^2 Newtons, where v is its velocity in metres per second and k is a positive constant. The limiting velocity of the car is *10m/s.*

Let x be the displacement of the car at time t seconds after it starts to move.

(i) Show that $2\dfrac{d^2x}{dt^2} = 100 - v^2$ **2**

(ii) Show that $v^2 = 100(1 - e^{-x})$ **2**

(iii) Find the time taken for the car to reach a velocity of *5 m/s*. **3**

(iv) When the car reaches a velocity of *5 m/s* the engine is switched off and a breaking force of F is applied. Find F, given that the distance travelled by the car to stop is equal to the distance to reach *5 m/s*. **3**

(b) The depth of water in a harbour on a particular day is *8.2* metres at low tide and *14.6* metres at high tide. Low tide is at 1:05 pm and high tide is at 7:20pm. The captain of a ship wants to leave the harbour after midday on that day. To leave the harbour the ship requires at least *13.3* metres of water. **5**

Find between what two times of that day the captain can leave the harbour.

QUESTION SEVEN. (15 MARKS) (START A NEW PAGE) **MARKS**

(a) The part of the curve $y = ln\frac{x}{e}$ between $x = e$ and $x = e^2$ is rotated about the $x - axis$ to form a solid.

(i) Draw the curve $y = ln\frac{x}{e}$ between $x = e$ and $x = e^2$ and show a sketch of the solid formed by the rotation of this curve. **1**

(ii) Use the method of cylindrical shells to find the volume of the solid. **4**

(b) A body is projected vertically upwards from the surface of the Earth with initial speed u. The acceleration due to gravity, g at any point on its path is inversely proportional to the square of the distance from the centre of the Earth. R is the radius of the Earth.

(i) Prove that the speed v at any position x is given by **3**

$v^2 = u^2 + 2gR^2(\frac{1}{x} - \frac{1}{R})$

(ii) Prove that the greatest height H above the Earth's surface is **2**

given by $H = \frac{u^2R}{2gR - u^2}$

(iii) Show that the body will escape from the Earth if $u \geq \sqrt{2gR}$ **1**

(iv) Find the minimum speed in *km/s* with which the body must be initially projected from the surface of the Earth so as to never return. **1**
(Take $R = 6400km, g = 10m/s^2$)

(v) If $u = \sqrt{2gR}$ prove that the time taken to reach a height $3R$, **3**

above the surface of the Earth is equal to $\frac{14}{3}\sqrt{\frac{R}{2g}}$.

QUESTION EIGHT. (15 MARKS) (START A NEW PAGE) **MARKS**

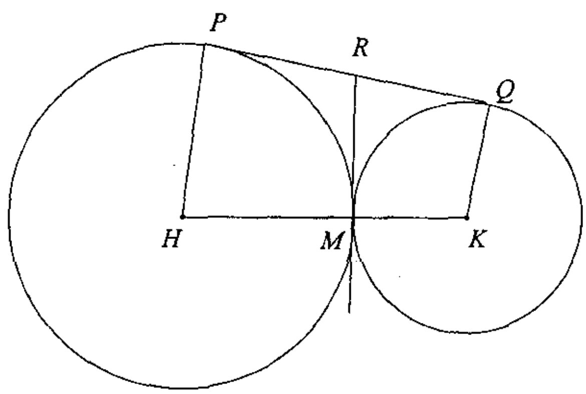
(a) (i) Sketch the graph of $y = px^2 + q$ where p and q are positive constants. **1**

(ii) By considering the area represented by $\int_1^2 (px^2 + q)dx$, show that **2**

$$p + q < \frac{7p + 3q}{3} < 4p + q$$

(b) Shown are two circles centres H and K which touch at M.

PQ and RM are common tangents.



(i) Show that quadrilaterals $HPRM$ and $MRQK$ are cyclic. **2**

(ii) Prove that triangles PRM and MKQ are similar. **3**

Question 8 continued on the next page.

QUESTION EIGHT CONTINUED.

(c) The Taylor series provides a way of expressing certain functions as infinite series. Using the Taylor series, it can be shown that

$$sinx = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots\ldots$$

$$cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots\ldots\ldots$$

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \ldots\ldots\ldots$$

(i) Using these infinite series, show that $e^{i\pi} = -1$ **2**

(ii) Show i^i is a real number. **2**

(iii) Find the general formula for lnz where $z = x + iy$, and state the interval for lnz that will give the principle value of lnz. **2**

(iv) Hence, find the principle value for the complex number $z = 1 + i$ **1**

END OF TEST.

Solutions: (1) 2009

Question One: (15 Marks)

(a) Let $u = 16 + x^2$

$\frac{du}{dx} = 2x$

$\therefore du = 2x\,dx$. (1)

If $\begin{cases} x = 0 & x = 3 \\ u = 16 & u = 25 \end{cases}$

$\therefore \int_0^3 \frac{x\,dx}{\sqrt{16+x^2}} = \int_{16}^{25} \frac{\frac{1}{2}\,du}{\sqrt{u}}$

$= \frac{1}{2}\int_{16}^{25} u^{-1/2}\,du$

$= \frac{1}{2}\left[2u^{1/2}\right]_{16}^{25}$ (1)

$= \left[\sqrt{u}\right]_{16}^{25}$

$= \sqrt{25} - \sqrt{16}$

$= 1.$ (1)

ALTERNATIVE:

OR $x = 4\tan\theta$, $dx = 4\sec^2\theta\,d\theta$

$\int_0^{\tan^{-1}\frac{3}{4}} \frac{4\tan\theta}{\sqrt{16+16\tan^2\theta}} \cdot 4\sec^2\theta\,d\theta$

$= \int_0^{\tan^{-1}\frac{3}{4}} 4\tan\theta\sec\theta\,d\theta$

$= 4\left[\sec\theta\right]_0^{\tan^{-1}3/4}$

$= 4\left\{\sec\left(\tan^{-1}\left(\frac{3}{4}\right)\right) - \sec 0\right\}$

$= 4\left\{\frac{5}{4} - 1\right\} = 1.$

(b) $\int \frac{x^2+x+1}{x(x^2+1)}\,dx = \int \frac{(x^2+1)+x}{x(x^2+1)}\,dx = \int \left(\frac{1}{x} + \frac{1}{x^2+1}\right)dx$ (1)

$= \ln|x| + \tan^{-1}x + C$ (1)

(c) $\begin{cases} t = \tan\theta/2 \\ \frac{dt}{d\theta} = \frac{1}{2}\sec^2\theta/2 \\ dt = \frac{1}{2}(1+t^2)\,d\theta \\ \text{Hence } d\theta = \frac{2}{(1+t^2)}\,dt. \end{cases}$

$\begin{cases} \text{When } \theta = 0 & t = 0 \\ \theta = \frac{\pi}{2} & t = 1 \end{cases}$

$\begin{cases} \text{Also } \sin\theta = \frac{2t}{1+t^2} \\ \text{So } \int_0^{\pi/2} \frac{d\theta}{2+\sin\theta} = \end{cases}$

$= \int_0^1 \frac{2}{1+t^2} \times \frac{1}{2+\frac{2t}{1+t^2}}\,dt$ (1)

$= \int_0^1 \frac{dt}{t^2+t+1}$

$= \int_0^1 \frac{dt}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{t+1/2}{\sqrt{3}/2}\right]_0^1$ (1)

$= \frac{2}{\sqrt{3}}\left[\tan^{-1}\sqrt{3} - \tan^{-1}\frac{1}{\sqrt{3}}\right] = \frac{2}{\sqrt{3}}\left[\frac{\pi}{3} - \frac{\pi}{6}\right] = \frac{\pi}{3\sqrt{3}}$

OR $= \frac{\sqrt{3}\pi}{9}$ (1) $\doteqdot 0.6045.$

(d) $\int xe^{-x}\,dx = \int x\frac{d}{dx}\left(-e^{-x}\right)dx$

$= -xe^{-x} - \int 1.\left(-e^{-x}\right)dx$ (1)

$= -xe^{-x} + \int e^{-x}\,dx$

$= -xe^{-x} - e^{-x} + C$ (1)

OR $e^{-x}(-x-1) + C$ or $-e^{-x}(x+1) + C$

(e) $I_n = \int_0^1 (1+x^2)^n\,dx$

(1) $= \left[x(1+x^2)^n\right]_0^1 - \int_0^1 x.\,n(1+x^2)^{n-1}.2x\,dx$

$= 2^n - 2n\int_0^1 x^2(1+x^2)^{n-1}\,dx$

(1) $= 2^n - 2n\int_0^1 (1+x^2-1)(1+x^2)^{n-1}\,dx$

$= 2^n - 2n\left\{\int_0^1 (1+x^2)^n\,dx - \int_0^1 (1+x^2)^{n-1}\,dx\right\}$

(1) $I_n = 2^n - 2nI_n + 2nI_{n-1}$

$\therefore (2n+1)I_n = 2^n + 2nI_{n-1},\ n = 1, 2, 3$

(ii) $u = \tan x$ $\quad x = 0 \quad u = 0$
$du = \sec^2 x\,dx$ $\quad x = \pi/4 \quad u = 1$

$$J_m = \int_0^{\pi/4} \sec^{2m} x \, dx$$

$$= \int_0^{\pi/4} (\sec^2 x)^{m-1} \sec^2 x \, dx$$

1 $\longrightarrow$ $$= \int_0^1 (1+u^2)^{m-1} \, du$$

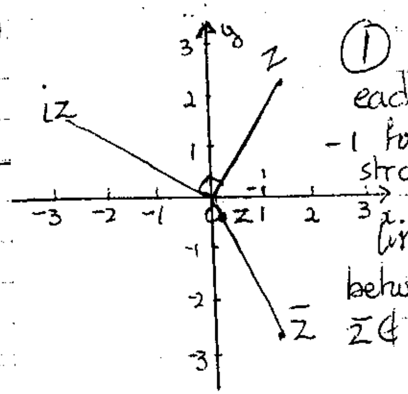
$$\therefore J_m = I_{m-1} \qquad m = 1, 2, 3 \ldots$$

$$\{2(m-1)+1\} J_m = 2^{m-1} + 2(m-1) I_{m-2}$$

1 $\longrightarrow$ $$\therefore (2m-1) J_m = 2^{m-1} + 2(m-1) J_{m-1}$$
$$m = 2, 3, 4 \ldots$$

Question Two: (15 Marks)

(a)



1 for each pt.
-1 for straight line between $\bar{z}$ & iz

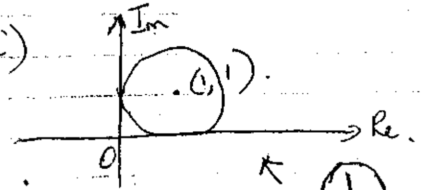
(i) $z = 3(\cos \pi/3 + i \sin \frac{\pi}{3})$
$= 3\left[\frac{1}{2} + i\frac{\sqrt{3}}{2}\right]$

(ii) $\bar{z} = \frac{3}{2} - \frac{i3\sqrt{3}}{2}$

(iii) $iz = i\left(3/2 + \frac{i3\sqrt{3}}{2}\right)$
$= \frac{3i}{2} - \frac{3\sqrt{3}}{2}$

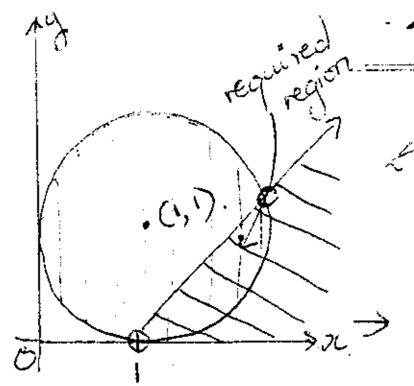
(iv) $z^{-1} = 3^{-1}\left[\cos \pi/3 + i \sin \frac{\pi}{3}\right]$
$z^{-1} = \frac{1}{3}\left[\cos \frac{-\pi}{3} + i \sin \frac{-\pi}{3}\right]$
& $\arg\left(\frac{1}{z}\right) = -\pi/3$
$\left(\frac{1}{6}, -\cdot 3\right)$

(b)(i)



1

(b)



(ii) 1 for correct region.

(i) 1 for circle, centre (1,1) radius 2.

(c)(i) $$\left(z + \frac{1}{z}\right)^4 + \left(z - \frac{1}{z}\right)^4 = 2\left(z^4 + 6z^2 \cdot \frac{1}{z^2} + \frac{1}{z^4}\right) \quad 1$$

$$= 2\left(z^4 + \frac{1}{z^4}\right) + 12$$

$$(2\cos\theta)^4 + (2i\sin\theta)^4 = 2(2\cos 4\theta) + 12$$

1 $\longrightarrow$ $$16(\cos^4\theta + \sin^4\theta) = 4(\cos 4\theta + 3)$$

$$\therefore \cos^4\theta + \sin^4\theta = 1/4(\cos 4\theta + 3)$$

(ii) $x = \cos\theta$
$8x^4 + 8(1-x^2)^2 = 7$
$1 - x^2 = \sin^2\theta$
$8(\cos^4\theta + \sin^4\theta) = 7$
$2(\cos 4\theta + 3) = 7$

Hence eqn. becomes
$x = \cos\theta$, $\cos 4\theta = 1/2$ $\longleftarrow$ 1

$4\theta = 2n\pi \pm \frac{\pi}{3}$

$\theta = \frac{(6n \pm 1)\pi}{12} \qquad n = 0, \pm 1, \pm 2 \ldots$

$x = \cos \frac{\pi}{12}, \cos \frac{5\pi}{12}, \cos \frac{7\pi}{12}, \cos \frac{11\pi}{12}$

1 $\longrightarrow$ $x = \cos \frac{\pi}{12}, \cos \frac{5\pi}{12}, \cos\left(\pi - \frac{5\pi}{12}\right) \cos\left(\pi - \frac{\pi}{12}\right)$

$\therefore x = \pm \cos \frac{\pi}{12}, \pm \cos \frac{5\pi}{12}$

(iii) $8x^4+8(1-x^2)^2=7$

$16x^4-16x^2+1=0$

roots: $\cos\frac{\pi}{12},\ -\cos\frac{\pi}{12},\ \cos\frac{5\pi}{12},\ -\cos\frac{5\pi}{12}$

(1) Then $\alpha\beta\gamma\delta=\cos^2\frac{\pi}{12}\cos^2\frac{5\pi}{12}=\frac{1}{16}$

$\sum\alpha\beta=-\cos^2\frac{\pi}{12}-\cos^2\frac{5\pi}{12}=-1$

(1) where $0<\frac{\pi}{12}<\frac{5\pi}{12}<\frac{\pi}{2}$

Then $\cos\frac{\pi}{12}\cos\frac{5\pi}{12}=+\sqrt{\frac{1}{16}}=\frac{1}{4}$ &

(1) $\cos^2\frac{\pi}{12}+\cos^2\frac{5\pi}{12}+2\cos\frac{\pi}{12}\cos\frac{5\pi}{12}=1+\frac{1}{2}$

$\therefore\left(\cos\frac{\pi}{12}+\cos\frac{5\pi}{12}\right)^2=\frac{3}{2}$

$\cos\frac{\pi}{12}+\cos\frac{5\pi}{12}=\frac{\sqrt{3}}{2}$

(iv) $\cos\frac{\pi}{12},\ \cos\frac{5\pi}{12}$ are roots of the quad. eqn.

(1) $x^2-\frac{\sqrt{3}}{\sqrt{2}}x+\frac{1}{4}=0$

$x=\dfrac{\sqrt{\frac{3}{2}}\pm\sqrt{\frac{3}{2}-1}}{2}=\dfrac{\sqrt{3}\pm 1}{2\sqrt{2}}$ (1)

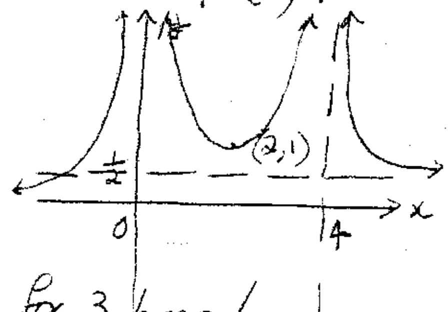
$\cos\frac{\pi}{12}>\cos\frac{5\pi}{12}\Rightarrow\cos\frac{\pi}{12}=\frac{\sqrt{3}+1}{2\sqrt{2}}$

OR $\cdot 9659$

OR $\frac{\sqrt{6}+\sqrt{2}}{4}$

QUESTION THREE:

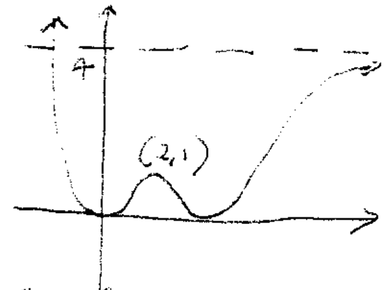
(a)(i) $y=\left|\frac{1}{f(x)}\right|$



(1) for 3 branches

(1) for asymptotes & turning pt.

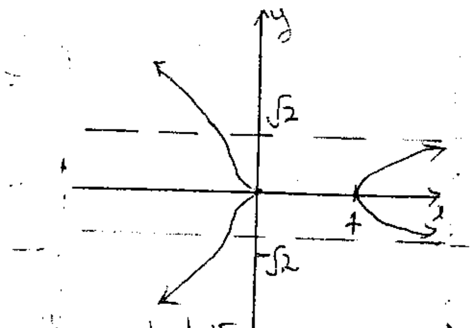
(ii) $y=[f(x)]^2$



(1) for shape

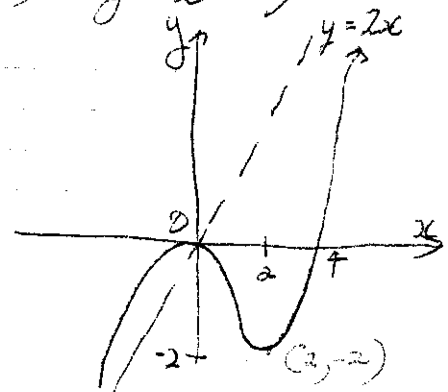
(1) for T.P & $y=4$ asympt.

(iii) $y^2=f(x)$



∴ needed $\pm\sqrt{2}$,

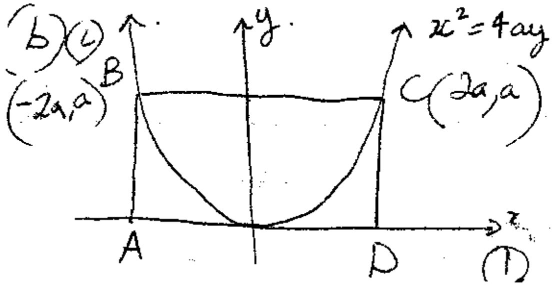
(iv) $y=x\,f(x)$



(1) for shape

(1) for asympt $y=2x$.

(b)(i)



(1)

Length of LR $=4a$

Area is area of rectangle ABCD - area under the curve between $x=-2a$ & $x=2a$

- 7 -

$$\text{Area} = 4a \cdot a - \int_{-2a}^{2a} \frac{x^2}{4a}\,dx$$ (1)

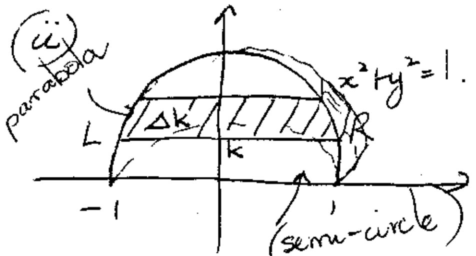
$$= 4a^2 - 2\left[\frac{x^3}{12a}\right]_0^{2a}$$ (1)

$$= 4a^2 - 2\left[\frac{8x^3}{12a} - 0\right]_0^{2a} = 4a^2 - \frac{4a^2}{3}$$

$$\therefore \boxed{A = \frac{8a^2}{3}u^2}$$

* Could do this question next to the y axis eg $2\int_0^a \sqrt{4ay}$.

(ii)



Let slice be Δk thick

When $y = k$

$$x = \pm\sqrt{1-k^2}$$

$\therefore$ length of $L\text{-}R = 2\sqrt{1-k^2}$

So $4a = 2\sqrt{1-k^2}$

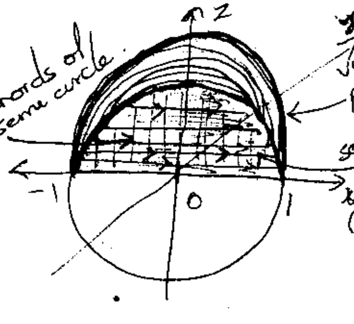
$$a = \frac{\sqrt{1-k^2}}{2}$$ (1)

Using (i) Area of Section $= \frac{8}{3}\left(\frac{\sqrt{1-k^2}}{2}\right)^2$

$$= \frac{8(1-k^2)}{12}$$

$$= \frac{2(1-k^2)}{3}$$ (1)

Volume of the slice $= \frac{2(1-k^2)}{3}\Delta k$



Vol. of Solid $= \frac{2}{3}\int_0^1 (1-k^2)\,dk$

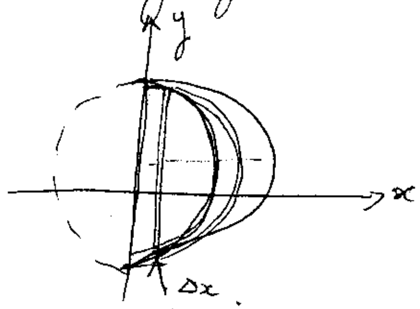
$$= \frac{2}{3}\left[k - \frac{k^3}{3}\right]_0^1$$ (1)

$$= \frac{2}{3}\left[1 - \frac{1}{3}\right] - 0$$

$$= \frac{4}{9}m^3$$ (1)

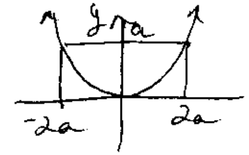
Alternate solution for (Q3b ii) (7a)

At height y, then thickness is Δx



Area $= \frac{8}{3}a^2$

"a" is half the y-distance



$$= \left(\frac{y}{2}\right)^2 \times \frac{8}{3}$$

$$= \frac{y^2}{4} \times \frac{8}{3}$$

$$= \frac{2}{3}y^2$$ (1)

(1) So $dV = \frac{2}{3}y^2\Delta x$ (Area of parabola $\times$ thickness)

$$V = \int_0^1 \frac{2}{3}\cdot(1-x^2)\,dx$$

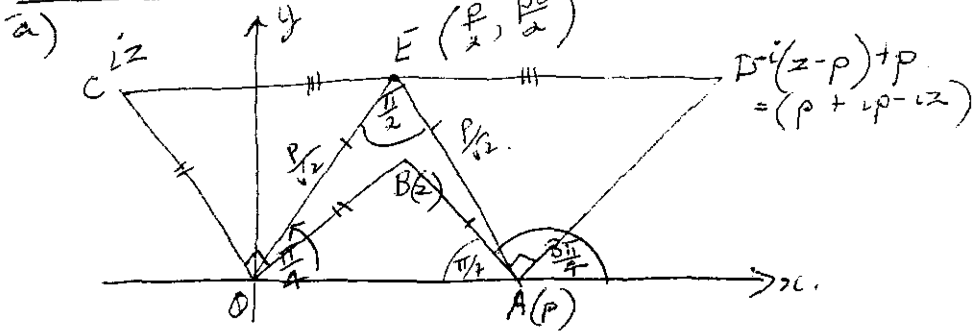
$$= \frac{2}{3}\left[x - \frac{x^3}{3}\right]_0^1$$ (1)

$$= \frac{2}{3}\times\left[1 - \frac{1}{3}\right]$$

$$= \frac{4}{9}u^3.$$ (1)

Question 4 Alternate Solution (8b)

a)



$\vec{AB} = z - p$.
$\vec{AD}$ is the pt $-i(z-p)$ So $\vec{OD} = \vec{OA} + \vec{AD}$
$= p + [-i(z-p)]$
$= -iz + ip + p$. (1)

Since E is the midpt of CD it has co ords. $\dfrac{iz - i(z-p) + p}{2} = \dfrac{pi + p}{2} = \dfrac{p}{2} + \dfrac{p}{2}i$

Distance of OE: $\sqrt{\left(\frac{p}{2}\right)^2 + \left(\frac{p}{2}\right)^2}$
$= \sqrt{\frac{p^2}{4} + \frac{p^2}{4}}$
$= \sqrt{\frac{p^2}{2}} = \frac{p}{\sqrt{2}}$

Distance of EA $= \sqrt{\left(p - \frac{p}{2}\right)^2 + \left(\frac{p}{2}\right)^2}$
$= \sqrt{\frac{p^2}{4} + \frac{p^2}{4}}$
$= \sqrt{\frac{p^2}{2}} = p/\sqrt{2}$ (1)

∴ OE = EA
∴ △OEA is isos △.

The arg(E) $= \tan^{-1}\left(\frac{p}{2} / \frac{p}{2}\right)$
$= \tan^{-1}(1) = \frac{\pi}{4}$ ∴ $\angle EAO = \frac{\pi}{4}$ (△OEA is isos)

∴ $\angle OEA = 90° = \frac{\pi}{2}$ (∠ sum of a triangle) (1)

QUESTION FOUR (8)

(a) $\vec{AB}$ represents $z - p$
$\vec{AD}$ is the clockwise rotation of $\vec{AB}$ by $90°$
& so represents $-i(z-p)$
$\vec{OD} = \vec{OA} + \vec{AD} = p + [-i(z-p)]$
$= -iz + ip + p$
$= p + y + i(p - x)$
Let $z = x + iy$ ∴ D is the pt $(p+y, p-x)$
$\vec{OC}$ is the anticlockwise rotation of $\vec{OB}$ by $90°$
and so represents $iz = i(x+iy) = -y + ix$
∴ C is the pt $(-y, x)$
Since E is the midpt of CD it has coord $\left(\frac{p}{2}, \frac{p}{2}\right)$
Let R be the midpt of OA, directly below E. As E is the ⊥ bisector of OA the △OAE is isos. and as a circle centred at R & radius $p/2$ can pass through the pts O, E & A then ∠OAE is rt ∠ at E.

b)(i) $\frac{PS}{PM} = e$ ∴ $PS = e\,PM = e\left(\frac{a}{e} - a\cos\theta\right)$
$= a(1 - e\cos\theta)$ (1)

(ii) Similarly $\frac{PS'}{PM'} = e$ ∴ $PS' = e\,PM' = e\left(\frac{a}{e} + a\cos\theta\right)$
(1) $= a(1 + e\cos\theta)$

(iii) Differentiating $\frac{2x}{a^2} + \frac{2y}{b^2} \times y' = 0$
$y' = \frac{-xb^2}{ya^2} = -\frac{b\cos\theta}{a\sin\theta}$ (1)
Equation of tangent P
$y - b\sin\theta = -\frac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$
(1) $ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$
$bx\cos\theta + ay\sin\theta - ab(\sin^2\theta + \cos^2\theta) = 0$
$bx\cos\theta + ay\sin\theta - ab = 0$

(iv) Distance from S $(ae, 0)$ to tangent is

$$SL = \frac{|abe\cos\theta - ab|}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} = \frac{ab(e\cos\theta - 1)}{\sqrt{b^2\cos^2\theta + a^2\sin^2}}$$

$$\text{or } \frac{ab\,|1 - e\cos\theta|}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} \leftarrow (1)$$

Similarly $S'L' = \dfrac{|-abe\cos\theta - ab|}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$ (1)

$$= \frac{ab\,(1 + e\cos\theta)}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

(v) In $\Delta PS'L'$ & ΔPSL

$$\frac{PS'}{PS} = \frac{a(1 + e\cos\theta)}{a(1 - e\cos\theta)} = \frac{1 + e\cos\theta}{1 - e\cos\theta}$$

$$\frac{SL'}{SL} = \frac{1 + e\cos\theta}{1 - e\cos\theta}$$

(1) $\rightarrow$ $\therefore \dfrac{PS'}{PS} = \dfrac{SL'}{SL}$ —— $\boxed{\text{eqn } 1}$

$\sin \angle L'PS' = \dfrac{S'L'}{PS'} = \dfrac{SL}{PS}$ from $\boxed{\text{eqn } 1}$

(1) $\rightarrow$ $= \sin \angle LPS$

$\therefore \angle L'PS' = \angle LPS$.

$\therefore \angle S'PN = 90° - \angle L'PS'$

$= 90° - \angle LPS$

(1) $\rightarrow$ $= \angle SPN$

$\therefore$ PN bisects $\angle SPS'$

(vi) PN has gradient $= \dfrac{a\sin\theta}{b\cos\theta}$

Eqn of PN is $y - b\sin\theta = \dfrac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$

x-intercept when $y = 0$

(1) $x = \dfrac{(a^2 - b^2)\sin\theta\cos\theta}{a\sin\theta} = \left[\dfrac{a^2 - a^2(1 - e^2)}{a}\right]\cos\theta$

$\rightarrow = ae^2\cos\theta$

$NS = ae - ON = ae - ae^2\cos\theta$

$= ae(1 - \cos\theta)$

$NS' = ae + ae^2\cos\theta = ae\,(1 + \cos\theta)$

$\therefore \dfrac{NS}{NS'} = \dfrac{1 - e\cos\theta}{1 + e\cos\theta} = \dfrac{PS}{PS'}$

(1) $\rightarrow$ (by (i) of (ii) parts)

$\therefore \dfrac{PS}{NS} = \dfrac{PS'}{NS'}$

QUESTION FIVE

(a) $100a + 10b + c = 99a + 9b + a + b + c$ —— (1)

$= 9(11a + b) + 3d$

$= 3\left(3[11a + b] + d\right)$ —— (1)

Hence divisible by 3.

(b)(i) $\alpha + \beta + \gamma = 0$, $\alpha\beta + \alpha\gamma + \beta\gamma = 3p$, $\alpha\beta\gamma = -q$.

$$\therefore \frac{\beta\gamma}{\alpha} + \frac{\alpha\gamma}{\beta} + \frac{\alpha\beta}{\gamma} = \frac{(\beta\gamma)^2 + (\alpha\gamma)^2 + (\alpha\beta)^2}{\alpha\beta\gamma}$$

$$= \frac{(\beta\gamma + \alpha\gamma + \alpha\beta)^2 - 2(\alpha\beta\gamma^2 + \alpha\beta^2\gamma + \alpha^2\beta\gamma)}{\alpha\beta\gamma}$$

(1)

$$= \frac{(\beta\gamma + \alpha\gamma + \alpha\beta)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)}{\alpha\beta\gamma}$$

$$= \frac{(3p)^2 + 2q(0)}{-q} = \frac{9p^2}{-q}$$

$$\frac{\beta\gamma}{\alpha}\cdot\frac{\alpha\gamma}{\beta} + \frac{\alpha\gamma}{\beta}\cdot\frac{\alpha\beta}{\gamma} + \frac{\beta\gamma}{\alpha}\cdot\frac{\alpha\beta}{\gamma} = \gamma^2 + \alpha^2 + \beta^2 \quad (1)$$

$$= (\gamma + \alpha + \beta)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$$

(11)

$$= (\gamma+\alpha+\beta)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$$
$$= 0 - 2\cdot 3p$$
$$= -6p$$

(1)

$$\frac{\beta\gamma}{\alpha}\cdot\frac{\alpha\gamma}{\beta}\cdot\frac{\alpha\beta}{\gamma} = \alpha\beta\gamma = -q \quad \text{— (1)}$$

$\therefore$ Required eqn is $\boxed{x^3 + \frac{9p^2}{q}x^2 - 6px + q = 0}$

(ii) $\frac{\alpha\beta}{\gamma} = 1$ is a root $\leftarrow$ (1)

$$\therefore 1 + \frac{9p^2}{q} - 6p + q = 0$$
$$q + 9p^2 - 6pq + q^2 = 0$$
$$\therefore (3p-q)^2 + q = 0 \quad \text{— (1)}$$

(c) Let $P(x) = x^5 + 2x^2 + ax + b$

If $(x+1)^2$ is a factor $P(-1) = P'(-1) = 0$

$\therefore P(-1) = -1 + 2 - a + b = 0 \leftarrow$ (1)

$\therefore -a + b = -1$

$P'(x) = 5x^4 + 5x + a \leftarrow$ (1)

$P'(-1) = 5 - 4 + a = 0$

$a = -1$

$\therefore b = -2 \leftarrow$ (1)

(d) (1)

$\sin(2k+1)\theta = \sin(2k\theta + \theta)$

$= \sin 2k\theta\cos\theta + \cos 2k\theta\sin\theta$

& $\sin(2k-1)\theta = \sin(2k\theta - \theta)$

$= \sin 2k\theta\cos\theta - \cos 2k\theta\sin\theta$

$(\sin 2k\theta\cos\theta + \cos 2k\theta\sin\theta - \sin 2k\theta\cos\theta + \cos 2k\theta\sin\theta)$

$$\therefore \frac{\sin(2k+1)\theta}{\sin\theta} - \frac{\sin(2k-1)\theta}{\sin\theta}$$

OR

$$\frac{\sin(2k\alpha+\alpha) - \sin(2k\alpha-\alpha)}{\sin\alpha} \qquad \frac{2\cos 2k\theta\sin\theta}{\sin\theta}$$

$*[\sin(A+B) - \sin(A-B) = 2\cos A\sin B]$ (1)

$$= \frac{2\sin\alpha\cos(2k\alpha)}{\sin\alpha} = 2\cos(2k\theta)$$

Alternate Solution for Q5 b ii) (10b)

$$\frac{\beta\gamma}{\alpha} = \frac{\alpha\beta\gamma}{\alpha^2}$$
$$= \frac{q}{\alpha^2}$$

Let $y = \frac{q}{\alpha^2}$ So $\alpha^2 = -\frac{q}{y}$, $\alpha = \sqrt{\frac{-q}{y}}$ — (1)

Sub (1) $\to$ eqn of polynomial: $x^3 + 3xp + q = 0$

i.e. $\left(\sqrt{\frac{-q}{y}}\right)^3 + 3\left(\sqrt{\frac{-q}{y}}\right)p + q = 0$

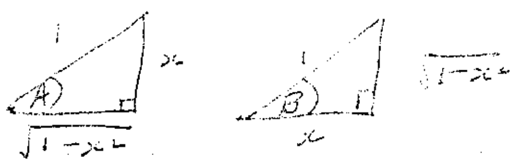
$$\sqrt{\frac{-q}{y}}\left[\left(\sqrt{\frac{-q}{y}}\right)^2 + 3p\right] + q = 0$$
$$\sqrt{\frac{-q}{y}}\left[\left(\sqrt{\frac{-q}{y}}\right)^2 + 3p\right] = -q$$
$$-\frac{q}{y}\left[\frac{q}{y} + 3p\right]^2 = q^2$$
$$-\frac{q}{y}\left[\frac{q^2}{y^2} - \frac{6pq}{y} + 9p^2\right] = q^2$$
$$-\frac{1}{y}\left[\frac{q^2}{y^2} - \frac{6pq}{y} + 9p^2\right] = q$$
$$\frac{-q^2}{y^3} + \frac{6pq}{y^2} - \frac{9p^2}{y} = q$$
$$-q^2 + 6pqy - 9p^2y^2 = y^3q$$

So $qy^3 + 9p^2y^2 - 6pqy + q^2 = 0$ Let $y = x$

$$\therefore x^3 + \frac{9py^2}{q} - 6px + q = 0$$

(e) let $A = \sin^{-1}x$ and $B = \cos^{-1}x$

$\therefore x = \sin A$ and $x = \cos B$



So $\sqrt{1-x^2} = \cos A$ $\sqrt{1-x^2} = \sin B$ (1)

$$\begin{aligned}\therefore LHS &= \sin(\sin^{-1}x - \cos^{-1}x)\\ &= \sin(A-B)\\ &= \sin A\cos B - \cos A\sin B\\ &= x.x - \sqrt{1-x^2}\,\sqrt{1-x^2} \quad (1)\\ &= x^2-(1-x^2)\\ &= 2x^2-1.\end{aligned}$$

QUESTION SIX:

(a)(i) $F = ma = 50 - kv^2$ (1) limiting vel = 10 $\therefore \ddot{x} = 0$ when $v = 0$, $k = \frac{1}{2}$

(1) so $2\ddot{x} = 100 - v^2$

ie $2\dfrac{d^2x}{dt^2} = 100 - v^2$

(ii) $\ddot{x} = v\dfrac{dv}{dx}$

$\therefore 2v\dfrac{dv}{dx} = 100 - v^2$ from part (i)

$$\int \frac{2v}{100-v^2}\,dv = \int dx$$

$\ln(100-v^2) = -x + c$

When $x=0$, $v=0$ — (1)

$\therefore c = \ln 100$

$\therefore -x = \ln(100-v^2) - \ln 100$

$-x = \ln\left(\dfrac{100-v^2}{100}\right)$

$e^{-x} = \dfrac{100-v^2}{100}$ (1)

$v^2 = 100(1-e^{-x})$

(iii) $\ddot{x} = \dfrac{dv}{dt}$

$\therefore 2\dfrac{dv}{dt} = 100 - v^2$

$$\int \frac{dv}{100-v^2} = \frac{1}{2}\int dt \quad (1)$$

$$\frac{1}{20}\int\left[\frac{1}{10-v} + \frac{1}{10+v}\right]dv = \frac{1}{2}\int dt$$

$\therefore -\ln(10-v) + \ln(10+v) = 10t + C$

When $t=0$, $v=0$, $\therefore c=0$

$t = \dfrac{1}{10}\ln\left(\dfrac{10+v}{10-v}\right)$ (1)

When $V=5$ $t = \dfrac{1}{10}\ln\left(\dfrac{15}{5}\right) = \dfrac{1}{10}\ln 3$ seconds. (1)

(iv) $\ddot{x} = -F - \dfrac{v^2}{2}$

$v\dfrac{dv}{dx} = -\left(F + \dfrac{v^2}{2}\right)$

$$\int \frac{v}{\left(F+\frac{v^2}{2}\right)}\,dv = -\int dx \quad \text{OR} \quad \int \frac{v}{\frac{2F+v^2}{2}}\,dv = \int \frac{2v}{2F+v^2}\,dv$$

(1) $\ln\left(F + \dfrac{v^2}{2}\right) = -x + c \Rightarrow = \ln\left(\dfrac{F+v^2/2}{F+25/2}\right)$

When $v=5$, $x=0$ $\therefore c = \ln\left(F+\dfrac{25}{2}\right)$

from part (ii) we obtain distance travelled when car reaches 5m/s.

$$25 = 100(1 - e^{-x})$$

$$\therefore e^{-x} = \frac{3}{4} \quad (1)$$

$$x = \ln\left(\frac{4}{3}\right)$$

Substitute $x = \ln\left(\frac{4}{3}\right)$ and $v = 0$ into eqn (1)

$$\ln\left(\frac{4}{3}\right) = \ln\left(\frac{F + \frac{25}{2}}{F}\right)$$

$$\therefore \frac{4}{3} = \frac{F + \frac{25}{2}}{F} \quad (1)$$

$$F = \frac{75}{2} = 37.5\,N$$

(b) Period $T = 2 \times [7{:}20 - 1{:}05] = 2 \times 375 = 750$ min ($6\frac{1}{4} = \frac{25}{4}$ hr)

$a = 0.5(14.6 - 8.2) = 3.2$ m

since motion is SHM

$$\ddot{x} = -n^2 x$$

$$\left\{ T = \frac{2\pi}{n} \text{ or } n = \frac{2\pi}{750} = \frac{\pi}{375} \right\} \quad \boxed{x = 13.3,\ t = ?}$$

Sol^n of this eqn is $x = a\cos(nt + \alpha)$ where $0 \le \alpha \le 2\pi$

High 14.6m 7:20pm

(1) → $x = 3.2\cos(nt + \alpha)$

Considering the initial conditions

Centre 11.4m

When $t = 0$ $x = -3.2$

(When $x = -3.2$, $t = 0$) → $-3.2 = 3.2\cos(0 + \alpha)$

$\therefore \cos\alpha = -1$ or $\alpha = \pi$

$x = 3.2\cos(nt + \pi)$

$= -3.2\cos nt$

Low 8.2m 1:05pm.

since $\cos(\theta + \pi) = -\cos\theta$

(1)

Min depth required = 13.3 (1)

i.e. $x = 13.3 - 11.4 = 1.9$

$\therefore 1.9 = -3.2\cos nt$

$$\therefore nt = \cos^{-1}\left(\frac{-19}{32}\right) = \pi - \cos^{-1}\left(\frac{19}{32}\right)$$

$$\text{or } t = \frac{1}{n}\left[\pi - \cos^{-1}\frac{19}{32}\right] = \frac{375}{\pi}\left[\pi - \cos^{-1}\left(\frac{19}{32}\right)\right]$$

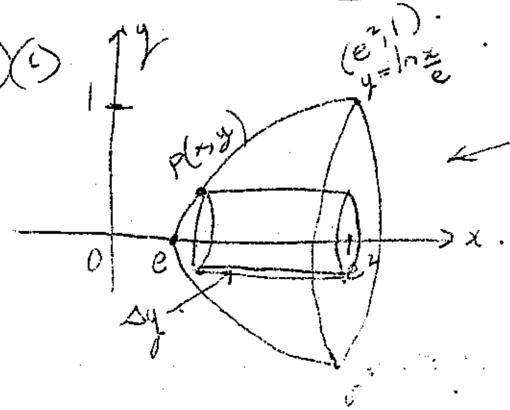
$\doteqdot 263$ min or 4h 23 min. (1)

$T - t = 750 - 263 = 487$ min $= 8$h 7min

Hence the ships can leave the harbour between $1{:}05 + 4{:}23 = 5{:}28$pm &
$1{:}05 + 8{:}07 = 9{:}12$pm. (1)

QUESTION SEVEN:

(a)(i)



(1) Given without the cylinder.

(ii) Considering hollow cylindrical shell: radius y, height $e^2 - x$ & thickness Δy.

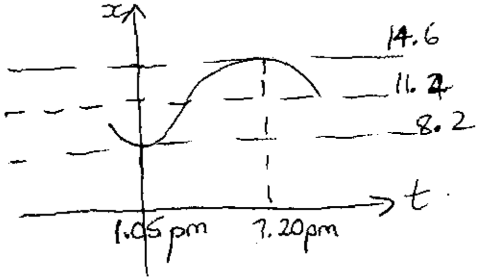
$$\therefore \Delta V = 2\pi y(e^2 - x)\Delta y \quad (1)$$

$$\text{or } \Delta V = 2\pi y(e^2 - e^{y+1})$$

Since $\ln\frac{x}{e} = y$ $\therefore \frac{x}{e} = e^y$ $\therefore x = e^{y+1}$

Also for $x = e$: $y = 0$ &

Alternate Solution for Q6(b) (14b)



Let x be the height of the tide at a particular time

Centre: $\frac{14.6+8.2}{2} = 11.4$

Amplitude $= 11.4 - 8.2 = 3.2$ — 1

Period: 375 mins × 2 $= 750$ mins or $12\frac{1}{2}$ hrs or $\frac{25}{4}$ hrs

1 → then $\frac{2\pi}{n} = T$

ie $\frac{2\pi}{n} = 750$

$\frac{2\pi}{750} = n = \frac{\pi}{375}$

or $n = \frac{4\pi}{25}$

- the height $x = 11.4 - 3.2\cos\frac{4\pi}{25}t$.
- the boat can leave when water height is greater than or equal to 13.3 ← 1

ie $13.3 \leq 11.4 - 3.2\cos\frac{4\pi}{25}t$ (2nd/3rd Quad)

$\cos\frac{4\pi}{25}t = -\frac{19}{32}$

then $\frac{4\pi}{25}t = 2.2065^c, 4.076$ ← 1

$\therefore t = 4.38, 8.11$

this converts to 4hrs 22.8mins, 8hrs 11mins

Hence times the boat can leave the harbour: 5:28pm and 9:12pm. ← 1

(16)

$\therefore V = 2\pi\int_0^1 y(e^2 - e^{y})\,dy$

$= 2\pi e^2\int_0^1 y\,dy - 2\pi\int_0^1 ye^y\,dy$.

Let $K = \int_0^1 ye^y\,dy$ use integration of parts

$u = y \quad dv = e^y\,dy$

$du = dy \quad v = e^y$

$K = \left[ye^y\right]_0^1 - \int_0^1 e^y\,dy$

$= e - \left[e^y\right]_0^1 = 1$ — 1

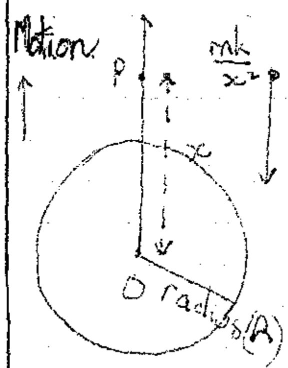
So $J = \int_0^1 y\,dy = \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2}$.

$\therefore V = 2\pi e^2\left(\frac{1}{2}\right) - 2\pi e$

$= \pi e^2 - 2\pi e$

$= \pi e(e-2)\ u^3$ — 1

(b)(i)



Upward is pos. motion

$m\ddot{x} = -\frac{mk}{x^2}$ or $\ddot{x} = -\frac{k}{x^2}$

On the surface of the earth

$x = R$

$-g = -\frac{k}{R^2}$, ie $k = R^2g$

Now $v\frac{dv}{dx} = -\frac{k}{x^2}$

ie $\frac{d}{dx}\left[\frac{1}{2}v^2\right] = -\frac{R^2g}{x^2}$

1 → So $\frac{1}{2}v^2 = \frac{R^2g}{x} + c$

When $v = u$

$x = R$

then $\frac{1}{2}v^2 = \frac{R^2g}{x} + \frac{1}{2}u^2 - gR$ ← (1)

$$v^2 = \frac{2R^2g}{x} + u^2 - 2gR$$

$$v^2 = u^2 + 2R^2g\left[\frac{1}{x} - \frac{1}{R}\right] \quad (1)$$

(vel of a body at distance x from centre)

(ii) When the body is at the highest point $v=0$ and hence $0 = \frac{2R^2g}{x} + u^2 - 2gR$

$$\frac{2R^2g}{x} = 2gR - u^2$$

$$x = \frac{2R^2g}{2gR-u^2} \quad (1)$$

∴ Greatest height H above the earth's surface is $H = \frac{2gR^2}{2gR-u^2} - R$ ← (1)

$$= \frac{2gR^2 - 2gR^2 + Ru^2}{2gR - u^2}$$

$$= \frac{Ru^2}{2gR-u^2}$$

(iii) Greatest height reached by the body will be infinite if $2gR - u^2 = 0$ or $u^2 = 2gR$

ie $u = \sqrt{2gR}$ (1)

Body will escape from the earth if $u \geq \sqrt{2gR}$

(iv) Min. escape velocity is given by $u = \sqrt{2gR}$

$$u = \sqrt{2 \times 0.01 \times 6400}$$

$= 11.3$ km/s correct to 1 dec place.

(v) $u = \sqrt{2gR}$

$v^2 = \frac{2R^2g}{x}$ ie $v = \frac{\sqrt{2gR^2}}{\sqrt{x}}$

$$\text{ie } \frac{dx}{dt} = \frac{\sqrt{2gR^2}}{\sqrt{x}}.$$

$$dt = \frac{\sqrt{x}}{\sqrt{2gR^2}}\,dx. \leftarrow (1)$$

time for body to reach a height of $3R$ above the Earth (then $x = 4R$) from initial time $t=0$ (when $x=R$) is given by

$$(1) \quad t = \int_R^{4R} \frac{x^{1/2}}{\sqrt{2gR^2}}\,dx$$

$$= \frac{1}{\sqrt{2gR^2}}\left[\frac{2}{3}x^{3/2}\right]_R^{4R} \leftarrow (1)$$

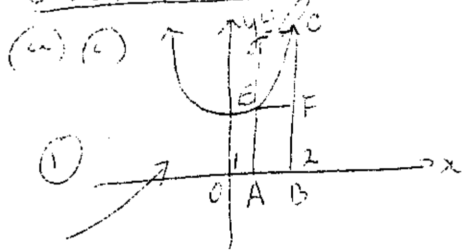
$$= \frac{1}{\sqrt{2gR^2}}\left[\frac{2}{3} \times 4^{3/2} R^{3/2} - \frac{2}{3}R^{3/2}\right]$$

$$= \frac{2}{3} \times \frac{1}{\sqrt{2gR^2}} \times 7R^{3/2}$$

$$= \frac{14}{3}\sqrt{\frac{R}{2g}}.$$

Question Eight

(a) (i)



1

(ii) Area of ABFE $< \int_1^2 (px^2+q) <$ Area of ABCD

1 $AB \times AE < \left[\frac{px^3}{3} + qx\right]_1^2 < AB \times BC$

$1 \times (p+q) < \left[\frac{8p}{3} + 2q\right] - \left(\frac{p}{3}+q\right)\Big] < 1 \times (4p+q)$

$\therefore (p+q) < \frac{7p+3q}{3} < 4p+q$ — 1

(b) (i) In HPRM, $\angle HPR = \angle RMH = 90°$

1 (Angle between radius & tangent at pt. of contact $= 90°$)

1 $\therefore$ HPRM is a cyclic quad, (opp $\angle$s are supple

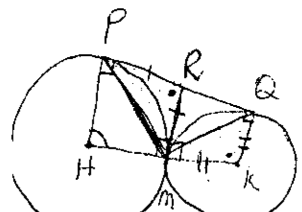
Similarly for MRQK.

(ii) Construction: Join PM and QM in

1 $\triangle PRM$ & $\triangle MQK$.

$\angle PRM = \angle MKQ$

(Ext. $\angle$ of cyclic quad. MRQK)



$PR = RM$ (tangent from ext. pt)

$KM = KQ$ (radii)

$\therefore$ triangle is isosceles.

$\therefore \angle RPM = \angle RMP = \angle KMQ$

$\therefore \triangle PRM \ ||| \ \triangle MQK$

1 (equiangular)

(c) (i) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} \ldots$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \ldots$

1 $e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)$

$= \cos x + i \sin x$

$e^{i\pi} = \cos\pi + i\sin\pi = \cos\pi$

1 $e^{i\pi} = -1$ since $\cos\pi = -1$ & $\sin\pi = 0$

(ii) $e^{\frac{i\pi}{2}} = \left(e^{i\pi}\right)^{\frac{1}{2}}$

$= (-1)^{\frac{1}{2}} = i$ — 1

from (i) $i^i = \left[e^{\frac{i\pi}{2}}\right]^i$

$= e^{-\pi/2}$ which is real. — 1

(iii) $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$\ln z = \ln re^{i\theta}$

$= \ln r + i\theta$ — 1

This will give an infinite number of results unless θ is restricted.

ie $= \ln r + i(\theta + 2\pi k)$ where k is an integer.

The principal value of $\ln z$ occurs when

$-\pi \leq \theta \leq \pi$. — 1

(iv) $z = 1+i$

$\ln z = \ln\sqrt{2} + \frac{i\pi}{4}$ — 1