AT

Name: ____________________

Class: 12MTZ1

Teacher: MR TONG

CHERRYBROOK TECHNOLOGY HIGH SCHOOL

2010 AP4

YEAR 12 TRIAL HSC EXAMINATION

MATHEMATICS EXTENSION 2

Time allowed - 3 HOURS
(Plus 5 minutes' reading time)

DIRECTIONS TO CANDIDATES:

- Attempt all questions.
- All questions are of equal value.
- Each question is to be commenced on a new page clearly marked Question 1, Question 2, etc on the top of the page. **
- All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used. Standard Integral Tables are provided.
- Your solutions will be collected in one bundle stapled in the top left corner.

Please arrange them in order, Q1 to 8.

Each page must show your name and your class. *

Total Marks – 120
Attempt Questions 1–8
All questions are of equal value

Question 1 (15 marks) Start a NEW PAGE. **MARKS**

a) Find $\displaystyle\int \frac{x^2}{\sqrt{x^3 - 1}}\,dx$ **2**

b) Find $\displaystyle\int \frac{dx}{\sqrt{8x^2 + 2}}$ **2**

c) Evaluate $\displaystyle\int_0^{\frac{1}{2}} \sin^{-1} x\,dx$ **3**

d) (i) Find A and B if $\dfrac{x^2 - 4x + 2}{(2x+1)(x^2+4)} = \dfrac{1}{2x+1} + \dfrac{Ax + B}{x^2 + 4}$ **2**

(ii) Hence evaluate $\displaystyle\int_0^2 \frac{x^2 - 4x + 2}{(2x+1)(x^2+4)}\,dx$, leave your answer in simplest exact form. **2**

e) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{1 + \sin\theta + \cos\theta}$ **4**

Question 2 (15 marks) Start a NEW PAGE. **MARKS**

a) Let $z = 5 - 12i$, $w = 3 + 4i$. Find, in the form of $a + bi$

(i) $z + \overline{w}$ 1

(ii) $\dfrac{z}{w}$ 2

b) (i) Show that $\tan\dfrac{\pi}{12} = \dfrac{\sqrt{3} - 1}{\sqrt{3} + 1}$. 1

(ii) Express $z = (\sqrt{3} + 1) + (\sqrt{3} - 1)i$ in modulus argument form. 2

(iii) Find the least positive integer n for which z^n is a real number. 1

c) (i) Shade the region $|z - 3 - 8i| \leq 3$, 2

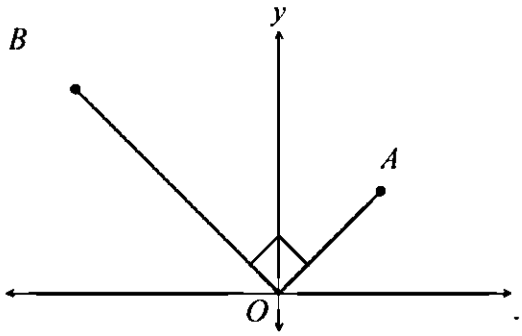
(ii) Find the minimum value of $|z + 3|$ where z satisfies the condition in (i). 2

d) Sketch in the complex plane, the locus of

(i) $\arg z = \dfrac{\pi}{4}$ 1

(ii) $\arg(\overline{z}) = \dfrac{\pi}{4}$. 1

e) On the Argand diagram, OA represents the complex number $z_1 = x + iy$, $\angle AOB = \frac{\pi}{2}$ and the length of OB is twice that of OA.

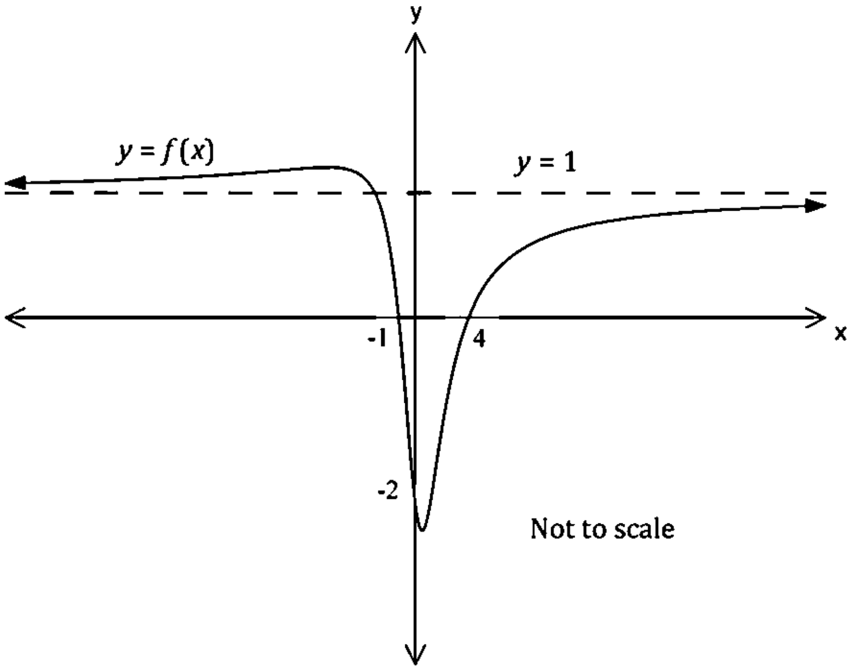


(i) Show that OB represents the complex number $-2y + 2ix$. 1

(ii) Given that AOBC is a rectangle, find the complex number represented by OC. 1

Question 3 (15 marks) Start a NEW PAGE. **MARKS**

a)



The diagram shows the graph of $y = f(x)$. $y = 1$ is the horizontal asymptote. The graph intersects the x-axis at the points $(-1,0)$ and $(4,0)$, and the y-axis at the point $(0,-2)$. Detach the printed graphs from the end of the question booklet and sketch on them the graphs of the following, without using calculus, indicating any asymptotes.

(i) $y = \sqrt{f(x)}$ **2**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = f(|x|)$ **2**

b) $P(x) = x^4 - 2x^3 + 3x^2 - 4x + 1$ and the equation $P(x) = 0$ has roots α, β, γ *and* δ.

(i) Show that the equation $P(x) = 0$ has no integer roots. **1**

(ii) Show that $P(x) = 0$ has a real root between 0 and 1. **1**

(iii) Evaluate $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$. **2**

(iv) Hence deduce that the $P(x) = 0$ has 2 real roots. **2**

(v) Find the equation whose roots are $\dfrac{1}{2\alpha\beta\gamma}, \dfrac{1}{2\alpha\beta\delta}, \dfrac{1}{2\alpha\gamma\delta}$ *and* $\dfrac{1}{2\beta\gamma\delta}$ **2**

(vi) Hence evaluate $\dfrac{1}{2\alpha\beta\gamma} + \dfrac{1}{2\alpha\beta\delta} + \dfrac{1}{2\alpha\gamma\delta} + \dfrac{1}{2\beta\gamma\delta}$. **1**

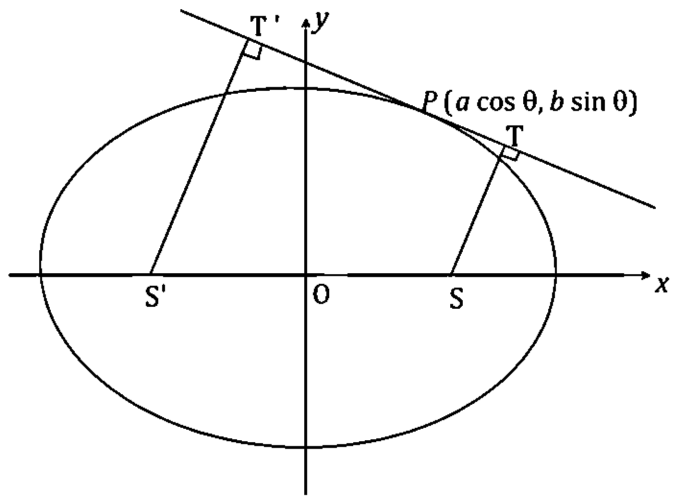
Question 4 (15 marks) Start a NEW PAGE. **MARKS**

a) The equation $|z - 2 + i| + |z - 10 + i| = 12$ corresponds to an ellipse on the Agrand diagram.

(i) Write down the complex number corresponding to the centre of the ellipse. **1**

(ii) Sketch the ellipse and state the lengths of the major and minor axes. **3**

b)



The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The points T and T' are the feet of the perpendiculars from the foci S and S' respectively to this tangent.

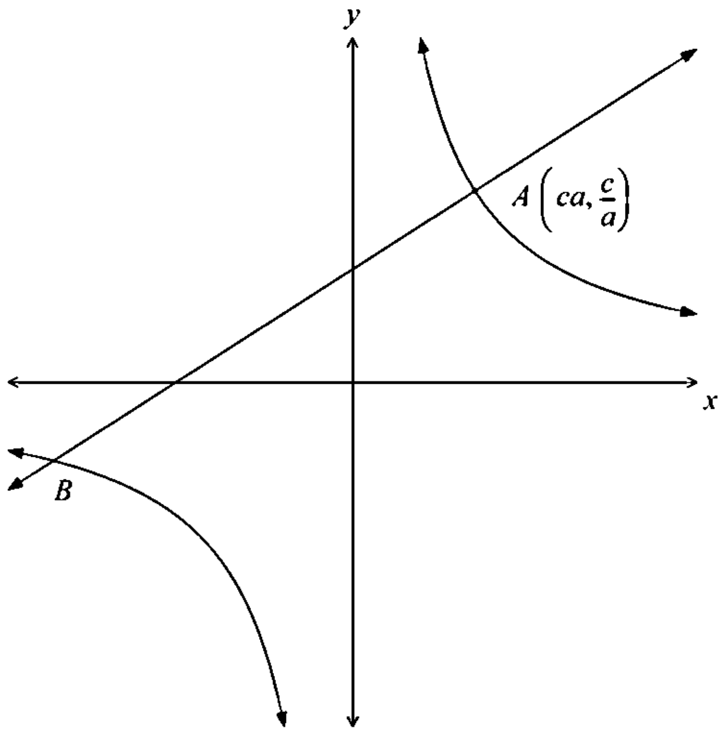
(i) Write down the equation of the tangent at P. **1**

(ii) Show that $ST = \dfrac{ab\,|e\cos\theta - 1|}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$. **2**

(ii) Hence prove that $ST \times S'T' = b^2$. **2**

Question 4 is continued on page 5.

c)



The point $A\left(ca, \frac{c}{a}\right)$, where $a \neq \pm 1$ lies on the hyperbola $xy = c^2$. The normal through A meets the other branch of the curve at B.

(i) Show that the equation of the normal through A is **2**

$$y = a^2 x + \frac{c}{a}\left(1 - a^4\right)$$

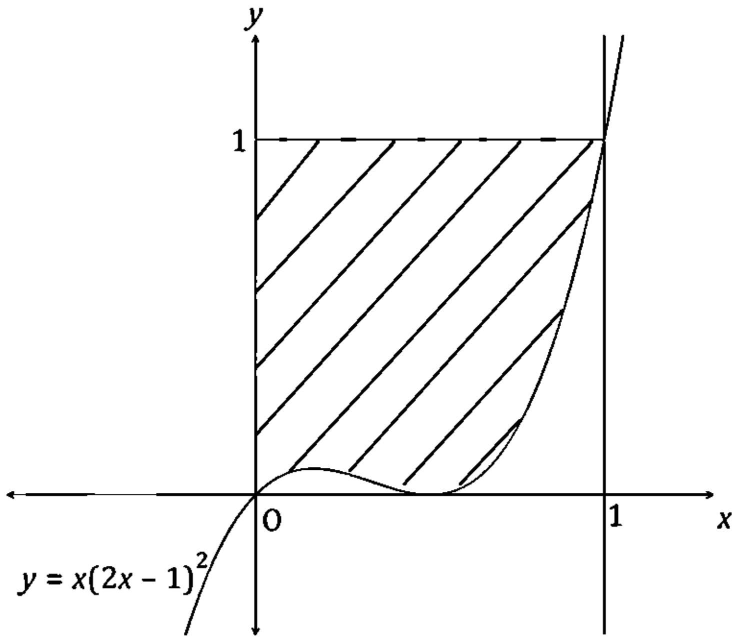
(ii) Hence if B has coordinates $\left(cb, \frac{c}{b}\right)$, show that $b = \frac{-1}{a^3}$. **1**

(iii) M (x, y) is the mid-point of AB. Show that at M, $a^2 = -\frac{y}{x}$. **1**

(iv) Hence show that the equation of the locus of M is $4x^3y^3 + c^2(x^2 - y^2)^2 = 0$. **2**

Question 5 (15 marks) Start a NEW PAGE. **MARKS**

a) 4



A solid is formed by rotating the shaded region bounded by the curve $y = x(2x-1)^2$, the y axis and the line $y = 1$ about the y axis. Use the method of cylindrical shells to find the volume of this solid.

b) If $z = \cos\theta + i\sin\theta$, show that

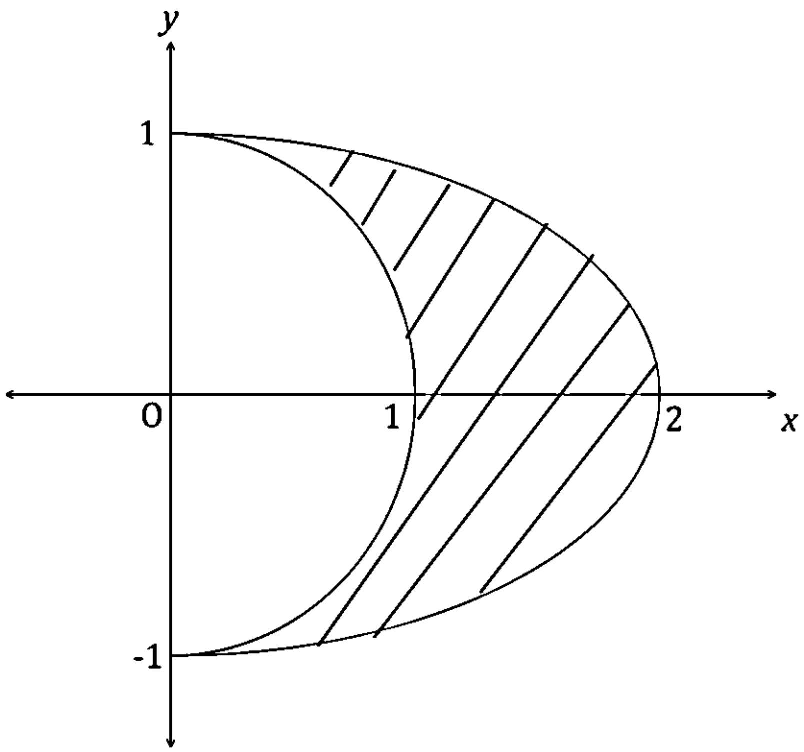
(i) $\cos\theta = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right)$ 1

(ii) $\cos n\theta = \dfrac{1}{2}\left(z^n + \dfrac{1}{z^n}\right)$ 1

(iii) $\cos^4\theta = \dfrac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$ 2

Question 5 continued on page 7

c)



The base of a solid is the shaded region between the semi-circle $x^2 + y^2 = 1$ and the semi-ellipse $\frac{x^2}{4} + y^2 = 1,\ x \geq 0.$ Vertical cross-sections taken parallel to the x-axis are rectangles with heights equal to the squares of their base lengths.

(i) Show that the volume of the solid is given by **2**

$$V = \int_{-1}^{1} (1 - y^2)^{\frac{3}{2}}\, dy$$

(ii) Find the value of V by using the substitution $y = \sin\theta$ and the result of part b). **2**

d) The sequence $\{ u_0, u_1, u_2, \ . \ . \ . \}$ is defined by the recurrence relation **3**

$$u_{n+2} - 4u_{n+1} + 4u_n = 0 \ , \quad n = 0, 1, 2, \ldots ; \text{ and } \ u_0 = 1\ , \ \ u_1 = 2\ .$$

Prove by mathematical induction that $u_n = 2^n$ for all integers n.

Question 6 (15 marks) Start a NEW PAGE. **MARKS**

a) (i) By using De Moivre's theorem or otherwise, show that **2**

$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(ii) Solve the equation $32x^5 - 40x^3 + 10x - 1 = 0$. **3**

(iii) Deduce that $\cos\frac{\pi}{15} + \cos\frac{7\pi}{15} + \cos\frac{13\pi}{15} + \cos\frac{19\pi}{15} = -\frac{1}{2}$. **2**

b) (i) By using the result of a) (i) and the identity **2**

$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$ (DO NOT prove it),

show that $\tan 5\theta = \dfrac{t^5 - 10t^3 + 5t}{5t^4 - 10t^2 + 1}$ where $t = \tan\theta$.

(ii) Deduce that $\tan\frac{\pi}{5}\tan\frac{2\pi}{5}\tan\frac{3\pi}{5}\tan\frac{4\pi}{5} = 5$. **2**

c) α, β are the roots of the equation $x^2 + px + q = 0$, and

$S_n = \alpha^n + \beta^n$ where n is a positive integer, (eg $S_2 = \alpha^2 + \beta^2$).

(i) Show that $S_{n+2} + pS_{n+1} + qS_n = 0$ **2**

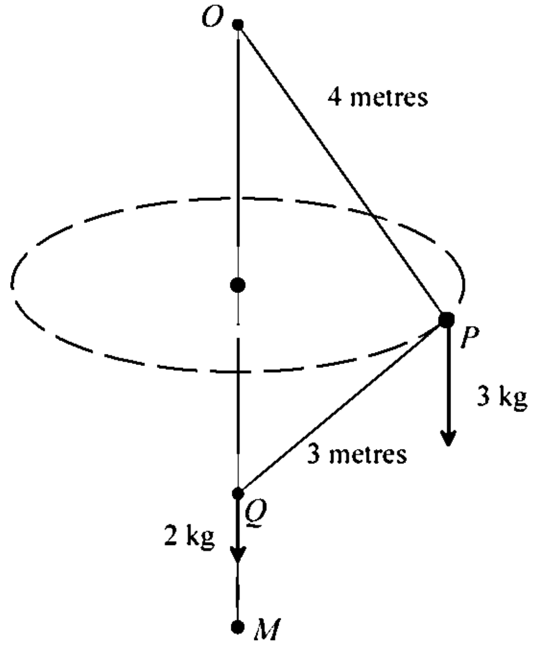
(ii) Hence show that **2**

$(\sqrt{2011} + \sqrt{2010})^4 + (\sqrt{2011} - \sqrt{2010})^4 = 64\,673\,762$

MARKS

Question 7 (15 marks) Start a NEW PAGE.

a)



The above sketch shows a smooth vertical rod OM. Light inextensible strings OP and QP are attached to the rod at O and a mass of 3kg at P. At Q, a 2kg mass is free to slide on the rod. P is rotating in a horizontal circle about the rod. The distance OQ is 5 metres.

(i) Calculate, in terms of g, the tension T_1 in PQ and T_2 in OP. **3**

(ii) Hence calculate the angular velocity of P in order to maintain this system. Give your answer correct to one decimal place. Take g as 10 ms^{-2} **3**

b) A particle of mass m kg is projected vertically upwards with an initial velocity $100\ ms^{-1}$ from O. It experiences air resistance during its motion equal to $0.1\ mv$, where is its speed in metres per second. Let x be the displacement, in metres, at time t seconds, of the particle measured from O. Take g as 10 ms^{-2}.

(i) Using a force diagram, explain why $\ddot{x} = -10 - 0.1v$. **1**

(ii) Show that $x = 1000 - 10v - 1000 \times \ln\left(\dfrac{20}{0.1v + 10}\right)$ **2**

(iii) Find the maximum height reached by the particle. **1**

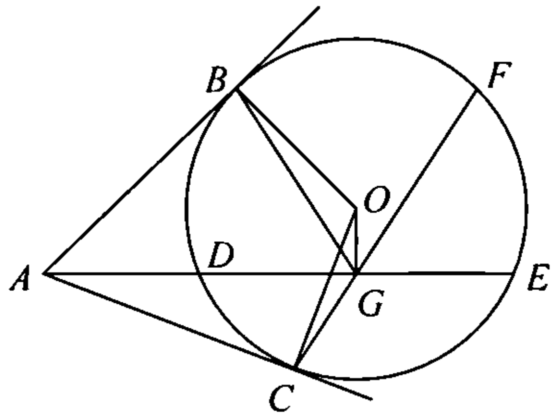
(iv) Show that $t = 10 \times \ln\left(\dfrac{20}{0.1v + 10}\right)$ **2**

(v) Hence find the time taken for the particle to reach the maximum height. **1**

(vi) Find the terminal velocity of the particle during its downward journey. **2**

Question 8 (15 marks) Start a NEW PAGE. **MARKS**

a)



In the diagram, AB and AC are tangents from A to the circle with centre O, meeting the circle at B and C. ADE is a straight line. G is the midpoint of DE. CG produced meets the circle at F.

(i) Detach the last page from the question booklet and attach it to your answer scripts.

(ii) Show that $ABOC$ and $AOGC$ are cyclic quadrilaterals. **3**

(iii) Show that $BF \parallel AE$. **2**

b) (i) Show that $\int \ln(1+x)\,dx = (1+x)\ln(1+x) - x + C$ **1**

(ii) Show that $(n+1)I_n = 2\ln 2 - \dfrac{1}{n+1} - nI_{n-1},\quad n = 1, 2, \ldots$ **2**

Where $I_n = \int_0^1 x^n \ln(1+x)\,dx,\quad n = 0, 1, 2, \ldots$

(iii) Hence, show that when n is odd, **2**

$(n+1)I_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{n+1}$

[You may assume $I_0 = 2\ln 2 - 1$.]

c) Let ω be a complex cube root of unity.

(i) By using the result $1 + \omega + \omega^2 = 0$, simplify $(1+\omega)^2$. **1**

(ii) Show that $(1+\omega)^3 = -1$. **1**

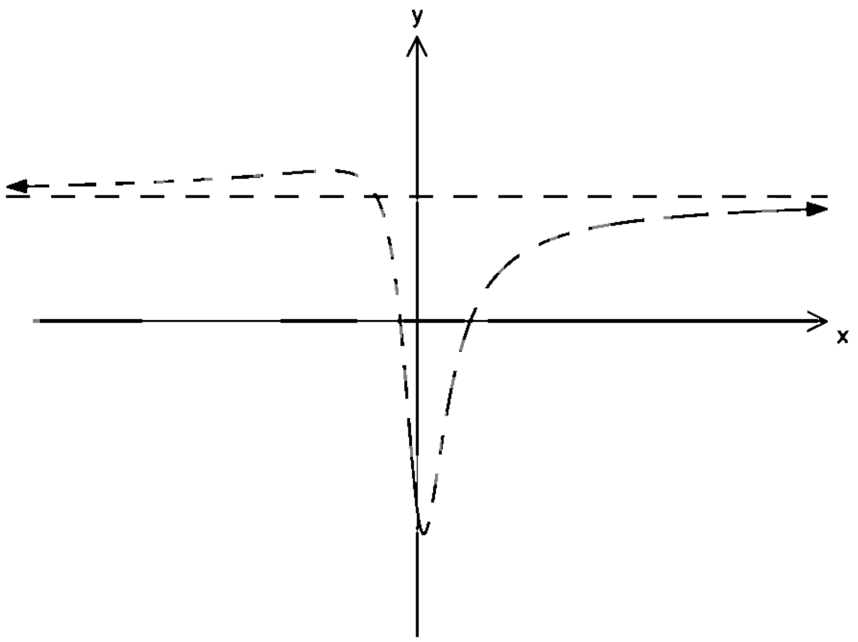
(iii) Use part (ii) to simplify $(1+\omega)^{3n}$ and hence show that **3**

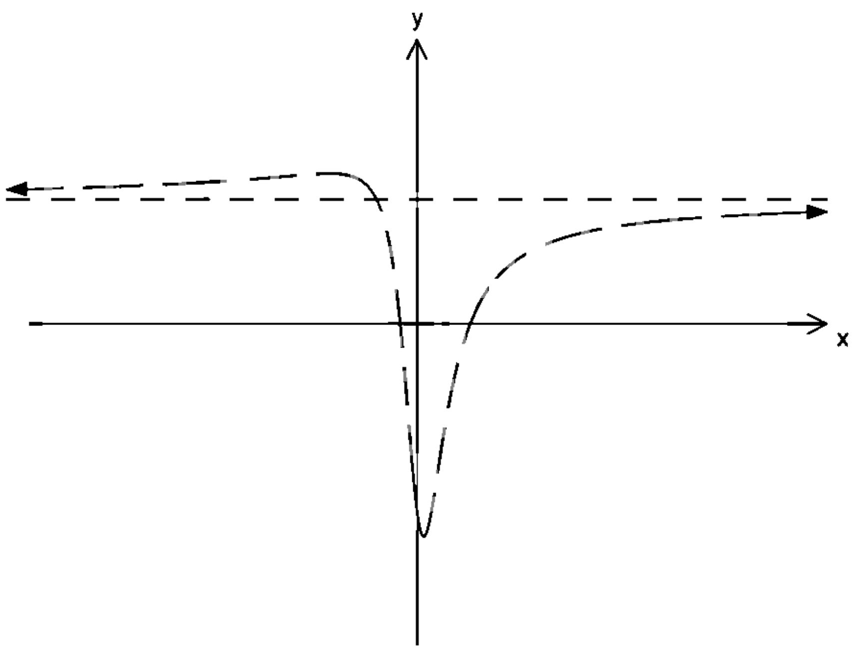
$${}^{3n}C_0 - \frac{1}{2}({}^{3n}C_1 + {}^{3n}C_2) + {}^{3n}C_3 - \frac{1}{2}({}^{3n}C_4 + {}^{3n}C_5) + {}^{3n}C_6 - \ldots + {}^{3n}C_{3n} = (-1)^n.$$

[You may assume $Re(\omega) = Re(\omega^2) = -\frac{1}{2}$]

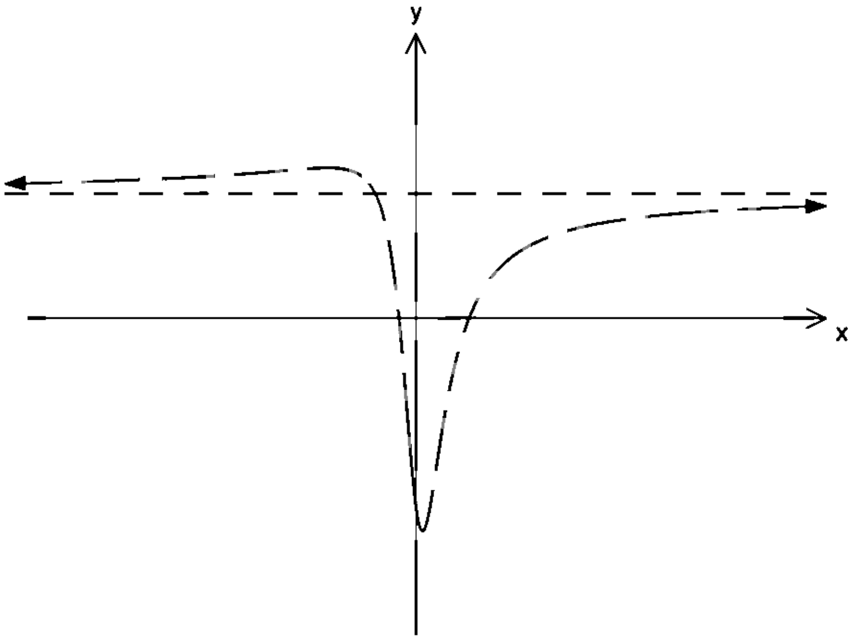
End of Paper

Answer sheet for Question 3 a) Name:___________________

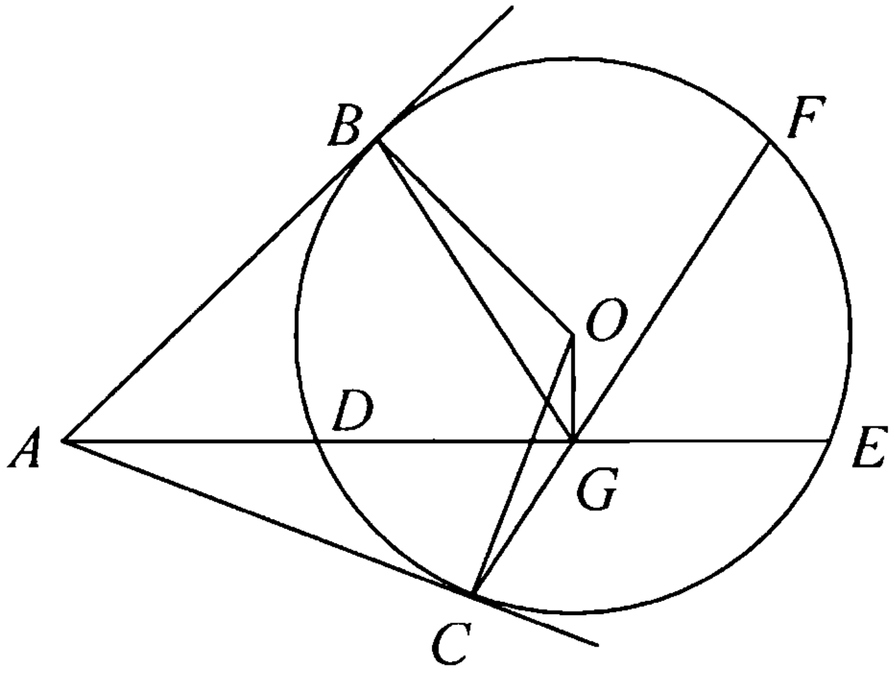




Answer sheet for Question 3 a) Name:____________________



For Question 8.
Detach this page and attach it to your answer script.



Solution to CTHS AP4 Ext 2 2010

Question 1

a) Let $u = x^3 - 1$

$\therefore du = 3x^2 dx$

$$\int \frac{x^2}{\sqrt{x^3-1}}\,dx = \frac{1}{3}\int \frac{du}{\sqrt{u}}$$ $\boxed{1}$

$$= \frac{2}{3}\sqrt{u} + c$$

$$= \frac{2}{3}\sqrt{x^3-1} + C$$ $\boxed{1}$

b) $$\int \frac{dx}{\sqrt{8x^2+2}} = \frac{1}{2\sqrt{2}}\int \frac{dx}{\sqrt{x^2+\frac{1}{4}}}$$ $\boxed{1}$

$$= \frac{1}{2\sqrt{2}} \ln\left[x + \sqrt{x^2+\tfrac{1}{4}}\right] + C$$ $\boxed{1}$

or $$\frac{1}{2\sqrt{2}} \ln\left[\frac{2x+\sqrt{x^2+1}}{2}\right] + c$$

$$= \frac{1}{2\sqrt{2}} \ln\left[2x+\sqrt{x^2+1}\right] - \frac{1}{2\sqrt{2}}\ln 2 + c$$

$$= \frac{1}{2\sqrt{2}} \ln\left[2x+\sqrt{x^2+1}\right] + c'$$

c) $$\int_0^{\frac{1}{2}} \sin^{-1}x\,dx = \left[x\sin^{-1}x\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\,dx$$ $\boxed{1}$

$$= \frac{\pi}{12} + \frac{1}{2}\int_0^{\frac{1}{2}} \frac{d(1-x^2)}{\sqrt{1-x^2}}$$ $\boxed{1}$

$$= \frac{\pi}{12} + \left[\sqrt{1-x^2}\right]_0^{\frac{1}{2}}$$

$$= \frac{\pi}{12} + \left[\sqrt{\tfrac{3}{4}} - 1\right]$$

$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$ $\boxed{1}$

d) (i) $$\frac{x^2-4x+2}{(2x+1)(x^2+4)} = \frac{1}{2x+1} + \frac{Ax+B}{x^2+4}$$

$\therefore (x^2+4) + (Ax+B)(2x+1) \equiv x^2 - 4x + 2$ (1)

Equating coeff. of x^2

$2A + 1 = 1$

$A = 0$ $\boxed{1}$

Put $x = 0$ into (1)

$4 + B = 2$

$B = -2$ $\boxed{1}$

(ii) $$\int_0^2 \frac{x^2-4x+2}{(2x+1)(x^2+4)}\,dx = \int_0^2 \frac{1}{2x+1} - \frac{2}{x^2+4}\,dx$$

$$= \frac{1}{2}\left[\ln(2x+1)\right]_0^2 - \left[\tan^{-1}\frac{x}{2}\right]_0^2$$ $\boxed{1}$

$$= \frac{1}{2}\ln 5 - \frac{\pi}{4}$$ $\boxed{1}$

Q1 e) Put $t = \tan\frac{\theta}{2}$

$\therefore d\theta = \dfrac{2\,dt}{1+t^2}$ $\boxed{1}$

when $\theta = \frac{\pi}{2}$, $t = 1$

$\theta = 0$, $t = 0$ $\}$ $\boxed{1}$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{d\theta}{1+\sin\theta+\cos\theta}$$

$$= \int_0^1 \frac{2\,dt}{(1+t^2)\left(1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}\right)}$$

$$= \int_0^1 \frac{2\,dt}{1+t^2+2t+1-t^2} \quad \boxed{1}$$

$$= \int_0^1 \frac{2\,dt}{2(1+t)}$$

$$= \int_0^1 \frac{dt}{1+t}$$

$$= \left[\ln(1+t)\right]_0^1 \quad \boxed{1}$$

$$= \ln 2$$

Question 2

a) (i) $z - \bar{w} = (5-12i) + \overline{(3+4i)}$

$= 5 - 12i + 3 - 4i$

$= 8 - 16i$ $\boxed{1}$

(ii) $\dfrac{z}{w} = \dfrac{5-12i}{3+4i}$

$= \dfrac{5-12i}{3+4i} \times \dfrac{3-4i}{3-4i}$ $\boxed{1}$

$= \dfrac{15-56i-48}{9+16}$

$= \dfrac{-33-56i}{25}$

$= -\dfrac{33}{25} - \dfrac{56}{25}i$ $\boxed{1}$

b) (i) $\tan\frac{\pi}{12} = \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

$= \dfrac{\tan\frac{\pi}{3} - \tan\frac{\pi}{4}}{1+\tan\frac{\pi}{3}\tan\frac{\pi}{4}}$ $\boxed{1}$

$= \dfrac{\sqrt{3}-1}{1+\sqrt{3}}$

(ii) $z = (\sqrt{3}+1) + (\sqrt{3}-1)i$

$|z| = \sqrt{(\sqrt{3}+1)^2 + (\sqrt{3}-1)^2}$

$= \sqrt{3+2\sqrt{3}+1+3-2\sqrt{3}+1}$

$= \sqrt{8}$ $\boxed{1}$

$$\arg z = \tan^{-1}\frac{\sqrt{3}-1}{\sqrt{3}+1}$$

$$= \frac{\pi}{12} \quad \text{from b) (i)} \qquad \boxed{1}$$

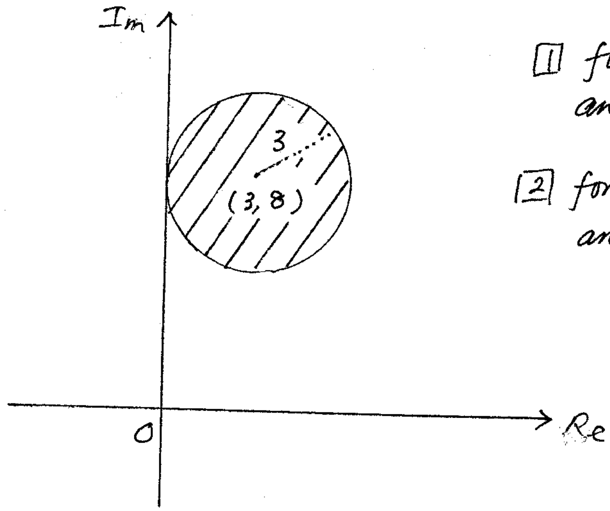
$$\therefore (\sqrt{3}+1)+(\sqrt{3}-1)i = \sqrt{8}\,\text{cis}\,\frac{\pi}{12}$$

(iii) $z^n = \sqrt{8}^{\,n}\,\text{cis}\,\frac{n\pi}{12}$

$Im(z^n) = \sqrt{8}^{\,n}\sin\frac{n\pi}{12}$

The least positive integer for which $\sin\frac{n\pi}{12}=0$ is 12 $\boxed{1}$

c) (i)



$\boxed{1}$ for correct centre and radius

$\boxed{2}$ for correct region and closed boundary.

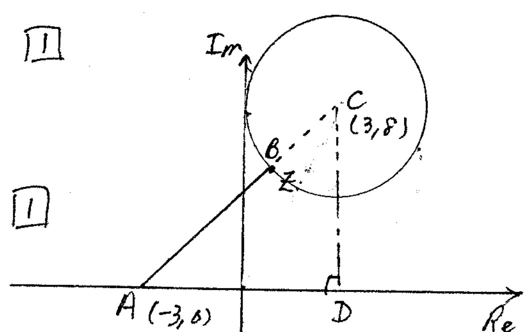
(ii) The min. value of $|z+3|$

$= AB$ $\boxed{1}$

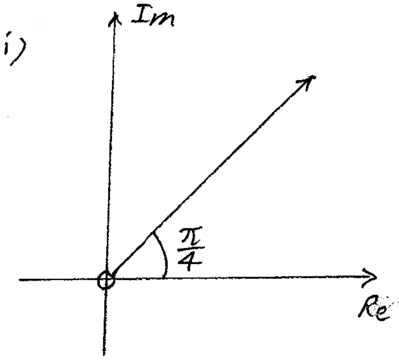
$= AC - BC$

$= \sqrt{6^2+8^2} - 3$

$= 7$ $\boxed{1}$

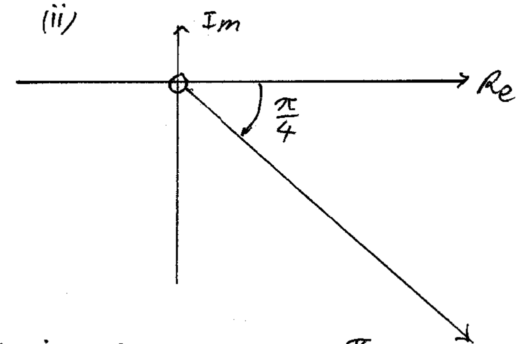


d) (i)



$\boxed{1}$

(ii)



$\boxed{1}$

e) (i) Since $\angle AOB = \frac{\pi}{2}$

$\therefore \vec{OB} = 2\vec{OA}\,\text{cis}\,\frac{\pi}{2}$ $\boxed{1}$

$= 2i\vec{OA}$

$= 2i(x+iy)$

$= 2ix - 2y$

$= -2y + 2ix$

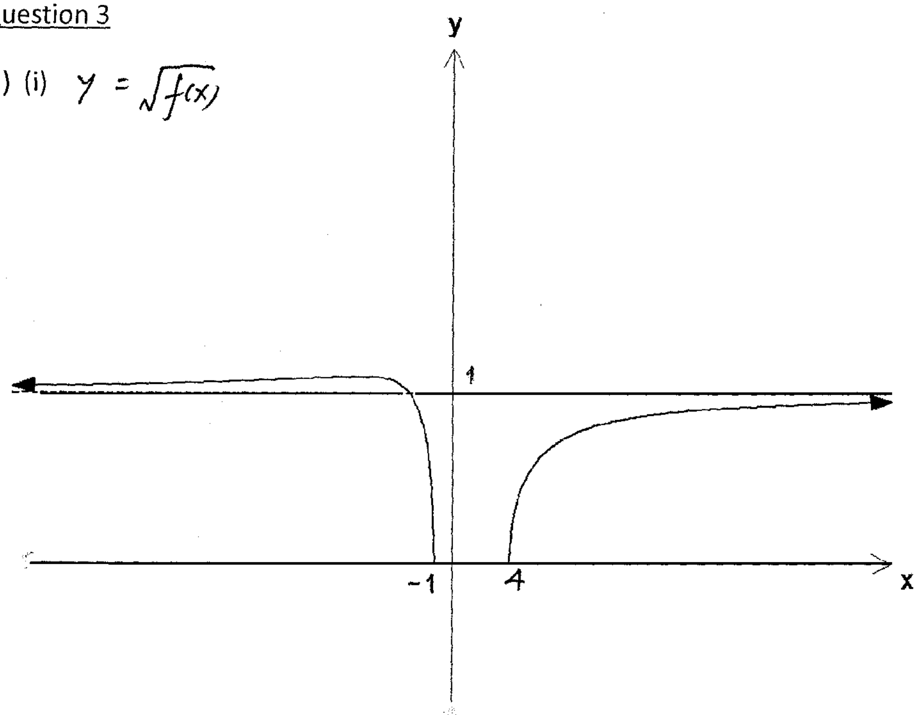
(ii) $\vec{OC} = \vec{OA} + \vec{OB}$

$= (x+iy) + (-2y+2ix)$

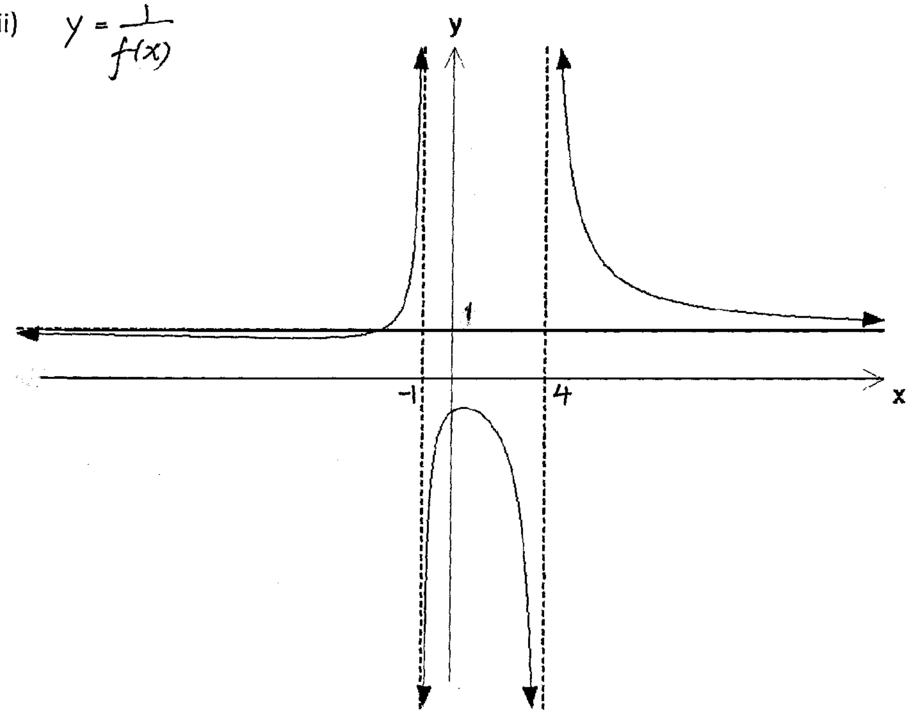
$= (x-2y) + (2x+y)i$ $\boxed{1}$

Question 3

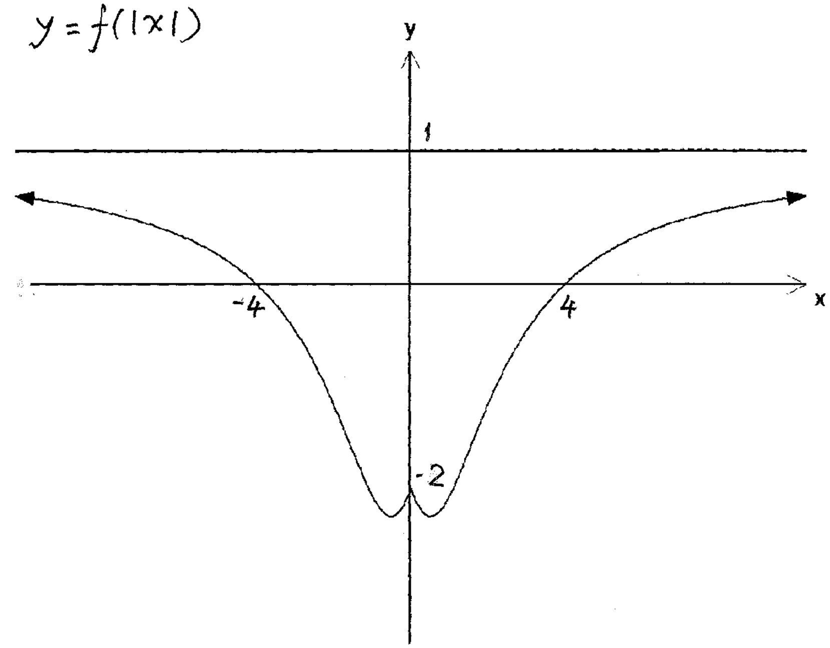
a) (i) $y = \sqrt{f(x)}$



(ii) $y = \frac{1}{f(x)}$



(iii) $y = f(|x|)$



b) (i) The only possible integer roots are 1 or -1.

But $(1) = -1 \neq 0$ $\boxed{1}$

$P(-1) = 10 \neq 0$

$\therefore$ $P(x) = 0$ has no integer roots.

(ii) $P(0) = 1$

$P(0)$ and $P(1)$ are opposite in sign, $\boxed{1}$

hence there must be a real root

(iii) $\alpha^2+\beta^2+\gamma^2+\delta^2 = (\sum \alpha)^2 - 2\sum \alpha\beta$ $\boxed{1}$

$= 2^2 - 2(3)$

$= -2$ $\boxed{1}$

(iv) Since $\sum \alpha^2 < 0$ $\boxed{1}$

$\therefore$ Not all root are real.

But there is a real root between 0 & 1 by (i)

and complex roots occur in pairs, there can only be 2 complex roots. $\boxed{1}$

Hence there must be 2 real roots.

Q 3 (b)

(v) $\dfrac{1}{2\alpha\beta\gamma} = \dfrac{\delta}{2\alpha\beta\gamma\delta}$

$= \dfrac{\delta}{2}$ since $\alpha\beta\gamma\delta = 1$

$\therefore$ let $y = \dfrac{x}{2}$

$x = 2y$

$P(x) = x^4 - 2x^3 + 3x^2 - 4x + 1 = 0$

$(2y)^4 - 2(2y)^3 + 3(2y)^2 - 4(2y) + 1 = 0$ $\boxed{1}$

$16y^4 - 16y + 12y - 8y + 1 = 0$

$\therefore$ The required equation is

$16x^4 - 16x^3 + 12y - 8y + 1 = 0$ $\boxed{1}$

(vi) $\dfrac{1}{2\alpha\beta\gamma} + \dfrac{1}{2\alpha\beta\delta} + \dfrac{1}{2\alpha\gamma\delta} + \dfrac{1}{2\beta\gamma\delta}$

= sum of roots of the new equation

$= 1$ $\boxed{1}$

Question 4

a) (i) Foci are $(2,-1)$ and $(10,-1)$

$\therefore$ Centre is $(6,-1)$

$\therefore$ The complex number corresponding to the centre of the ellipse is $6 - i$ $\boxed{1}$

(ii) $2a = 12$

$a = 6$

$ae = 4$

$6e = 4$

$e = \frac{2}{3}$

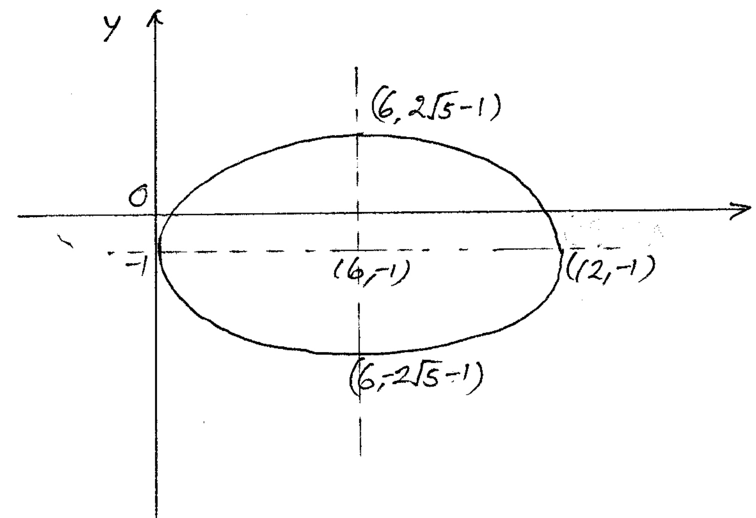
$b = a\sqrt{1-e^2}$

$= 6\sqrt{1-\frac{4}{9}}$

$= 2\sqrt{5}$

$\therefore$ major axis $= 12$ $\boxed{1}$

minor axis $= 4\sqrt{5}$ $\boxed{1}$



$\boxed{1}$

b) (i) The tangent at P is $\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$ $\boxed{1}$

(ii) S is $(ae, 0)$

$\therefore ST = \dfrac{\left|\dfrac{ae\cos\theta}{a} - 1\right|}{\sqrt{\left(\dfrac{\cos\theta}{a}\right)^2 + \left(\dfrac{\sin\theta}{b}\right)^2}}$ $\boxed{1}$

Q 4 (b)(ii) cont'd

$$ST = \frac{ab\,|e\cos\theta - 1|}{ab\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}}$$

$$= \frac{ab\,|e\cos\theta - 1|}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}} \quad \boxed{1}$$

(iii) Similarly $S'T' = \dfrac{ab\,|-e\cos\theta-1|}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}$

$$= \frac{ab\,|e\cos\theta+1|}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}$$

$$\therefore ST \times S'T' = \frac{ab\,|e\cos\theta-1|}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}} \times \frac{ab\,|e\cos\theta+1|}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}$$

$$= \frac{a^2b^2\,(1-e\cos\theta)\,(1+e\cos\theta)}{a^2\sin^2\theta+b^2\cos^2\theta}$$

$$= \frac{a^2b^2(1-e^2\cos^2\theta)}{a^2\sin^2\theta+b^2\cos^2\theta} \quad \boxed{1}$$

$$= \frac{b^2(a^2-a^2e^2\cos^2\theta)}{a^2\sin^2\theta+b^2\cos^2\theta}$$

$$= \frac{b^2[a^2-(a^2-b^2)\cos^2\theta]}{a^2\sin^2\theta+b^2\cos^2\theta}$$

$$= \frac{b^2[(a^2-a^2\cos^2\theta)+b^2\cos^2\theta]}{a^2\sin^2\theta+b^2\cos^2\theta} \quad \boxed{1}$$

$$= \frac{b^2(a^2\sin^2\theta+b^2\cos^2\theta)}{a^2\sin^2\theta+b^2\cos^2\theta}$$

$$= b^2$$

(c) (i) $xy = c^2$

$$y = \frac{c^2}{x}$$

$$\therefore y' = -\frac{c^2}{x^2}$$

$\therefore$ gradient of tangent at A is $m_1 = -\dfrac{c^2}{c^2a^2}$

$$= -\frac{1}{a^2}$$

gradient of normal at B is $m_2 = a^2$. $\boxed{1}$

$\therefore$ Equation of normal at A is

$$y - \frac{c}{a} = a^2(x-ca)$$

$$= a^2x - ca^3 \quad \boxed{1}$$

ie $y = a^2x - ca^3 + \dfrac{c}{a}$

$$y = a^2x + \frac{c}{a}(1-a^4) \qquad (1)$$

(ii) B lies on the normal at A

$\therefore (cb, \frac{c}{b})$ satisfies (1)

$$\frac{c}{b} = a^2cb + \frac{c}{a}(1-a^4)$$

$$ac = a^3cb^2 + bc(1-a^4)$$

$$a = a^3b^2 + b - a^4b$$

$$a - b = a^3b(b-a) \quad \boxed{1}$$

$$1 = -a^3b$$

$$\therefore b = -\frac{1}{a^3}$$

[alternatively]

when $y = a^2x + \frac{c}{a}(1-a^4)$ meets $xy = c^2$

$$\frac{c^2}{x} = a^2x + \frac{c}{a}(1-a^4)$$

$$ac^2 = a^3x^2 + c(1-a^4)x$$

a^2 $$a^3x^2 + c(1-a^4)x - ac^2 = 0$$

The x-coordinates of A and B are solution of this equation

$$\text{Product of roots} = -\frac{ac^2}{a^3}$$

$$= -\frac{c^2}{a^2}$$

$$\therefore (ca)(cb) = -\frac{c^2}{a^2} \qquad \boxed{1}$$

$$ab = -\frac{1}{a^2}$$

$$b = -\frac{1}{a^3}$$

(iii) At M $$x = \frac{c}{2}(a+b) \qquad (1)$$

$$y = \frac{c}{2}\left(\frac{1}{a}+\frac{1}{b}\right)$$

$$y = \frac{c}{2}\,\frac{a+b}{ab} \qquad (2)$$

$\frac{(1)}{(2)}$ $$\frac{x}{y} = ab \qquad (3)$$

but $$b = -\frac{1}{a^3}$$

Put into (3) $$\frac{x}{y} = -\frac{1}{a^2}$$

or $$a^2 = -\frac{y}{x} \qquad \boxed{1} \qquad (4)$$

(iv) From (2) $$y = \frac{c}{2}\left(\frac{1}{a}+\frac{1}{b}\right)$$

$$= \frac{c}{2}\left(\frac{1}{a} - a^3\right)$$

$$= \frac{c}{2}\left(\frac{1-a^4}{a}\right)$$

$$y^2 = \frac{c^2}{4}\,\frac{(1-a^4)^2}{a^2} \qquad (5)$$

Put (4) into (5)

$$y^2 = \frac{c^2}{4}\,\frac{\left[1-\left(-\frac{y}{x}\right)^2\right]^2}{\left(-\frac{y}{x}\right)} \qquad \boxed{1}$$

$$= \frac{c^2}{4}\,\frac{\left[1-\frac{y^2}{x^2}\right]^2}{-\left(\frac{y}{x}\right)}$$

$$= -\frac{c^2}{4}\,\frac{(x^2-y^2)^2}{x^4}\cdot\frac{x}{y}$$

$$= -\frac{c^2(x^2-y^2)^2}{4x^3y}$$

ie $$4x^3y^3 + c^2(x^2-y^2)^2 = 0 \qquad \boxed{1}$$

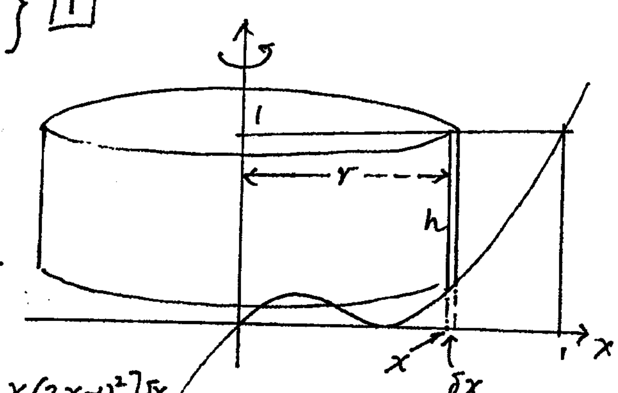
Question 5

a) $$\left.\begin{aligned} h &= 1 - x(2x-1)^2 \\ r &= x \end{aligned}\right\} \boxed{1}$$

Vol of the cylindrical shell

$$\delta V = 2\pi r h\,\delta x$$

$$= 2\pi x[1 - x(2x-1)^2]\delta x$$



$\therefore$ Vol of solid

$$V = \lim_{\delta x \to 0} \sum_{x=0}^{1} 2\pi x[1 - x(2x-1)^2]\delta x$$

Q5 (cont'd)

$$V = \int_0^1 2\pi x\left[1 - x(2x-1)^2\right]dx \qquad \boxed{1}$$

$$= 2\pi\int_0^1 x - x^2(2x-1)^2\,dx$$

$$= \pi\left[x^2\right]_0^1 - 2\pi\int_0^1 x^2(4x^2-4x+1)\,dx$$

$$= \pi - 2\pi\int_0^1 (4x^4 - 4x^3 + x^2)\,dx \qquad \boxed{1}$$

$$= \pi - 2\pi\left[\frac{4x^5}{5} - x^4 + \frac{x^3}{3}\right]_0^1$$

$$= \pi - 2\pi\left(\frac{4}{5} - 1 + \frac{1}{3}\right)$$

$$= \pi - \frac{4\pi}{15}$$

$$= \frac{11\pi}{15}\ \text{unit}^3 \qquad \boxed{1}$$

b) (i) $z = \cos\theta + i\sin\theta$ (1)

$\frac{1}{z} = \cos\theta - i\sin\theta$ (2)

(1)+(2) $z + \frac{1}{z} = 2\cos\theta$ $\boxed{1}$

$\therefore\ 2\cos\theta = z + \frac{1}{z}$

$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

(ii) $z^n = \cos n\theta + i\sin n\theta$ (3)

$\frac{1}{z^n} = \cos n\theta - i\sin n\theta$ (4)

(3)+(4) $z^n + \frac{1}{z^n} = 2\cos n\theta$ $\boxed{1}$

ie $\cos n\theta = \frac{1}{2}\left(z^n + \frac{1}{z^n}\right)$

(iii) $(2\cos\theta)^4 = \left(z + \frac{1}{z}\right)^4$

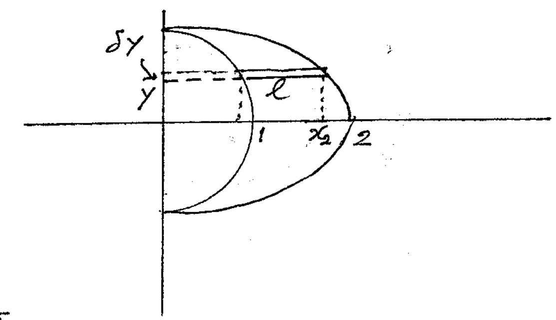
$16\cos^4\theta = z^4 + 4z^2 + 6 + \frac{4}{z^2} + \frac{1}{z^4}$ $\boxed{1}$

$= \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$

$= 2\cos 4\theta + 8\cos 2\theta + 6$

$\therefore\ \cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$ $\boxed{1}$

(c) For the circle $x^2 + y^2 = 1$



(i) $x_1 = \sqrt{1-y^2}$

For the ellipse $\frac{x^2}{4} + y^2 = 1$

$x_2 = 2\sqrt{1-y^2}$

$\therefore\ l = x_2 - x_1$

$= 2\sqrt{1-y^2} - \sqrt{1-y^2}$

$= \sqrt{1-y^2}$

Height of the rectangle $= l^2$

ie $h = 1 - y^2$ $\boxed{1}$

$\therefore$ Area of cross-section $= hl$

$= (1-y^2)\sqrt{1-y^2}$

$= (1-y^2)^{3/2}$

$\therefore$ Volume of the vertical slice is

$\delta V = (1-y^2)^{3/2}\,\delta y$ $\boxed{1}$

Q 5 (cont'd)

Volume of the solid

$$V = \lim_{\delta y \to 0} \sum \delta V$$

$$= \lim_{\delta y \to 0} \sum_{y=-1}^{1} (1-y^2)^{3/2} \delta y$$

$$= \int_{-1}^{1} (1-y^2)^{3/2} dy$$

(ii) Let $y = \sin\theta$

$dy = \cos\theta \, d\theta$

when $y=1$ $\quad \theta = \frac{\pi}{2}$

when $y=0$ $\quad \theta = 0$

$\therefore V = 2\int_0^1 (1-y^2)^{3/2} dy$ $\quad \because (1-y^2)^{\frac{3}{2}}$ is an even function of y.

$$= 2\int_0^{\frac{\pi}{2}} (1-\sin^2\theta)^{3/2} \cos\theta \, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \cos^3\theta \cdot \cos\theta \, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \cos^4\theta \, d\theta \qquad \boxed{1}$$

$$= \frac{2}{8}\int_0^{\frac{\pi}{2}} (\cos 4\theta + 4\cos 2\theta + 3)\, d\theta$$

$$= \frac{1}{4}\left[\frac{1}{4}\sin 4\theta + 2\sin 2\theta + 3\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{4}\left[0 + 0 + \frac{3\pi}{2}\right]$$

$$= \frac{3\pi}{8} \text{ unit}^3 \qquad \boxed{1}$$

5d) $u_2 = 4u_1 - 4u_0$

$= 4(2) - 4(1)$

$= 4$

$= 2^2$

$\therefore$ It is true for $n=2$

$u_3 = 4u_2 - 4u_1$

$= 4(4) - 4(2)$

$= 8$

$= 2^3$

$\therefore$ True for $n=3$. $\qquad \boxed{1}$

Assume it is true for $n=k$ and $k+1$

ie $\quad u_k = 2^k, \quad u_{k+1} = 2^{k+1}$ $\qquad \boxed{1}$

then $\quad u_{k+2} = 4u_{k+1} - 4u_k$

$= 4(2^{k+1}) - 4(2^k)$ $\quad$ by assumption

$= 4(2^{k+1} - 2^k)$

$= 4(2 \cdot 2^k - 2^k)$

$= 4(2^k)$ $\qquad \boxed{1}$

$= 2^{k+2}$

ie true for $n=k+2$ if true for $n=k$ and $n=k+1$, since it is proved true for $n=2$ and 3, hence it is true for $n=4,5,6,\dots$ ie true for all positive integers.

Question 6

a)(i) $\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$

$= \cos^5\theta + 5(\cos^4\theta)(i\sin\theta) + 10(\cos^3\theta)(i\sin\theta)^2 + 10(\cos^2\theta)(i\sin\theta)^3 + 5(\cos\theta)(i\sin\theta)^4 + (i\sin\theta)^5$ $\boxed{1}$

$= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$

Equating real parts

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$ $\boxed{1}$

$= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(ii) Let $x = \cos\theta$

$32x^5 - 40x^3 + 10x - 1 = 0$ (1)

$16x^5 - 20x^3 + 5x = \frac{1}{2}$

$16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = \frac{1}{2}$

$\cos 5\theta = \frac{1}{2}$ $\boxed{1}$

$5\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \frac{13\pi}{3}$

$\theta = \frac{\pi}{15}, \frac{\pi}{3}, \frac{7\pi}{15}, \frac{11\pi}{15}, \frac{13\pi}{15}$ $\boxed{1}$

$\therefore$ Roots are $\cos\frac{\pi}{15}, \cos\frac{\pi}{3}, \cos\frac{7\pi}{15}, \cos\frac{11\pi}{15}, \cos\frac{13\pi}{15}$ $\boxed{1}$

(iii) Sum of roots of (1)

$\cos\frac{\pi}{15} + \cos\frac{\pi}{3} + \cos\frac{7\pi}{15} + \cos\frac{11\pi}{15} + \cos\frac{13\pi}{15} = 0$ $\boxed{1}$

$\cos\frac{\pi}{15} + \frac{1}{2} + \cos\frac{7\pi}{15} + \cos\left(2\pi - \frac{19\pi}{15}\right) + \cos\frac{13\pi}{15} = 0$ $\boxed{1}$

$\cos\frac{\pi}{15} + \cos\frac{7\pi}{15} + \cos\frac{13\pi}{15} + \cos\frac{19\pi}{15} = -\frac{1}{2}$

b)

$\tan 5\theta = \dfrac{\sin 5\theta}{\cos 5\theta}$

$= \dfrac{16\sin^5\theta - 20\sin^3\theta + 5\sin\theta}{16\cos^5\theta - 20\cos^3\theta + 5\cos\theta}$

$= \dfrac{(16\sin^5\theta - 20\sin^3\theta + 5\sin\theta)\times\frac{1}{\cos^5\theta}}{(16\cos^5\theta - 20\cos^3\theta + 5\cos\theta)\times\frac{1}{\cos^5\theta}}$ $\boxed{1}$

$= \dfrac{16\tan^5\theta - 20\tan^3\theta\sec^2\theta + 5\tan\theta\cdot\sec^4\theta}{16 - 20\sec^2\theta + 5\sec^4\theta}$

$= \dfrac{16\tan^5\theta - 20\tan^3\theta(1+\tan^2\theta) + 5\tan\theta(1+\tan^2\theta)^2}{16 - 20(1+\tan^2\theta) + 5(1+\tan^2\theta)^2}$ $\boxed{1}$

$= \dfrac{16\tan^5\theta - 20\tan^3\theta - 20\tan^5\theta + 5\tan\theta + 10\tan^3\theta + 5\tan^5\theta}{16 - 20 - 20\tan^2\theta + 5 + 10\tan^2\theta + 5\tan^4\theta}$

$= \dfrac{\tan^5\theta - 10\tan^3\theta + 5\tan\theta}{5\tan^4\theta - 10\tan^2\theta + 1}$

Q6 (cont'd)

b) (ii) Consider $\tan 5\theta = 0$

$\therefore 5\theta = \pi, 2\pi, 3\pi, 4\pi, 5\pi \ldots\ldots$

$\theta = \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, \pi, \ldots\ldots$

but $\tan 5\theta = \dfrac{t^5 - 10t^3 + 5t}{5t^4 - 10t^2 + 1}$ where $t = \tan\theta$

$\therefore$ The solution to $\dfrac{t^5 - 10t^3 + 5t}{5t^4 - 10t^2 + 1} = 0$

ie $t^5 - 10t^3 + 5t = 0$

are $t = \tan\frac{\pi}{5}, \tan\frac{2\pi}{5}, \tan\frac{3\pi}{5}, \tan\frac{4\pi}{5}, \tan\pi$ $\boxed{1}$

$t = \tan\pi$ corresponding to $t = 0$

$\therefore$ Roots of $t^4 - 10t^2 + 5 = 0$

are $\tan\frac{\pi}{5}, \tan\frac{2\pi}{5}, \tan\frac{3\pi}{5}, \tan\frac{4\pi}{5}$ $\boxed{1}$

Product of roots:

$\tan\frac{\pi}{5}\tan\frac{2\pi}{5}\tan\frac{3\pi}{4}\tan\frac{4\pi}{5} = 5$

c) (i) Since α, β are roots of $x^2 + px + q = 0$

$\therefore \alpha^2 + p\alpha + q = 0$ (1)

$\beta^2 + p\beta + q = 0$ (2)

(1) $\cdot \alpha^n$ $\quad \alpha^{n+2} + p\alpha^{n+1} + q\alpha^n = 0$ (3)

(2) $\cdot \beta^n$ $\quad \beta^{n+2} + p\beta^{n+1} + q\beta^n = 0$ (4) $\Big\}\ \boxed{1}$

(3) + (4) $(\alpha^{n+2} + \beta^{n+2}) + p(\alpha^{n+1} + \beta^{n+1}) + q(\alpha^n + \beta^n) = 0$ $\boxed{1}$

$S_{n+2} + pS_{n+1} + qS_n = 0$

(ii) Let $\alpha = \sqrt{2011} + \sqrt{2010}$, $\beta = \sqrt{2011} - \sqrt{2010}$

$\therefore \alpha + \beta = 2\sqrt{2011}$, $\alpha\beta = 2011 - 2010 = 1$

$\therefore$ α and β are roots of the equation

$x^2 - 2\sqrt{2011}\,x + 1 = 0$ $\boxed{1}$

$S_2 = \alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= 4(2011) - 2$

$= 8042$

$S_3 - 2\sqrt{2011}\,S_2 + S_1 = 0$

$\therefore S_3 = 2\sqrt{2011}\,S_2 - S_1$

$= 16084\sqrt{2011} - 2\sqrt{2011}$

$= 16082\sqrt{2011}$

$S_4 - 2\sqrt{2011}\,S_3 + S_2 = 0$ $\boxed{1}$

$S_4 = 2\sqrt{2011}\,S_3 - S_2$

$= 2\sqrt{2011}(16082\sqrt{2011}) - 8042$

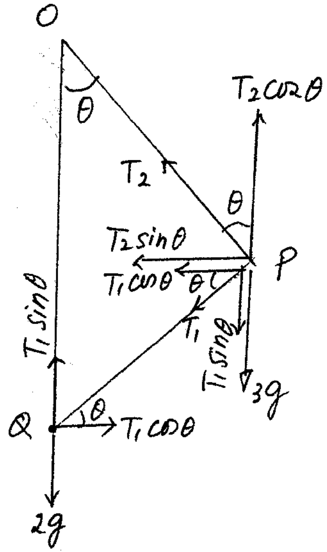
$= 64673762.$

Question 7

a) (i) $OP = 4$, $PQ = 3$, $OQ = 5$

$\therefore \angle OPQ = 90^\circ$

$\cos\theta = \frac{4}{5}$, $\sin\theta = \frac{3}{5}$



Resolving forces vertically at Q:

$T_1 \sin\theta = 2g$

$\frac{3T_1}{5} = 2g$

$\therefore T_1 = \frac{10g}{3}$ $\boxed{1}$

Resolving forces vertically at P

$T_2\cos\theta = T_1 \sin\theta + 3g$ $\boxed{1}$

$\frac{4T_2}{5} = \frac{3T_1}{5} + 3g$

$= \frac{3}{5}\left(\frac{10g}{3}\right) + 3g$

$= 5g$

$\therefore T_2 = \frac{25g}{4}$ $\boxed{1}$

(ii) Radius of the horizontal circle

$r = 4\sin\theta$

$= \frac{12}{5}$ $\boxed{1}$

Resolving forces horizontally at P

$T_1\cos\theta + T_2 \sin\theta = 3\omega^2 r$ $\boxed{1}$

$\frac{10g}{3}\cdot\frac{4}{5} + \frac{25g}{4}\cdot\frac{3}{5} = 3\cdot\frac{12}{5}\omega^2$

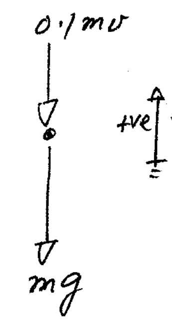
$\therefore \omega^2 = \frac{385g}{432}$

$\therefore \omega = \sqrt{\frac{385g}{432}}$

$= 2.985$

$= 3.0 \text{ rad s}^{-1}$ (1 dp) $\boxed{1}$

b) (i) Take upward direction as positive



$m\ddot{x} = -mg - 0.1mv$

$\therefore \ddot{x} = -g - 0.1v$ $\boxed{1}$

(ii) $v\frac{dv}{dx} = -(g + 0.1v)$

$\therefore \int_{100}^{v} \frac{v\,dv}{g + 0.1v} = -\int_0^x dx$ $\boxed{1}$

$\int_{100}^{v}\left(10 - \frac{10g}{g+0.1v}\right)dv = -[x]_0^x$

$\left[10v - 100g\ln(g+0.1v)\right]_{100}^{v} = -x$ $\boxed{1}$

$10v - 100g\ln(g+0.1v) - 1000 + 100g\ln(g+10) = -x$

$\therefore x = 1000 - 10v - 100g\ln\left(\frac{g+10}{g+0.1v}\right)$

Q7 (cont'd)

$x = 1000 - 10v - 1000 \ln\left(\frac{20}{10+0.1v}\right)$ $\qquad g=10$

(iii) At max height, $v=0$

$x_{max} = 1000 - 1000 \ln\left(\frac{20}{10}\right)$

$= 1000 - 1000 \ln 2$ $\qquad \boxed{1}$

(iv) From $\ddot{x} = -(g+0.1v)$

$\frac{dv}{dt} = -(10+0.1v)$

$\int_{100}^{v} \frac{dv}{10+0.1v} = -\int_0^t dt$ $\qquad \boxed{1}$

$10 \ln[10+0.1v]_{100}^{v} = -[t]_0^t$

$10 \ln\left(\frac{10+0.1v}{20}\right) = -t$

$\therefore \quad t = 10 \ln\left(\frac{20}{10+0.1v}\right)$ $\qquad \boxed{1}$

(v) when $v=0$, $t = 10 \ln \frac{20}{10}$

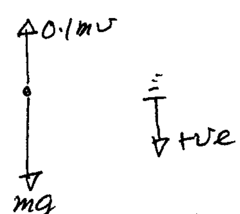
$= 10 \ln 2$ $\qquad \boxed{1}$

(vi) During downward journey

$m\ddot{x} = mg - 0.1mv$

$\ddot{x} = g - 0.1v$ $\qquad \boxed{1}$

$= 10 - 0.1v$

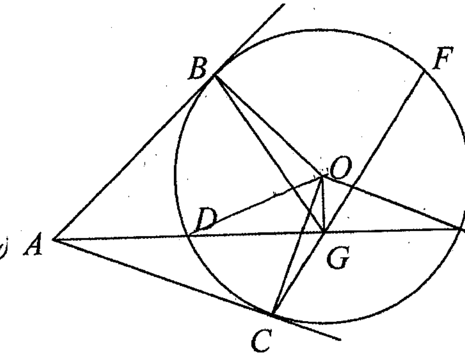


when $v = v_T$, $\ddot{x} = 0$

$\therefore 0 = 10 - 0.1v_T$

$v_T = 100 \text{ ms}^{-1}$ $\qquad \boxed{1}$

Question 8



a) (ii) $\angle ABO = \angle ACO = 90°$

(radius ⊥ tangent at pt of contact)

$\angle ABO + \angle ACO = 2$ rt $\angle$'s

$\therefore ABOC$ is a cyclic quad. $\boxed{1}$

(opp $\angle$'s of cyclic quad are supplementary)

In $\triangle ODG$, $\triangle OEG$

$OD = OE$ (radii)

$DG = EG$ (given)

$OG = OG$ (common)

$\therefore \triangle DGO \equiv \triangle EGO$ (SSS)

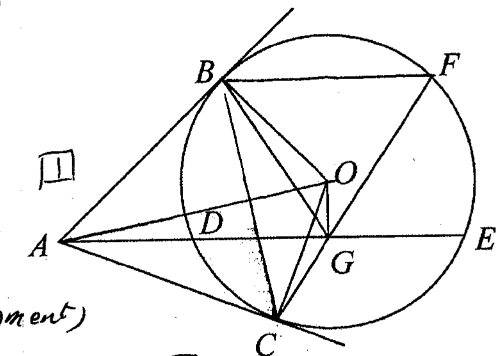
$\therefore \angle DGO = \angle EGO$ (corr. $\angle$s of congruent $\triangle$s)

but $\angle DGO + \angle EGO = 180°$ ($\angle$s on a str. line)

$\therefore \angle DGO = 90°$ $\qquad \boxed{1}$

$= \angle OCA$

$\therefore AOGC$ is a cyclic quad. ($\angle$'s in same segment are equal) $\boxed{1}$



(iii) $\angle FGE = \angle AGC$ (vert. opp $\angle$'s)

$\angle AGC = \angle AOC$

($\angle$'s in same segment, AOGC is a cyclic quad) $\boxed{1}$

$\angle AOC = \angle ABC$

($\angle$'s in same segment, ABOC is cyclic quad)

$\angle ABC = \angle BFC$ ($\angle$'s in alt segment)

$\therefore \angle FGE = \angle BFG$ $\qquad \boxed{1}$

$BF \parallel AE$ (alt $\angle$'s on $\parallel$ lines are equal)

Q 8 (cont'd)

b) (i) $\frac{d}{dx}[(1+x)\ln(1+x)-x]$

$= \ln(1+x) + \frac{1+x}{1+x} - 1$

$= \ln(1+x)$ $\boxed{1}$

$\therefore \int \ln(1+x)\,dx = (1+x)\ln(1+x) - x + C$

(ii) $I_n = \int_0^1 x^n \ln(1+x)\,dx$

$= \int_0^1 x^n \,d[(1+x)\ln(1+x)-x]$

$= \left[x^n[(1+x)\ln(1+x)-x]\right]_0^1 - n\int_0^1 x^{n-1}[(1+x)\ln(1+x)-x]\,dx$ $\boxed{1}$

$= (2\ln 2 - 1) - n\int_0^1 x^{n-1}\ln(1+x)dx - n\int_0^1 x^n \ln(1+x)dx + n\int_0^1 x^n dx$

$= 2\ln 2 - 1 - nI_{n-1} - nI_n + n\left[\frac{x^{n+1}}{n+1}\right]_0^1$

$= 2\ln 2 - 1 - nI_{n-1} - nI_n + \frac{n}{n+1}$ $\boxed{1}$

$\therefore (n+1)I_n = 2\ln 2 - nI_{n-1} - 1 + \frac{n}{n+1}$

$= 2\ln 2 - \frac{1}{n+1} - nI_{n-1}$

(iii)

$(n+1)I_n + nI_{n-1} = 2\ln 2 - \frac{1}{n+1}$ (1)

$nI_{n-1} + (n-1)I_{n-2} = 2\ln 2 - \frac{1}{n}$ (2) $\boxed{1}$

$(n-1)I_{n-2} + (n-2)I_{n-3} = 2\ln 2 - \frac{1}{n-1}$ (3)

$(n-2)I_{n-3} + (n-3)I_{n-4} = 2\ln 2 - \frac{1}{n-2}$ (4)

- - - - - - - -

$4I_3 + 3I_2 = 2\ln 2 - \frac{1}{4}$ (n-2)

$3I_2 + 2I_1 = 2\ln 2 - \frac{1}{3}$ (n-1)

$2I_1 + I_0 = 2\ln 2 - \frac{1}{2}$ (n)

n is odd:

(1) − (2) + (3) − (4) + ... + (n−2) − (n−1) + n $\boxed{1}$

$(n+1)I_n + I_0 = -\frac{1}{n+1} + \frac{1}{n} - \frac{1}{n-1} + \frac{1}{n-2} - \cdots - \frac{1}{4} + \frac{1}{3} - \frac{1}{2} + 2\ln 2$

$(n+1)I_n + 2\ln 2 - 1 = -\frac{1}{n+1} + \frac{1}{n} - \frac{1}{n-1} + \frac{1}{n-2} - \cdots - \frac{1}{4} + \frac{1}{3} - \frac{1}{2} + 2\ln 2$

$\therefore (n+1)I_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{n} - \frac{1}{n+1}$

c) (i) $1 + \omega + \omega^2 = 0$

$\therefore 1 + \omega = -\omega^2$

$(1+\omega)^2 = (-\omega^2)^2$

$= \omega^4$

$= \omega$ $\quad \because \omega^3 = 1$ $\boxed{1}$

(ii) $(1+\omega)^3 = (-\omega^2)^3$

$= -\omega^6$

$= -(\omega^3)^2$ $\boxed{1}$

$= -1$

(iii) $(1+\omega)^{3n} = [(1+\omega)^3]^n$ $\boxed{1}$

$= (-1)^n$

But $(1+\omega)^{3n} = {}^{3n}C_0 + {}^{3n}C_1\omega + {}^{3n}C_2\omega^2 + {}^{3n}C_3\omega^3 + \cdots + {}^{3n}C_{3n}\omega^{3n}$

$\therefore (-1)^n = {}^{3n}C_0 + {}^{3n}C_1\omega + {}^{3n}C_2\omega^2 + {}^{3n}C_3 + {}^{3n}C_4\omega + {}^{3n}C_5\omega^2 + {}^{3n}C_6$

$+ \cdots + {}^{3n}C_{3n}$ $\boxed{1}$

Equating real parts

$(-1)^n = {}^{3n}C_0 + {}^{3n}C_1(-\frac{1}{2}) + {}^{3n}C_2(-\frac{1}{2}) + {}^{3n}C_3 + {}^{3n}C_4(-\frac{1}{2}) + {}^{3n}C_5(-\frac{1}{2})$ $\boxed{1}$

$+ {}^{3n}C_6 + \cdots + {}^{3n}C_{3n}$ [By assumption]

$\therefore {}^{3n}C_0 - \frac{1}{2}({}^{3n}C_1 + {}^{3n}C_2) + {}^{3n}C_3 - \frac{1}{2}({}^{3n}C_4 + {}^{3n}C_5) + {}^{3n}C_6 - \cdots + {}^{3n}C_{3n} = (-1)^n$