JG

Name:

Class: 12MTZ1

Teacher: MRS GOBERT

CHERRYBROOK TECHNOLOGY HIGH SCHOOL

2012 AP4

YEAR 12 TRIAL HSC EXAMINATION

MATHEMATICS EXTENSION 2

Time allowed - 3 HOURS
(Plus 5 minutes reading time)

Directions to candidates

- Attempt all questions
- Approved calculators may be used.
- Standard Integral Tables are provided at the back of this paper.
- Write your name and class in the space provided at the top of this question paper.

Section I - TOTAL MARKS 10

- To be answered on the removable answer grid at the back of the exam paper.
- Allow about 15 minutes for this section.

Section II - TOTAL MARKS 90

- All answers to be completed on the writing paper provided. Each question is to be commenced on a new page clearly marked Question 11, Question 12, etc on the top of the page. Write your name and class at the top of each page.
- All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.
- Allow about 2 hours and 45 minutes for this section.

YOUR ANSWERS WILL BE COLLECTED IN ONE BUNDLE. THE MULTIPLE CHOICE SECTION I ON TOP AND THEN WRITTEN ANSWERS TO SECTION II AND THEN THE QUESTION PAPER.

SECTION I 10 MARKS

INSTRUCTIONS

- Attempt all questions
- Allow about 15 minutes for this section
- Section I answers are to be completed on the multiple-choice answer sheet attached to the back of this question paper.
- Select the alternative A, B, C or D that best answers the question

1. Find $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^9$

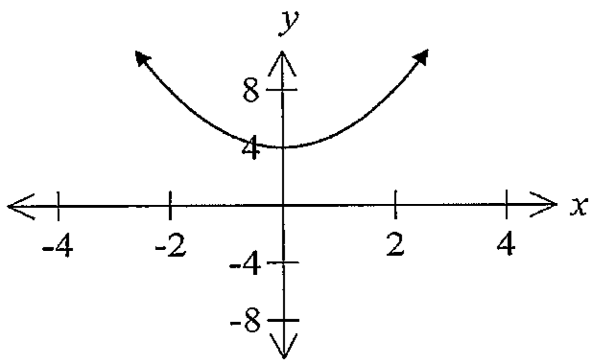
(A) -1 (B) 1 (C) 0 (D) 2

2. The eccentricity of the ellipse $\frac{x^2}{25}+\frac{y^2}{16}=1$ is

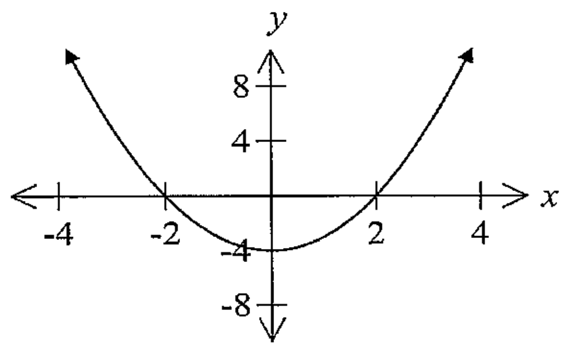
(A) $\frac{3}{5}$ (B) $\frac{4}{5}$ (C) $\frac{5}{3}$ (D) $\frac{5}{4}$

3. The graph of $y=|f(x)|$, given that $f(x)=4-x^2$, is

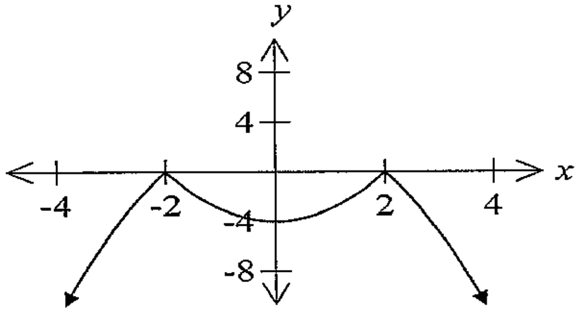
(A)



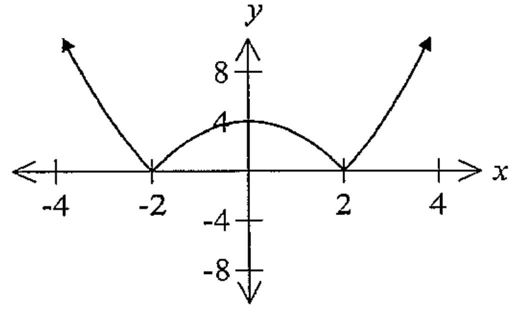
(B)



(C)



(D)

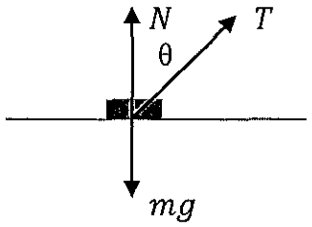


4. The parametric equations of the hyperbola $x^2 - y^2 = 4$ are

(A) $x = 2\tan\theta$ and $y = 2\sec\theta$ (B) $x = 2\sec\theta$ and $y = 2\tan\theta$

(C) $x = 4\tan\theta$ and $y = 4\sec\theta$ (D) $x = 4\sec\theta$ and $y = 4\tan\theta$

5. A body of mass m kg is being pulled along a smooth horizontal table by means of a string inclined at θ to the vertical. The diagram below indicates the forces acting on the body. Which one of the following is true?



(A) $N - mg = 0$
(B) $N + T\sin\theta - mg = 0$
(C) $N - T\sin\theta - mg = 0$
(D) $N + T\cos\theta - mg = 0$

6. The equation $x^3 + 2x - 1 = 0$ has roots α, β and γ.
Find the monic equations with roots α^2, β^2 and γ^2

(A) $x^3 + 2x^2 + 2x - 3 = 0$

(B) $x^3 + 4x^2 + 2x - 1 = 0$

(C) $x^3 + 4x^2 + 4x - 1 = 0$

(D) $4x^3 + 4x^2 + 4x - 1 = 0$

7. If $x = \theta - \sin\theta$ and $y = 1 - \cos\theta$, then the expression for $\frac{dy}{dx}$ is

(A) $\cot^2\frac{\theta}{2}$ (B) $\cot\frac{\theta}{2}$ (C) $\tan\frac{\theta}{2}$ (D) $\tan^2\frac{\theta}{2}$

8. Find $\int \frac{1}{x^2 + 2x + 2}\,dx$

(A) $\tan^{-1}(x+2) + c$

(B) $\tan^{-1}(x+1) + c$

(C) $\sin^{-1}(x+1) + c$

(D) $\cos^{-1}(x+1) + c$

9. Which expression is equal to $\int 3\sqrt{x}\ln x\,dx$

(A) $2x\sqrt{x}\left(\ln x - \frac{2}{3}\right) + c$

(B) $2x\sqrt{x}\left(\ln x + \frac{2}{3}\right) + c$

(C) $\frac{1}{\sqrt{x}}\left(\frac{3}{2}\ln x - 1\right) + c$

(D) $\frac{1}{\sqrt{x}}\left(\frac{3}{2}\ln x + 1\right) + c$

10. What is the solution to the inequation $\frac{x(5-x)}{x-4} \geq -3$?

(A) $2 \leq x < 4$ or $x \geq 6$

(B) $1 \leq x < 4$ or $x \geq 5$

(C) $4 < x \leq 6$ or $x \leq 2$

(D) $4 < x \leq 5$ or $x \leq 1$

END OF SECTION I

SECTION II 90 MARKS

INSTRUCTIONS

- Answer all questions on the writing paper provided.
- Allow about 2 hours and 45 minutes for this section.
- Begin each question on a new page and write your name clearly.
- Show all necessary working.

Question 11 (15 marks) BEGIN A NEW PAGE **Marks**

(a) Find $\int \frac{e^{2x}}{e^x + 1}\, dx$. **2**

(b) Using the substitution $u = \pi - x$,

(i) Show that $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}\, dx$ **2**

(ii) Hence deduce that $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx = \frac{\pi^2}{4}$ **3**

(c) (i) Find real numbers a, b and c such that **2**

$$\frac{8-2x}{(1+x)(4+x^2)} = \frac{a}{1+x} + \frac{bx+c}{4+x^2}$$

(ii) Hence evaluate in simplest form **2**

$$\int_0^4 \frac{8-2x}{(1+x)(4+x^2)}\, dx$$

(d) Let $I_n = \int_1^e (\ln t)^n\, dt$ for $n = 0, 1, 2, \ldots$

(i) Show that $I_n = e - nI_{n-1}$ for $n = 1, 2, 3, \ldots$ **2**

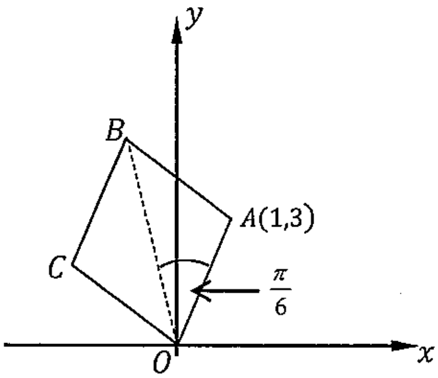
(ii) Hence or otherwise, find the exact value I_3. **2**

Question 12 (15 marks) BEGIN A NEW PAGE **Marks**

(a) (i) Show that $(1-3i)^2 = -8-6i$. **1**

(ii) Hence solve the equation $2z^2 - 8z + (12+3i) = 0$ **2**

(b) OABC is a rhombus in the Argand diagram, where O is the origin and point B is in the second quadrant. The point A is (1,3) and the angle AOB is $\frac{\pi}{6}$.



Copy the diagram onto your writing paper and find the complex numbers represented by the points B and C in the form $a + ib$. **3**

(c) Given $|z| < \frac{1}{2}$, show that $|(1+i)z^3 + iz| < \frac{3}{4}$. **2**

(d) Given that w is a root of $z^3 + iz^2 + ikz + 2i = 0$, where k is real, and $(1-i)w$ is real. Find the possible values of k. **3**

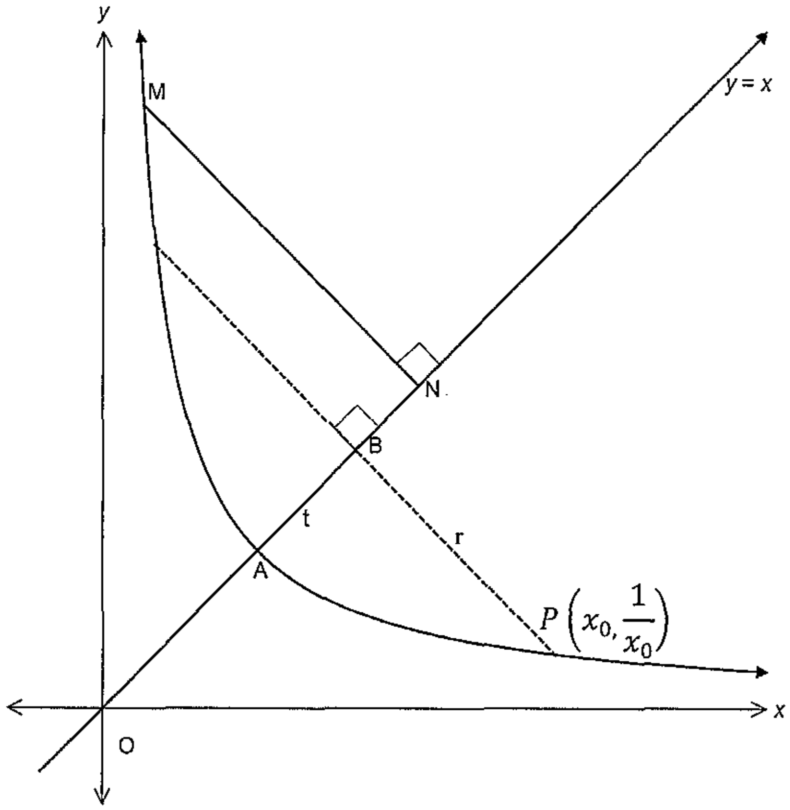
(e) Let $z = cos\theta + isin\theta$.

(i) Show that $z^n + \frac{1}{z^n} = 2\cos n\theta$ and find similar expression for $z^n - \frac{1}{z^n}$. **2**

(ii) Hence prove that $2^5 sin^4\theta cos^2\theta = cos6\theta - 2cos4\theta - cos2\theta + 2$. **2**

Question 13 (15 marks) BEGIN A NEW PAGE **Marks**

(a)



The diagram shows the curve $xy = 1$ for x > 0 where the points A and B lie on the line $y = x$. Given PB and MN are perpendicular to AN, where $AN = h$, let $AB = t$ and $BP = r$.

(i) Show that $2r^2 = \left(x_o - \frac{1}{x_o}\right)^2$. **1**

(ii) Hence show that $OP^2 = 2(1 + r^2)$. **1**

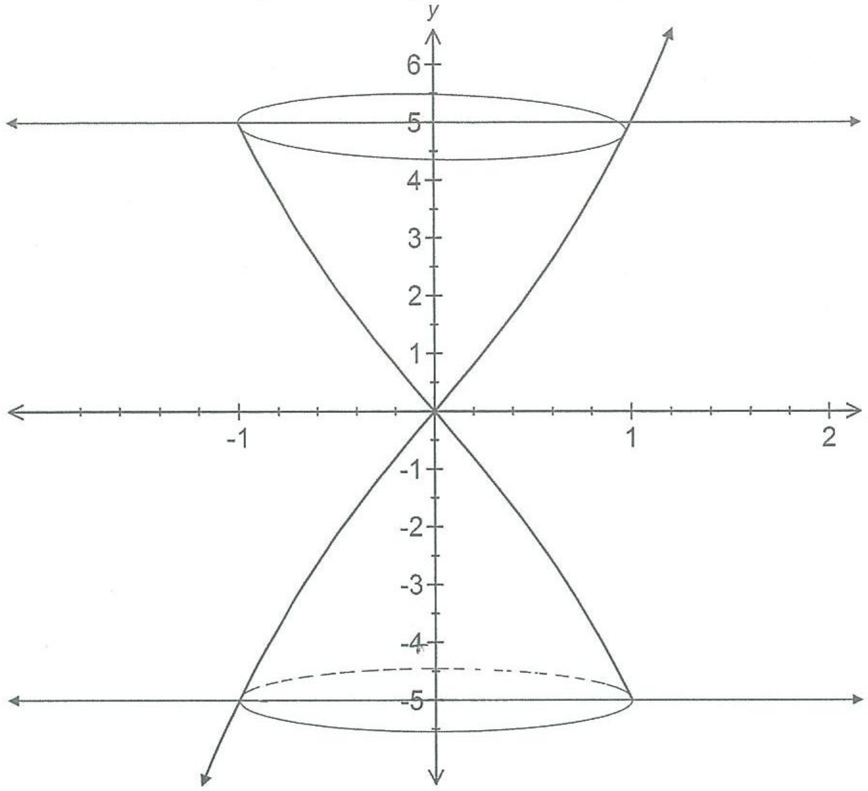
(iii) Show that $r^2 = (t + \sqrt{2})^2 - 2$. **2**

(iv) Hence, by taking slices perpendicular to $y = x$, of thickness δt, show that the volume V of the solid formed by rotating the area enclosed by the curve and the lines $y = x$ and MN, about $y = x$, is given by **4**

$$V = \frac{\pi h^2}{3}\left(h + 3\sqrt{2}\right).$$

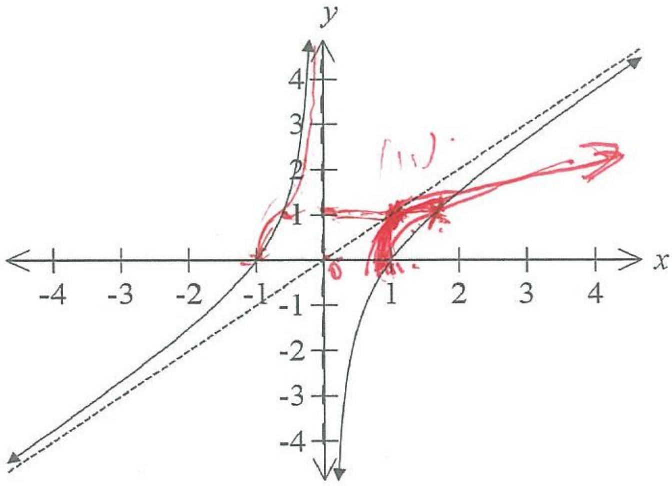
Question 13 continued next page……

(b) A solid is formed by rotating the region bounded by the curve $y = x^3 + 4x$, the y-axis and the ordinates $y = -5$ and $y = 5$ about the y-axis. **5**



By using the method of cylindrical shells, find the volume of this solid.

(c)



Copy the graph of $y = f(x)$ shown above. **2**

On the same number plane draw the graph of $y^2 = f(x)$.

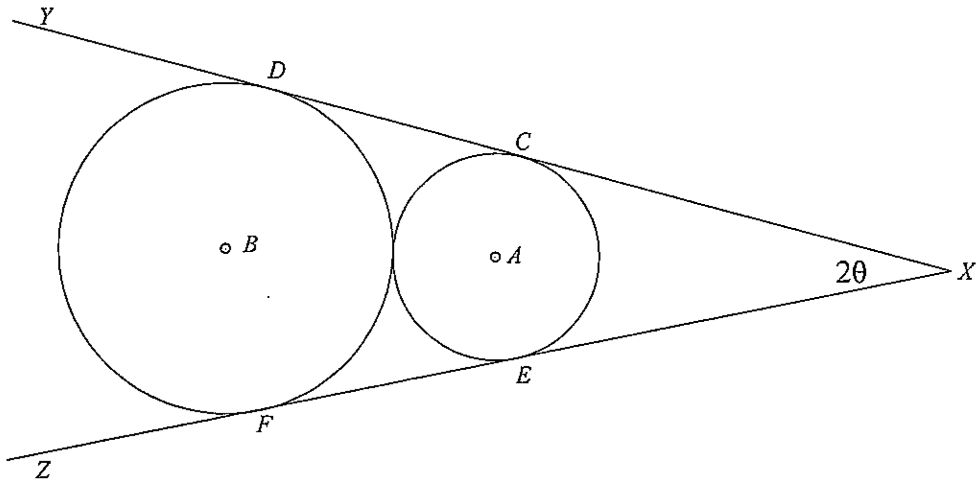
Question 14 (15 marks) BEGIN A NEW PAGE **Marks**

(a) (i) Given x is real and $x > 0$ show that $x + \frac{1}{x} \geq 2$. **1**

(ii) Hence show by Mathematical Induction that **3**

$$x^n + \frac{1}{x^n} \geq x^{n-1} + \frac{1}{x^{n-1}} \quad \text{for all positive integers } n \geq 1.$$

(b) Two circles with centres A and B are in contact with each other.
Two straight lines XY and XZ are tangents to the circle with $\angle YXZ = 2\theta$.



Copy the diagram onto your writing paper and

(i) Prove that line AX bisects $\angle YXZ$ **2**

(ii) Prove that the points X, A and B are collinear. **4**

(c) Given that

$$\tan(\alpha + \beta + \gamma) = \frac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}{1 - \tan\alpha\tan\beta - \tan\beta\tan\gamma - \tan\gamma\tan\alpha},$$ **2**

evaluate $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{13}$.

(d) (i) Show that $\sin x + \sin 3x = 2\sin 2x \cos x$. **1**

(ii) Hence solve $\sin x + \sin 2x + \sin 3x = 0$ for $0 \leq x \leq 2\pi$. **2**

Question 15 (15 marks) BEGIN A NEW PAGE **Marks**

(a) A particle is projected vertically upwards under gravity in a medium where the resistance to motion is proportional to the square of its speed. The speed of projection is equal to the terminal velocity V *m/s*, of the particle when it falls in the same medium.

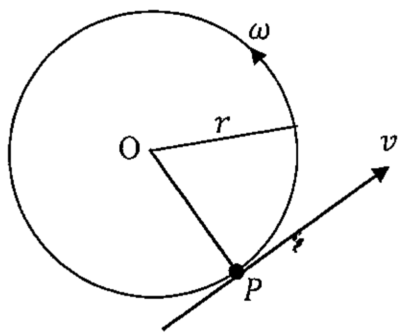
(i) Let x be the height of the particle above its point of projection and v *m/s* is its velocity at time t. Show that $\ddot{x} = -\frac{g}{V^2}(V^2 + v^2)$, where g is the acceleration due to gravity. **2**

(ii) Given H is the maximum height above the point of projection, show that $H = \frac{V^2 \ln 2}{2g}$. **2**

(iii) Given T is the time taken to achieve maximum height, show that for $0 \le t \le T$, $t = \frac{V}{g}\left\{\frac{\pi}{4} - \tan^{-1}\left(\frac{v}{V}\right)\right\}$. **1**

(iv) Given that $\frac{2V^2}{V^2+v^2} = 1 + \sin\left(\frac{2g}{V}t\right)$, show that the time for the particle to reach half its maximum height is $\frac{V}{2g}\sin^{-1}(\sqrt{2} - 1)$ seconds. **1**

(b) The diagram below shows a particle moving in a circular path of radius r with an angular speed of ω.



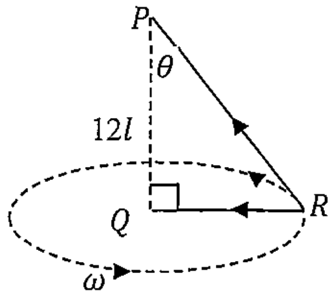
(i) State, without proof, the tangential and normal components of the acceleration a, of the particle. **1**

(ii) What effect, if any, is there on these components if ω is constant? **1**

Question 15 continued next page

Question 15 (Continued) **Marks**

(c) The ends of a piece of string of length $18l$ m are attached to two fixed points P and Q. Q is at a depth $12l$ m below P and $\angle PQR = 90^o$.
A small, smooth ring R of mass m kg is threaded onto the string. The ring revolves with constant speed ω rads/sec in a horizontal circle with Q as centre and the string is taut.



(i) Copy the diagram and indicate on your diagram the tensions in the string at R. Hence show that the tension in the rope is $\frac{13mg}{12}$. **2**

(ii) Find the period of the motion. **2**

(iii) The ring R is now **tied** to the string so that $QR = 5l$, and the ring revolves with constant speed in a horizontal circle with Q as centre, with the string remaining taut. **3**
Let the acceleration along QR be Ω rads/sec.

Copy the diagram again and indicate the active tension(s) for this situation. If the period of motion is $\sqrt{2}$ times the period of the motion when the ring was free to move on the string, show that the ratio of the tensions along PR to QR in the string is $13:4$.

Question 16 (15 marks) BEGIN A NEW PAGE **Marks**

(a) Find the polynomial $P(x)$, given that it is monic, of degree 4, with -1 as a single zero and 3 as a zero of multiplicity 3. **1**

(b) Show that when the polynomial $f(x)$ is divided by $(x-a)(x-b)$, where $a \neq b$, the remainder is $\dfrac{(x-b)f(a)-(x-a)f(b)}{a-b}$ **3**

(c) The roots of $z^3 - 1 = 0$ are 1, Ω and $\overline{\Omega}$, where Ω is one of the complex roots.

(i) Explain why $1 + \Omega + \Omega^2 = 0$ **1**

(ii) Show that $\overline{\Omega} = \Omega^2$ **2**

(d) The tangent to the rectangular hyperbola $xy = 4$ at the point $P\left(2t, \frac{2}{t}\right)$ has equation $x + t^2 y - 4t = 0$.

(i) The tangent at P intersects the x-axis at $Q(4t, 0)$ **1**
Show that the equation of the line through Q, perpendicular to the tangent at P is $t^2 x - y - 4t^3 = 0$

(ii) The line of equation in part (i) cuts the rectangular hyperbola at the points R and S. **2**
Show that the midpoint M of RS is $(2t, -2t^3)$.

(iii) Hence find the equation of the locus of M as P moves on the rectangular hyperbola. State any restrictions. **2**

(e) (i) Show that $\frac{x}{e} > \log_e x$ for $x > e$. **2**

(ii) Hence explain why $e^{\pi} > \pi^{e}$. **1**

END OF PAPER

NAME:________________________________

CTHS AP4 EXTENSION 2 MATHEMATICS 2012

SECTION 1 10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Select the correct response, A, B, C, or D . Fill in the response circle completely.

1	O A	O B	O C	O D
2	O A	O B	O C	O D
3	O A	O B	O C	O D
4	O A	O B	O C	O D
5	O A	O B	O C	O D
6	O A	O B	O C	O D
7	O A	O B	O C	O D
8	O A	O B	O C	O D
9	O A	O B	O C	O D
10	O A	O B	O C	O D

Markers use only

Q1-Q10	/10
Q11	
Q12	
Q13	
Q14	
Q15	
Q16	
TOTAL	/100

Solutions AP4 2012

Section 1

Q1 A. $\left(\text{cis}\frac{\pi}{3}\right)^9$

$= \text{cis}\left(\frac{\pi}{3}\times 9\right)$

$= \text{cis}\frac{\pi}{3}$

$= \cos\pi + i\sin\pi$

$= -1.$

Q2. A $a=5,\ b=4.$

$b^2 = a^2(1-e^2)$

$16 = 25(1-e^2)$

$e^2 = 1-\frac{16}{25}$

$= \frac{3}{5}$

Q3 D

Q4 B. $x^2-y^2=4$

$\frac{x^2}{4}-\frac{y^2}{4}=1$

$a=2,\ b=2$

$x=2\sec\theta,\ y=\tan\theta$

$-\pi<\theta\le\pi,\ \theta\ne\pm\frac{\pi}{2}$

Q5 D

Q6. C $(\sqrt{x})^3+2(\sqrt{x})-1=0$

$x\sqrt{x}+2\sqrt{x}=1$

$[\sqrt{x}(x+2)]^2=1$

$x(x^2+4x+4)=1$

$x^3+4x^2+4x-1=0$

Q7 B $x=\theta-\sin\theta$

$\frac{dx}{d\theta}=1-\cos\theta$

$y=1-\cos\theta$

$\frac{dy}{d\theta}=\sin\theta$

$\frac{dy}{dx}=\frac{\sin\theta}{1-\cos\theta}$

$=\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$

$=\cot\frac{\theta}{2}$

Q8 B $x^2+2x+2=(x+1)^2+1$

Let $x+1=u$

$dx=du$

$\int\frac{1}{x^2+2x+2}\,dx=\int\frac{1}{(x+1)^2+1}\,dx$

$=\int\frac{du}{u^2+1}$

$=\tan^{-1}u+C$

$=\tan^{-1}(x+1)+C$

Q9. A $\int 3\sqrt{x}\ln x\,dx$

$=3\int\ln x\frac{d}{dx}\left(\frac{2x}{3}\right)^{\frac{3}{2}}dx$

$=3\left[\frac{2x^{3/2}}{3}\ln x-\frac{2}{3}\int x^{3/2}\frac{1}{x}\,dx\right]$

$=2x^{3/2}\ln x-2\times\frac{2x^{3/2}}{3}+C$

$=2x\sqrt{x}\left(\ln x-\frac{2}{3}\right)+C$

Q10. c

$$\frac{(x-4)^2 x(5-x)}{x-4} \geqslant -3(x-4)^2$$

$x(x-4)(5-x) \geqslant -3(x-4)^2$

$x(x-4)(5-x)+3(x-4)^2 \geqslant 0$

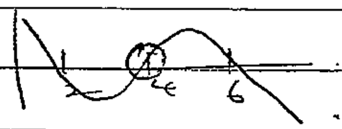
$(x-4)[x(5-x)+3(x-4)] \geqslant 0$

$(x-4)(5x-x^2+3x-12) \geqslant 0$

$(x-4)(-x^2+8x-12) \geqslant 0$

$(4-x)(x^2-8x+12) \geqslant 0$

$(4-x)(x-6)(x-2) \geqslant 0$



$4 < x \leq 6$ or $x \leq 2$.

Q11

(a) $\int \frac{e^{2x}}{e^x+1}\,dx$

$= \int \frac{e^x(e^x+1)-e^x}{e^x+1}\,dx$

$= \int \left(e^x - \frac{e^x}{e^x+1}\right)dx$

$= e^x - \ln(e^x+1) + C$

(b). When $x = \pi - u$

(i) $dx = -du$

$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$

$= \int_{\pi}^{0} \frac{(\pi-u)\sin(\pi-u)}{1+\cos^2(\pi-u)} - du$

$= \int_0^{\pi} \frac{(\pi-u)\sin u}{1+(-\cos u)^2}\,du$

$= \int_0^{\pi} \frac{(\pi-u)\sin u}{1+\cos^2 u}\,du$

$= \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}\,dx$

(ii) $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$

$= \frac{1}{2}\left[\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx + \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}\,dx\right]$

$= \frac{1}{2}\left[\int_0^{\pi} \frac{\pi\sin x}{1+\cos^2 x}\,dx\right]$

$= \frac{1}{2}\left[-\pi\tan^{-1}(\cos x)\right]_0^{\pi}$

$= \frac{1}{2}\left[-\frac{\pi}{4} - \frac{\pi}{4}\right] = \frac{1}{2} \times \frac{\pi^2}{2}$

$= \frac{\pi^2}{4}$

(c) $\frac{8-2x}{(1+x)(4+x^2)} = \frac{a}{1+x} + \frac{bx+c}{4+x^2}$

(i) $8-2x = a(4+x^2) + (bx+c)(1+x)$

Let $x = -1$, $10 = 5a$

$\therefore a = 2$

Let $x = 0$, $8 = 2(4+0) + c(1+0)$

$\therefore c = 0$

Equating coefficients of x^2

$0 = 2+b$

$\therefore b = -2$

$\therefore a = 2,\ b = -2,\ c = 0.$

(ii) $\int_0^4 \left[\frac{2}{1+x} + \frac{-2x}{4+x^2}\right]dx$

$= \left[2\ln(1+x) - \ln(4+x^2)\right]_0^4$

$= 2\ln 5 - \ln 20 - (2\ln 1 - \ln 4)$

$= 2(\ln 5 - \ln 1) - (\ln 20 - \ln 4)$

$= \ln 5$

d) (i) $I_n = \int_1^e (\ln t)^n dt$

$= \left[t(\ln t)^n\right]_1^e - \int_1^e t\, n(\ln t)^{n-1} \frac{1}{t} dt$

$= e(\ln e)^n - 1(\ln 1)^n - n\int_1^e (\ln t)^{n-1} dt$

$= e - 0 - n I_{n-1}$

$= e - n I_{n-1}$

$n = 1, 2, 3, \ldots$

(ii) $I_0 = \int_0^e (\ln t)^0 dt$

$= [t]_0^1 = e - 1$

$I_1 = e - I_0$

$= e - (e-1)$

$= 1$

$I_2 = e - 2I_1$

$= e - 2$

$I_3 = e - 3I_2$

$= e - 3(e-2)$

$= -2e + 6$

Q12.

(a) (i) $(1-3i)^2 = 1 - 6i + 9i^2$

$= 1 - 6i - 9$

$= -8 - 6i$

(ii) $z = \dfrac{8 \pm \sqrt{64 - 4\times 2(12+3i)}}{4}$

$= \dfrac{8 \pm \sqrt{64 - 96 - 24i}}{4}$

$= \dfrac{8 \pm \sqrt{-32 - 24i}}{4}$

$= \dfrac{8 \pm 2\sqrt{-8-6i}}{4}$

$= \dfrac{4 \pm (1-3i)}{2}$

b) $C = \left(\text{cis}\frac{\pi}{3}\right)(1+3i)$

$= \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)(1+3i)$

$= \frac{1}{2} + \frac{3}{2}i + \frac{\sqrt{3}}{2}i + \frac{3\sqrt{3}}{2}i^2$

$= \frac{1-3\sqrt{3}}{2} + i\left(\frac{3+\sqrt{3}}{2}\right)$

$\vec{OB} = \vec{OA} + \vec{OB}$

$= (1+3i) + \left(\frac{1-3\sqrt{3}}{2}\right) + i\left(\frac{3+\sqrt{3}}{2}\right)$

$= \frac{3-3\sqrt{3}}{2} + \left(\frac{3+3+\sqrt{3}}{2}\right)i$

$= \frac{3-3\sqrt{3}}{2} + \frac{9+\sqrt{3}}{2}i$

(c) By Δ inequality property

$|(1+i)z^3+iz| \le |(1+i)z^3|+|iz|$

$LHS = |1+i||z|^3+|i||z|$

$< \sqrt{2}\left(\frac{1}{2}\right)^3+\frac{1}{2}$ since $|z|<\frac{1}{2}$

$< \frac{\sqrt{2}}{8}+\frac{1}{2}$

$< \frac{2}{8}+\frac{4}{8}$

$< \frac{3}{4}$

$\therefore |(1+i)z^3+iz| < \frac{3}{4}$

d) Let $w = x+iy$

$(1-i)w = (1-i)(x+iy)$

$= x+y+i(y-x)$

is real when $y=x$.

$\therefore w = (1+i)x$ *

$w^2 = 2ix^2$

$w^3 = (2i-2)x^3$

$(2i-2)x^3-2x^2+ik(1+i)x+2i=0$

Since k and x are real

Equating real & imaginary parts

$2x^3+2x^2+k=0$

$2x^3+kx+2=0$

$\therefore 2x^2 = 2$

$x = \pm 1$

when $x=1$, $k=-4$

$x=-1$, $k=0$

* $(1-i)w = 2x$

$\cancel{w} = \frac{2x}{1-i} \times \frac{1+i}{1+i}$

$= (1+i)x$.

(e)

(i) $z = \cos\theta + i\sin\theta$

$\frac{1}{z} = \cos\theta - i\sin\theta$

$z^n+\frac{1}{z^n} = \cos n\theta + \cancel{i\sin n\theta}$

$+ \cos n\theta - \cancel{i\sin n\theta}$

$= 2\cos n\theta$

~~ii) $\sin^4\theta\cos^2\theta = \sin^4\theta(1-\sin^2\theta)$~~

~~$= \sin^4\theta - \sin^6\theta$~~

$=$

$z^n - \frac{1}{z^n} = 2i\sin n\theta$

ii) $\sin^4\theta\cos^2\theta$

$= \sin^4\theta(1-\sin^2\theta)$

$= \sin^4\theta - \sin^6\theta$

$= \frac{1}{2^4}\left(z-\frac{1}{z}\right)^4 - \frac{1}{2^6}\left(z-\frac{1}{z}\right)^6$

$= \frac{1}{2^4}\left[z^4-4z^2+6-\frac{4}{z^2}+\frac{1}{z^4}\right]$

$+\frac{1}{2^6}\left[z^6-6z^4+15z^2-20+\frac{15}{z^2} - \frac{6}{z^4}-\frac{1}{z^6}\right]$

$= \frac{1}{2^4}\left[\left(z^4+\frac{1}{z^4}\right)-4\left(z^2+\frac{1}{z^2}\right)+6\right]$

$+\frac{1}{2^6}\left[\left(z^6+\frac{1}{z^6}\right)-6\left(z^4+\frac{1}{z^4}\right)+15\left(z^2+\ldots\right) - 20\right]$

$= \frac{1}{2^4}(\cos 4\theta - 4\cos 2\theta + 3)$

~~$+\frac{1}{2^5}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\ldots - 10)$~~

$= \frac{1}{2^5}(\cos 6\theta - 2\cos 4\theta - \cos 2\theta - \ldots$

$2^5\sin^4\theta\cos^2\theta$

$= \cos 6\theta - 2\cos 4\theta - \cos 2\theta - 2$

5

Q13 (a)

(i) Eqn of AN is $x - y = 0$

$PB = \dfrac{|x_0 - \frac{1}{x_0}|}{\sqrt{1+1}}$

$r^2 = \dfrac{\left(x_0 - \frac{1}{x_0}\right)^2}{2}$

$2r^2 = \left(x_0 - \frac{1}{x_0}\right)^2$

(ii) $OP^2 = x_0^2 + \dfrac{1}{x_0^2}$

$= \left(x_0 - \dfrac{1}{x_0}\right)^2 + 2$

$= 2r^2 + 2$

$= 2(r^2+1)$

(iii) ΔOPB is right angled at B

$OP^2 = r^2 + (OA + AB)^2$

$OA = \sqrt{2}$ since $A(1,1)$

$AB = t$.

$\therefore OP^2 = r^2 + (t+\sqrt{2})^2$

$2(r^2+1) = r^2 + (t+\sqrt{2})^2$

$r^2 = (t+\sqrt{2})^2 - 2$

(iv) radius $= r$, distance t from A

thickness of disc $= \delta t$

$\delta V = \pi r^2 \delta t$

$= \pi\left[(t+\sqrt{2})^2 - 2\right]\delta t$

$V = \lim_{\delta t \to 0} \sum_0^h \pi\left[(t-\sqrt{2})^2 - 2\right]\delta t$

$= \pi \int_0^h \left[(t+\sqrt{2})^2 - 2\right] dt$

$= \pi\left[\frac{1}{3}(t+\sqrt{2})^3 - 2t\right]_0^h$

$= \pi\left[\frac{1}{3}(h+\sqrt{2})^3 - 2h - \frac{1}{3}(\sqrt{2})^3\right]$

$= \frac{\pi}{3}\left[h^3 + 3h^2\sqrt{2} + \cancel{3h(\sqrt{2})^2} + \cancel{(\sqrt{2})^3} - \cancel{6h} - \cancel{2\sqrt{2}}\right]$

$= \frac{\pi}{3}\left[h^3 + 3\sqrt{2}h^2\right]$

$= \frac{\pi}{3}h^2(h + 3\sqrt{2})$

b) Inner radius x, outer radius $x+\delta x$

height $5-y$

Volume of shell

$\delta V = \pi\left[(x+\delta x)^2 - x^2\right](5-y)$

$= \pi\left[2x\,\delta x + \delta x^2\right](5-y)$

$= \pi(2x+\delta x)(5-x^3-4x)\delta x$

$= 2 \times \lim_{\delta x \to 0} \sum_0^1 \pi(2x+\delta x)(5-x^3-4x)\delta x$

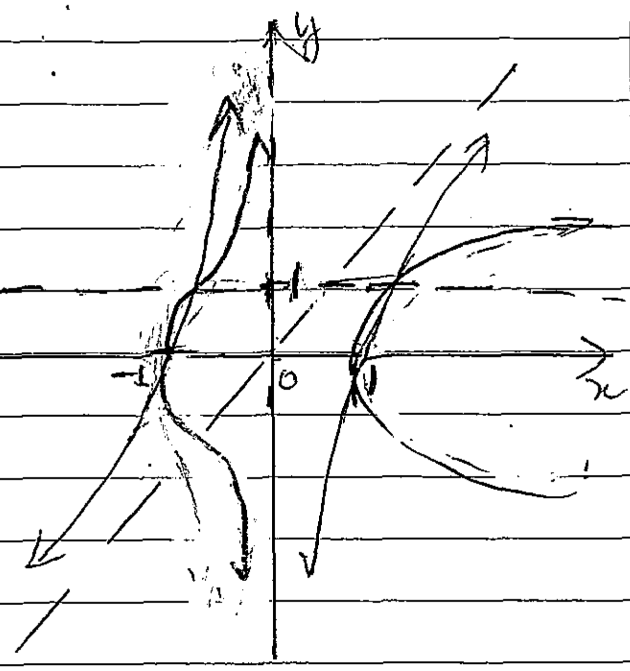
$= 2\pi \int_0^1 2x(5-x^3-4x)\,dx$

$= 4\pi \int_0^1 (5x - x^4 - 4x^2)\,dx$

$= 4\pi\left[\frac{5x^2}{2} - \frac{x^5}{5} - \frac{4x^3}{3}\right]_0^1$

$= \frac{58\pi}{15}$

c)



14)

a) (i) $\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 \geq 0$

But $x+\frac{1}{x} = \left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 + 2$

for $x>0$, $x+\frac{1}{x} \geq 2$ (1).

(ii) When $n=1$,

$x^1+\frac{1}{x^1} \geq x^0+\frac{1}{x^0}$ by (i).

≥ 2

$\therefore$ Statement true for $n=1$

Assume statement is true for $n=k$

$x^k+\frac{1}{x^k} \geq x^{k-1}+\frac{1}{x^{k-1}}$

$\left(x^k+\frac{1}{x^k}\right)-\left(x^{k-1}+\frac{1}{x^{k-1}}\right) \geq 0$

RTP statement is true when $n=k+1$.

~~Prove that~~

$x^{k+1}+\frac{1}{x^{k+1}} \geq x^k+\frac{1}{x^k}$

$LHS = \left(x+\frac{1}{x}\right)\left(x^k+\frac{1}{x^k}\right)-\left(x^{k-1}+\frac{1}{x^{k-1}}\right)$

$\geq 2\left(x^k+\frac{1}{x^k}\right)-\left(x^{k-1}+\frac{1}{x^{k-1}}\right)$ by (i)

$= \left(x^k+\frac{1}{x^k}\right)+\left[\left(x^k+\frac{1}{x^k}\right)-\left(x^{k-1}+\frac{1}{x^{k-1}}\right)\right]$

$\geq x^k+\frac{1}{x^k}$

Since $\left[\left(x^k+\frac{1}{x^k}\right)-\left(x^{k-1}+\frac{1}{x^{k-1}}\right)\right] \geq$ by assumption.

$\therefore$ Statement is true by MI

9

14(b)

(i) In $\triangle AXC$ and $\triangle AXE$

$AC = CE$ (Radii of circle)

$XC = XE$ (Tangents to circle from external point are equal)

XA is common

$\therefore \triangle AXC \equiv \triangle AXE$ (SSS)

$\angle CXA = \angle AXE$ (corresponding $\angle$'s in congruent $\triangle$'s)

AX bisects $\angle CXE$

$\therefore AX$ bisects $\angle YXZ$ (same angle as $\angle CXE$)

(ii) $\angle XCA = \angle ACD = \angle BDC = 90°$

($\angle$ between tangent and radius at point of contact $= 90°$)

$\angle CXA = \frac{1}{2} \times \angle CXE$ (by (i))

Let $\angle CXA = \theta$

$\angle CXA + \angle XCA + \angle XAC = 180°$

($\angle$ sum of $\triangle$)

$\theta + 90 + \angle XAC = 180°$

$\angle XAC = 90° - \theta$

In $\triangle BXD$ and $\triangle BXF$

$DB = BF$ (radii of circle)

$DX = FX$ (Tangents from external point are equal)

BX is common

$\therefore \triangle BXD \equiv \triangle BXF$ (SSS)

$\angle DXB = \frac{1}{2} \angle DXF$ (by (i))

$\angle DXB + \angle XDB + \angle XBD = 180°$

($\angle$ sum of $\triangle$)

$\theta + 90 + \angle XBD = 90 - \theta$

In quadrilateral $CDAB$

$\angle DCA + \angle CAB + \angle ABD + \angle CDB = 360°$

($\angle$ sum of quadrilateral)

$90 + \angle CAB + (90 - \theta) + 90 = 360$

$\angle CAB = 90 + \theta$

$\angle XAC + \angle CAB = (90 - \theta) + (90 +$

$= 180°$

$\therefore X, A, B$ are collinear

(c) Let $\alpha = \tan^{-1}\frac{1}{2}$

$\therefore \tan\alpha = \frac{1}{2}$

$\beta = \tan^{-1}\frac{1}{4}$

$\tan\beta = \frac{1}{4}$

$\gamma = \tan^{-1}\frac{1}{13}$

$\tan\gamma = \frac{1}{13}$

$\tan(\alpha+\beta+\gamma) = 1$

$= \dfrac{\frac{1}{2} + \frac{1}{4} + \frac{1}{13} - \frac{1}{2} \times \frac{1}{4} \times \frac{1}{13}}{1 - \frac{1}{2} \times \frac{1}{4} - \frac{1}{4} \times \frac{1}{13} - \frac{1}{13} \times \frac{1}{2}}$

$= 1$

$\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{13}$

$= \tan^{-1} 1$

$= \frac{\pi}{4}.$

d)

(i) $\sin x + \sin 3x$

$= \sin(2x-x) + \sin(2x+x)$

$= \sin 2x \cos x - \cos 2x \sin x$
$\quad + \sin 2x \cos x + \cos 2x \sin x$

$= 2\sin 2x \cos x$

(ii) $\sin x + \sin 2x + \sin 3x = 0$

$2\sin 2x \cos x + \sin 2x = 0$

$\sin 2x(2\cos x + 1) = 0$

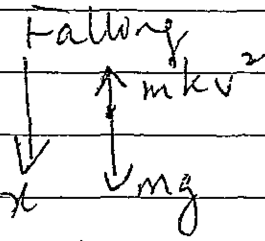
$\sin 2x = 0, \quad \cos x = -\frac{1}{2}$

$2x = 0, \pi, 2\pi, 3\pi, 4\pi$

$x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \frac{2\pi}{3}, \frac{4\pi}{3}$

15

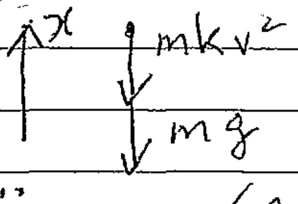
a) (i) Falling



$m\ddot{x} = m(g - kv^2)$

$\ddot{x} \to 0, \; V^2 = \frac{g}{k}$

$k = \frac{g}{V^2}$



$m\ddot{x} = -m(g + kv^2)$

$\ddot{x} = -(g + \frac{g}{V^2}v^2)$

$\ddot{x} = -\frac{g}{V^2}(V^2 + v^2)$

(ii) $\frac{1}{2}\frac{d(v^2)}{dx} = -\frac{g}{V^2}(V^2 + v^2)$

$-\frac{2g}{V^2}\frac{dx}{d(v^2)} = \frac{1}{V^2 + (v^2)}$

$\int -\frac{2g}{V^2}\,dx = \int \frac{1}{V^2 + (v^2)}\,d(v^2)$

$-\frac{2g}{V^2}x = \ln(V^2 + v^2) + c$

when $t = 0, x = 0, v = V$

$0 = \ln(V^2 + v^2) + c$

$c = -\ln 2V^2$

$-\frac{2g}{V^2}x = \ln(V^2 + v^2) - \ln 2V^2$

$-\frac{2g}{V^2}x = \ln\left(\frac{V^2 + v^2}{2V^2}\right)$

$x = -\frac{V^2}{2g}\ln\left(\frac{V^2 + v^2}{2V^2}\right)$

(9)

at greatest height, $x=H$, $v=0$

$$\therefore H = -\frac{V^2}{2g}\left(\ln\frac{1}{2}\right)$$

$$= \frac{V^2\ln 2}{2g}$$

(iii) $t \le T$, particle is moving upward.

$$\frac{dv}{dt} = -\frac{g}{V^2}(V^2+v^2)$$

$$-\frac{g}{V^2}\frac{dt}{dv} = \frac{1}{V^2+v^2}$$

$$\int -\frac{g}{V^2}dt = \int \frac{1}{V^2+v^2}dv$$

$$-\frac{g}{V^2}t = \frac{1}{V}\tan^{-1}\left(\frac{v}{V}\right)+c$$

when $t=0$, $v=V$

$$0 = \tan^{-1}1 + C$$

$$\therefore C = -\frac{\pi}{4}$$

$$t = \frac{V}{g}\left\{\frac{\pi}{4} - \tan^{-1}\left(\frac{v}{V}\right)\right\}$$

for $t \le T$

(iv) $\ln\left(\frac{2V^2}{V^2+v^2}\right) = \frac{2g}{V^2}x$

$$x = \frac{1}{2}H = \frac{1}{2}\times\frac{V^2\ln 2}{2g}$$

$$= \frac{V^2\ln 2}{4g}$$

$$\ln\left(\frac{2V^2}{V^2+v^2}\right) = \frac{\cancel{2g}}{\cancel{V^2}}\times\frac{\cancel{V^2}\ln 2}{\cancel{4g}_{2}}$$

$$= \frac{1}{2}\ln 2$$

$$= \ln\sqrt{2}$$

$$\therefore \frac{2V^2}{V^2+v^2} = \sqrt{2}$$

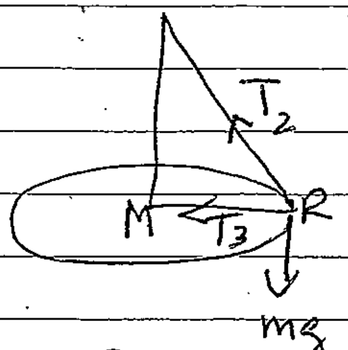
$$\therefore 1+\sin\left(\frac{2g}{V}t\right) = \sqrt{2}$$

$$\sin\left(\frac{2g}{V}t\right) = \sqrt{2}-1$$

$$\frac{2g}{V}t = \sin^{-1}(\sqrt{2}-1)$$

$$t = \frac{V}{2g}\sin^{-1}(\sqrt{2}-1)$$

iii) vertically ↓



$m \times 0 = mg - T_2 \cos\theta$

$T_2 = \cos\theta \, mg \quad (3)$

towards M,

$m \times 5l \, \Omega^2 = T_3 + T_2 \sin\theta \quad (4)$

$\cos\theta = \frac{12}{13}, \ \sin\theta = \frac{5}{13}.$

From(3) $T_2 = \frac{mg}{\cos\theta} = \frac{13mg}{12}$

Period $\sqrt{2}$ time T in (ii)

Period $= \sqrt{2} \times \frac{2\pi}{\omega}$

$\therefore \frac{2\pi}{\Omega} = \sqrt{2} \times \frac{2\pi}{\omega}$

$\Omega = \frac{\omega}{\sqrt{2}}$

$= \frac{1}{\sqrt{2}}\sqrt{\frac{3g}{10l}}$

From (4)

$m \times \cancel{5} l \times \frac{1}{2} \cdot \frac{3g}{\cancel{10}_2 l} = T_3 + \frac{\cancel{13}mg}{12} \times \frac{5}{\cancel{13}}$

$T_3 = \frac{3}{4}mg - \frac{5}{12}mg$

$= \frac{mg}{3}$

$\therefore T_2 : T_3 = \frac{13mg}{12} : \frac{mg}{3}$

$= \frac{13}{12} : \frac{1}{3}$

$= 13 : 4$

Q16.

(a) $P(x) = (x+1)(x-3)^3$

$= (x+1)(x^3 - 9x^2 + 27x - 27)$

$= x^4 - 8x^3 + 18x^2 - 27$

(b) $f(x) = T(x)(x-a)(x-b) + cx + d$

$f(a) = ac + d \quad -(1)$

$f(b) = bc + d \quad -(2)$

$(1)-(2) \ f(a) - f(b)$

$= c(a-b)$

$\therefore c = \frac{f(a) - f(b)}{a-b}.$

Sub into (1).

$f(a) = a\left[\frac{f(a) - f(b)}{a-b}\right] + d.$

$d = f(a) - a\left[\frac{f(a) - f(b)}{a-b}\right]$

$= \frac{\cancel{af(a)} - bf(a) - \cancel{af(a)} + af(b)}{a-b}$

$= \frac{af(b) - bf(a)}{a-b}.$

Sub into $cx + d$

Remainder $=$

$\frac{f(a) - f(b)}{a-b}x + \frac{af(b) - bf(a)}{a-b}.$

$= \frac{xf(a) - xf(b) + af(b) - bf(a)}{a-b}.$

$= \frac{(x-b)f(a) - (x-a)f(b)}{a-b}.$

15b)

i) tangential acceleration

$$\frac{dv}{dt} = \frac{d}{dt}(r\omega) = r\frac{d\omega}{dt}$$

normal acceleration along radius towards O $= r\omega^2$

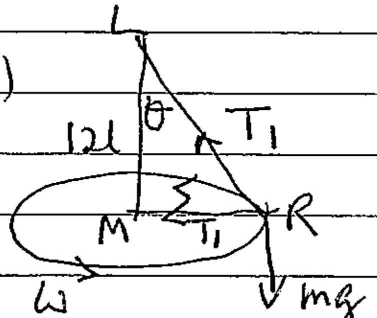
ii) ω is constant

$$\frac{d\omega}{dt} = 0$$

$\therefore$ tangential acceleration $= 0$

normal acceleration $= r\omega^2$

(c)(i)



Vertical forces at R downwards

$$m\ddot{x} = mg - T_1\cos\theta$$

$$\ddot{x} = 0$$

$$\therefore T_1\cos\theta = mg \quad (1)$$

towards M, $MR = 5l$

$LR = 13l$

$$m \cdot 5l\omega^2 = T_1 + T_1\sin\theta \quad -(2)$$

$$\cos\theta = \frac{12l}{13l} = \frac{12}{13}$$

$$\sin\theta = \frac{5l}{13l} = \frac{5}{13}$$

From (1) $T_1 = \frac{mg}{\cos\theta} = \frac{13mg}{12}$

ii) From (2)

$$m \times 5l\omega^2 = \frac{13mg}{12} + \frac{13mg}{12} \times \frac{5}{13}$$

$$= \frac{18mg}{12}$$

$$\therefore \omega^2 = \frac{18mg}{12m \times 5l}$$

$$= \frac{3g}{10l}$$

$$\omega = \sqrt{\frac{3g}{10l}}$$

Period $T = \frac{2\pi}{\omega}$

$$= 2\pi\sqrt{\frac{10l}{3g}}$$

c) $z^3 - 1 = 0$

(i) Ω is a root

$\therefore \Omega^3 - 1 = 0$

$(\Omega - 1)(\Omega^2 + \Omega + 1) = 0$

$\therefore \Omega^2 + \Omega + 1 = 0$

(ii) $(z - 1)(z^2 + z + 1) = 0$

for $z^2 + z + 1 = 0$

$$z = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$\Omega = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$$

$$\bar{\Omega} = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$$

$$\bar{\Omega}^2 = \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)^2$$

$$= \frac{1}{4} - \frac{3}{4} - \frac{\sqrt{3}}{2} i$$

$$= -\frac{1}{2} - \frac{\sqrt{3}}{2} i$$

$$= \bar{\Omega}$$

(d)

(i) grad of tangent $= -\frac{1}{t^2}$

$\perp$ grad $= t^2$

Equation through $Q(4t, 0)$

$y - 0 = t^2(x - 4t)$

$y = t^2x - 4t^3$

$t^2x - y - 4t^3 = 0$.

(ii) $xy = 4$ (1)

$t^2x - y - 4t^3 = 0$ (2)

$x \times (2)$

$t^2x^2 - xy - 4xt^3 = 0$. (3)

Sub (1) into (3).

$t^2x^2 - 4t^3x - 4 = 0$.

Sum of roots

$x_1 + x_2 = -\frac{-4t^3}{t^2}$

$= 4t$

midpoint of RS

$x = \frac{x_1 + x_2}{2} = \frac{4t}{2}$

$= 2t$

Sub $x = 2t$ in eqn in part(ii)

$t^2 \times 2t - y - 4t^3 = 0$

$y = -2t^3$

$M(2t, -2t^3)$

Alternate method solve by substitution of simultaneous equations.

(13)

16d)

(iii) $x = 2t$ (1)

$y = -2t^3$ (2).

Sub $t = \frac{x}{2}$ from (1) into (2)

$y = -2\left(\frac{x}{2}\right)^3$

$y = -\frac{x^3}{4}$

$4y + x^3 = 0$

$x^3 + 4y = 0, \ t \neq 0$

16(e)

(i) Let $f(x) = \frac{x}{e} - \log_e x$

$f'(x) = \frac{1}{e} - \frac{1}{x} > 0, \; x > e$

$f'(e) = 0$ and $f(x)$ is increasing function $x > e$

$\therefore f(x) > 0$ for $x > e$

$\therefore \frac{x}{e} > \log_e x, \; x > e.$

(ii) $\pi > e$

$\frac{\pi}{e} > \log_e \pi$ by (i)

$\pi > e \log_e \pi$

$= \log_e \pi^e$

$\therefore e^{\pi} > \pi^e.$