Name:	______________
Class:	12MTZ1
Teacher:	MR WOO

CHERRYBROOK TECHNOLOGY HIGH SCHOOL

2014 AP4

YEAR 12 TRIAL HSC EXAMINATION

MATHEMATICS EXTENSION 2

Time allowed - 3 HOURS
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES:

- Attempt all questions.
- Multiple choice questions are to be answered on the multiple choice sheet provided at the back of the paper.
- Each question in Section 2 is to be commenced in a new booklet clearly marked Question 11, Question 12, etc on the top of the page and must show your name and class.
- All necessary working should be shown in every question in Section 2. Full marks may not be awarded for careless or badly arranged work.
- Board of Studies approved calculators may be used. Standard Integral Tables are provided.
- Write your name and class in the space provided at the top of this question paper.

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10

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1 Consider the hyperbola with the equation $\frac{x^2}{144}-\frac{y^2}{25}=1$.

What are the equations of the directrices?

(A) $y=\pm\frac{25}{13}$ (B) $y=\pm\frac{144}{13}$

(C) $x=\pm\frac{25}{13}$ (D) $x=\pm\frac{144}{13}$

2 The points $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ lie on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the chord PQ subtends a right angle at $(0,0)$. Which of the following is the correct expression?

(A) $\tan\theta\tan\phi=-\frac{b^2}{a^2}$ (B) $\tan\theta\tan\phi=-\frac{a^2}{b^2}$

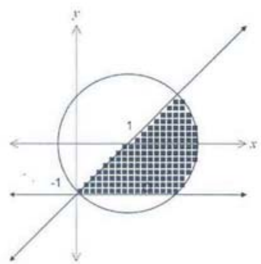
(C) $\tan\theta\tan\phi=\frac{b^2}{a^2}$ (D) $\tan\theta\tan\phi=\frac{a^2}{b^2}$

3 What is $-\sqrt{3}+i$ expressed in modulus-argument form?

(A) $\sqrt{2}(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})$ (B) $2(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})$

(C) $\sqrt{2}(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6})$ (D) $2(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6})$

4 Consider the Argand diagram below.



Which inequality could define the shaded area?

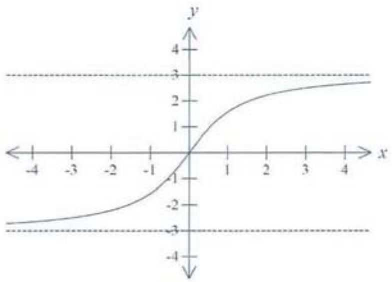
(A) $|z-1|\le\sqrt{2}$ and $0\le\arg(z-i)\le\frac{\pi}{4}$

(B) $|z-1|\le\sqrt{2}$ and $0\le\arg(z+i)\le\frac{\pi}{4}$

(C) $|z-1|\le 1$ and $0\le\arg(z-i)\le\frac{\pi}{4}$

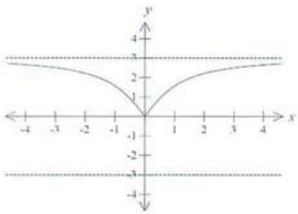
(D) $|z-1|\le 1$ and $0\le\arg(z+i)\le\frac{\pi}{4}$

5 The diagram shows the graph of the function $y = f(x)$.

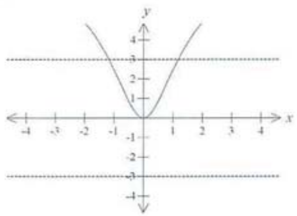


Which of the following is the graph of $y = \sqrt{f(x)}$?

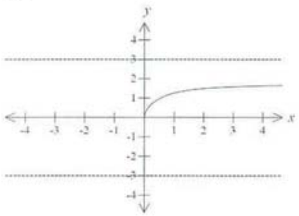
(A)



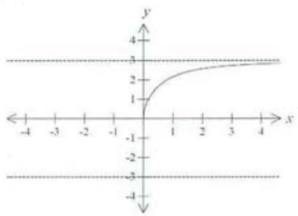
(B)



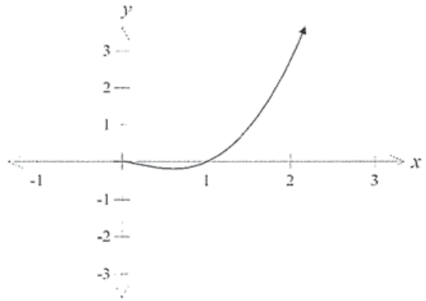
(C)



(D)

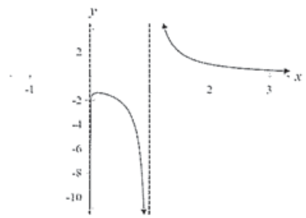


6 The diagram shows the graph of the function $y = f(x)$.

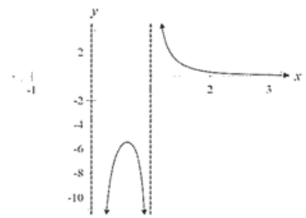


Which of the following is the graph of $y = \dfrac{1}{f(x)}$?

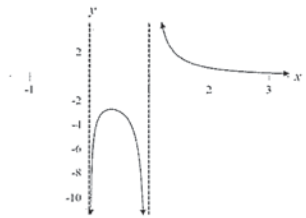
(A)



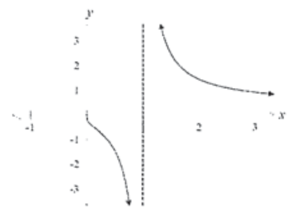
(B)



(C)



(D)



7 Which of the following is an expression for $\int \frac{1}{\sqrt{7-6x-x^2}}dx$?

(A) $\sin^{-1}\left(\frac{x-3}{2}\right)+c$ (B) $\sin^{-1}\left(\frac{x+3}{2}\right)+c$

(C) $\sin^{-1}\left(\frac{x-3}{4}\right)+c$ (D) $\sin^{-1}\left(\frac{x+3}{4}\right)+c$

8 Which of the following is an expression for $\int \frac{1}{\sqrt{x^2-6x+10}}dx$?

(A) $\ln\left(x-3-\sqrt{x^2-6x+10}\right)+c$

(B) $\ln\left(x+3-\sqrt{x^2-6x+10}\right)+c$

(C) $\ln\left(x-3+\sqrt{x^2-6x+10}\right)+c$

(D) $\ln\left(x+3+\sqrt{x^2-6x+10}\right)+c$

9 The equation $4x^3-27x+k=0$ has a double root.
What are the possible values of k?

(A) ± 4

(B) ± 9

(C) ± 27

(D) $\pm\frac{81}{2}$

10 Given that $(x-1)p(x) = 16x^5 - 20x^3 + 5x - 1$, then if $p(x) = (4x^2 + ax - 1)^2$, the value of a is:

(A) 1

(B) 2

(C) $\frac{1}{2}$

(D) 0

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section

Answer each question in a new answer booklet.

All necessary working should be shown in every question.

Question 11 (15 marks) – Use a separate booklet **Marks**

(a) Evaluate $\int_0^{\ln 3} \frac{e^x}{1+e^x} dx$. **2**

(b) Use the completion of squares method to find $\int \frac{-2}{\sqrt{3+2x-x^2}} dx$. **2**

(c) Consider the function $f(x) = \frac{x^2-11}{(3x-1)(x+2)^2}$.

(i) Find real numbers *a*, *b* and *c* such that $f(x) \equiv \frac{a}{3x-1} + \frac{b}{x+2} + \frac{c}{(x+2)^2}$. **3**

(ii) Hence, or otherwise find $\int f(x)\,dx$. **2**

(d) A point $P(a \sec\theta, b \tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with foci $S(ae, 0)$ and $S'(-ae, 0)$.

(i) Show that $PS = a(e\sec\theta - 1)$ and $PS' = a(e\sec\theta + 1)$. **3**

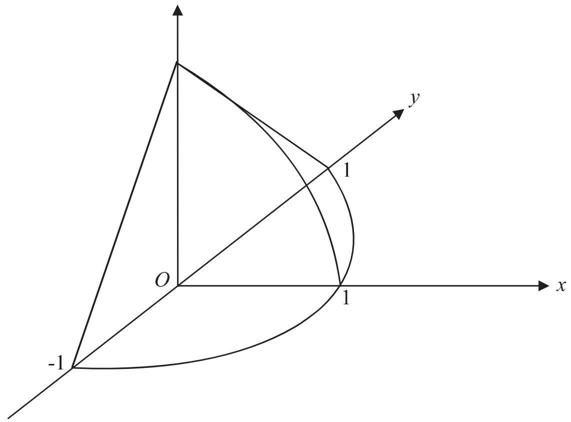
(ii) Deduce $PS - PS' = 2a$. **3**

Question 12 (15 marks) – Use a separate booklet **Marks**

(a) For what values of *k* does the equation $x^3 - 3x^2 - 24x + k = 0$ have one real root? **4**

(b) Find all the complex numbers $z = a + ib$, where *a* and *b* are real, such that $|z|^2 - 7 = 2i(z+2)$. **4**

(c)



The base of a solid is the semi-circular region in the x - y plane with the straight edge running from the point $(0,-1)$ to the point $(0,1)$ and the point $(1,0)$ on the curved edge of the semicircle.

Each cross-section perpendicular to the *x*-axis is an isosceles triangle, in which the two equal side lengths are equal to three quarters the length of the third side.

(i) Show that the area of the triangular cross-section at $x = a$ is given by: **2**

$$A(x) = \frac{\sqrt{5}}{2}\left(1 - a^2\right)$$

(ii) Hence find the volume of the solid. **2**

(d) A polynomial $P(x)$ gives remainders –2 and 1 when divided by $2x-1$ and $x-2$ respectively. What is the remainder when $P(x)$ is divided by $2x^2 - 5x + 2$? **3**

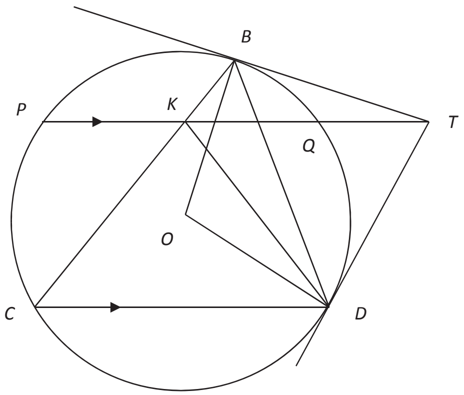
Question 13 (15 marks) – Use a separate booklet **Marks**

(a) The region bounded by the curve $y = \cos x$ and the axes between $x = 0$ and $x = \dfrac{\pi}{2}$ is rotated about the line $x = 3$ to form a solid of revolution.

(i) Draw a neat diagram showing the resultant volume. **1**

(ii) Using the method of cylindrical shells, find the volume of the solid. **4**

(b)



In the diagram, *TB* and *TD* are tangents to the circle (centre *O*), at *B* and *D*. *P*, *K*, *Q* and *T* are collinear such that $PT \parallel CD$.

Copy or trace the diagram into your writing booklet.

(i) Prove that *TBKD* is a cyclic quadrilateral. **3**

(ii) Prove that *TBOD* is a cyclic quadrilateral. **2**

(iii) Show that *K* is the midpoint of *PQ*. **2**

(c) *z* is a complex number satisfying the equation $|z + 2 - 4i| = 2$.

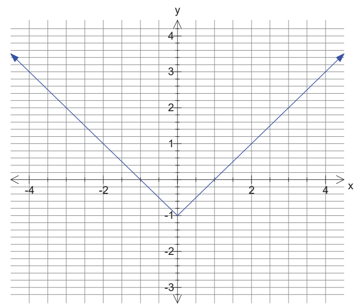
(i) Sketch the locus of *z*. **2**

(ii) Find the maximum value of $|z - 1|$. **1**

Question 14 (15 marks) – Use a separate booklet **Marks**

(a) Use the Table of Standard Integrals to find $\displaystyle\int \frac{2x\,dx}{\sqrt{x^4 + 16}}$. **2**

(b) Consider the graph of the function $f(x) = |x| - 1$ shown below.



Transformations of $y = f(x)$ are shown below. Draw the resultant graphs on the answer page provided with your multiple choice answer sheet. Show distinguishing features with an accurate scale.

(i) $y = x\,f(x)$ **2**

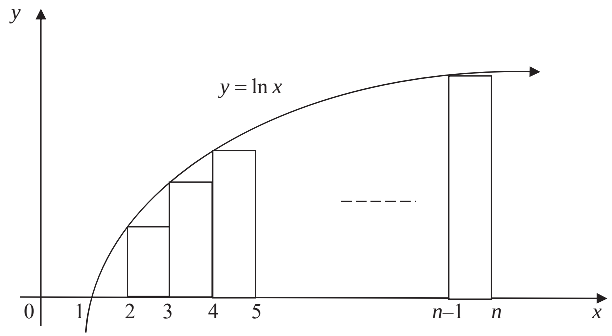
(ii) $|y| = f(x)$ **2**

(iii) $y = e^{f(x)}$ **2**

Question 14 continues on the next page

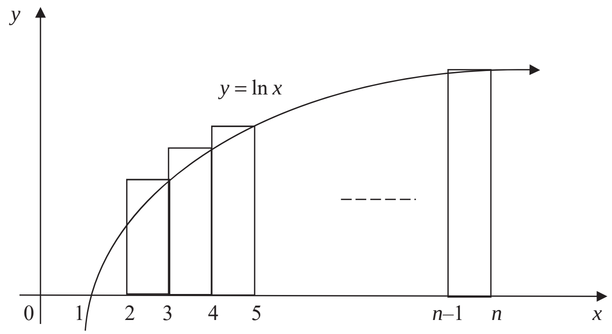
(c) (i) Evaluate exactly the integral $\int_1^n \ln x \, dx$. **2**

(ii) Consider the curve $y = \ln x$. The area under the curve for $1 \le x \le n$ is approximated by dividing it into rectangles under the curve each of width 1 unit. See diagram below (*not to scale*). **2**



Show that the sum of the rectangles, S_u, is given by $S_u = \ln\left((n-1)!\right)$.

(iii) Another approximation, S_a, is made by dividing the area into rectangles that lie above the curve. See diagram below (*not to scale*). **1**



Find a similar expression for this area, S_a.

(iv) Hence or otherwise show that $(n-1)! < n^n e^{1-n} < n!$. **2**

Question 15 (15 marks) – Use a separate booklet **Marks**

(a) Let $z = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$. Find z^6. **2**

(b) (i) Show that the tangent to the rectangular hyperbola $xy = 4$ at the point $P\left(2t, \frac{2}{t}\right)$ has the equation $x + t^2 y - 4t = 0$. **2**

(ii) The tangent at *P* cuts the *x* axis at point *Q*. Show that the line through *Q* which is perpendicular to the tangent at *P* has the equation $t^2x - y - 4t^3 = 0$. **1**

(iii) The line $t^2x - y - 4t^3 = 0$ cuts the rectangular hyperbola at the points *R* and *S*. Show the midpoint *M* of *RS* has coordinates $M(2t, -2t^3)$. **2**

(iv) Find the equation of the locus of *M* as *P* moves on the rectangular hyperbola. State any restrictions. **2**

(c) The equation $x^4 + bx^3 + cx^2 + dx + e = 0$ has two roots which are reciprocals and two other roots which are opposites.

(i) Show that $c = 1 + e$. **2**

(ii) Hence, deduce that $d = be$. **2**

(d) Show that $e^x > 1 + x$ for $x > 0$. **2**

Question 16 (15 marks) – Use a separate booklet **Marks**

(a) **Without** using calculus, draw the graph of: **3**

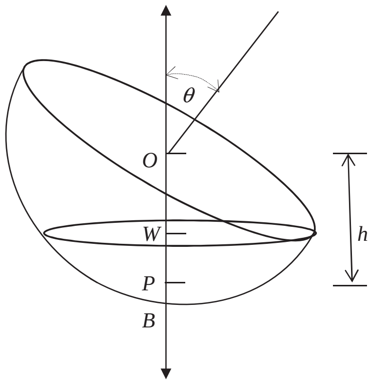
$y = \frac{1}{9}(x-1)^2(x^3 - 4x^2 - 15x + 18)$

Show all major features of the curve.

(b) (i) Explain why the cubic $f(x) = x^3 - 3x + 1$ has **exactly** one root, $x = \alpha$, between 0 and 1. **2**

(ii) Beginning with the approximation $x = 0$, use two applications of Newton's method to show that $\alpha \approx 0.347$ (to 3 decimal places). **2**

(iii) The diagram below shows a hemispherical bowl of radius r. The bowl has been tilted so that its axis is no longer vertical, but at an angle θ to the vertical. At this angle, it can hold a volume V of water. **2**



The vertical line from the centre O meets the surface of the water at W and meets the bottom of the bowl at B. Let P be any point between W and B and let h be the distance OP.

Explain why $V = \int_{r\sin\theta}^{r} \pi(r^2 - h^2)\,dh$.

(iv) Hence show $V = \frac{r^3\pi}{3}(2 - 3\sin\theta + \sin^3\theta)$ **3**

(v) The bowl has been tilted so that it is exactly half full. Using the results above, find the angle θ correct to one tenth of a degree. **3**

END OF ASSESSMENT TASK

2014

X2 Trial HSC ~ Multiple Choice Answers

1. $b^2 = a^2(e^2 - 1)$ $\quad\quad$ $a^2 = 144$ and $b^2 = 25$.

$25 = 144(e^2 - 1)$ $\quad\quad$ $a = 12$ $\quad$ $b = 5$

$(e^2 - 1) = \frac{25}{144}$ or $e^2 = \frac{169}{144}$ or $e = \frac{13}{12}$

Equation of the directrices are $x = \pm\frac{a}{e} = \pm\frac{144}{13}$. **(D)**

2. POQ is a right-angled triangle. Therefore $OP^2 + OQ^2 = PQ^2$.

$a^2\cos^2\theta + b^2\sin^2\theta + a^2\cos^2\phi + b^2\sin^2\phi = a^2(\cos\theta - \cos\phi)^2 + b^2(\sin\theta - \sin\phi)^2$

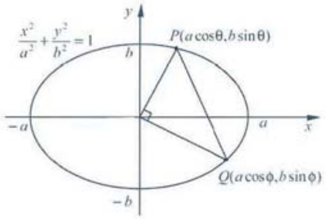
$a^2(\cos^2\theta + \cos^2\phi) + b^2(\sin^2\theta + \sin^2\phi) = a^2(\cos\theta - \cos\phi)^2 + b^2(\sin\theta - \sin\phi)^2$

Hence

$0 = -2a^2\cos\theta\cos\phi - 2b^2\sin\theta\sin\phi$

$2b^2\sin\theta\sin\phi = -2a^2\cos\theta\cos\phi$

$\frac{\sin\theta\sin\phi}{\cos\theta\cos\phi} = \frac{-2a^2}{2b^2}$ or $\tan\theta\tan\phi = -\frac{a^2}{b^2}$



(B)

3. $\tan\theta = \frac{1}{-\sqrt{3}}$

$\theta = \frac{5\pi}{6}$

$r^2 = x^2 + y^2$

$= (\sqrt{3})^2 + 1^2$

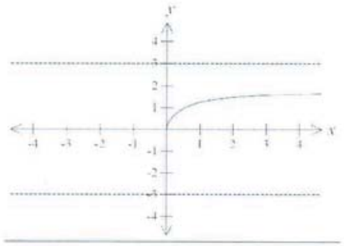
$r = 2$

$-\sqrt{3} + i = 2(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6})$ **(D)**

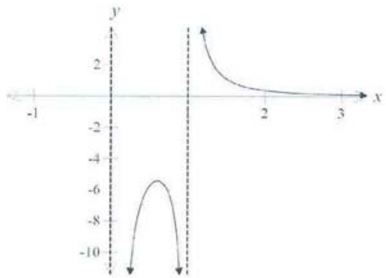
4. $|z - 1| \le \sqrt{2}$ represents a region with a centre is (1, 0) and radius is less than or equal to $\sqrt{2}$.

$0 \le \arg(z + i) \le \frac{\pi}{4}$ represents a region between angle 0 and $\frac{\pi}{4}$ whose vertex is $(-1, 0)$ not including the vertex $\quad \therefore |z - 1| \le \sqrt{2}$ and $0 \le \arg(z + i) \le \frac{\pi}{4}$ **(B)**

5. **(C)**



6. (B)



7. $\int\frac{1}{\sqrt{7 - 6x - x^2}}dx = \int\frac{1}{\sqrt{16 - 9 - 6x - x^2}}dx$ **(D)**

$= \int\frac{1}{\sqrt{16 - (x + 3)^2}}dx$

$= \sin^{-1}\left(\frac{x + 3}{4}\right) + c$

8. (C)

$$\int \frac{dx}{\sqrt{x^2-6x+10}} = \int \frac{dx}{\sqrt{x^2-6x+9+1}} = \frac{dx}{\sqrt{(x-3)^2+1}}$$

$$= \ln\left(x-3+\sqrt{(x-3)^2+1}\right)+c$$

$$= \ln\left(x-3+\sqrt{x^2-6x+10}\right)+c$$

9. Let $P(x)=4x^3-27x+k$ (C)

$P'(x)=12x^2-27$

Let α be the double root.

Hence $P(\alpha)=0$ and $P'(\alpha)=0$

When $P'(\alpha)=0$ then $12\alpha^2-27=0$

$$\alpha^2=\frac{9}{4}$$

$$\alpha=\pm\frac{3}{2}$$

When $P(\alpha)=0$ then $4\alpha^3-27\alpha+k=0$

$$k=27\alpha-4\alpha^3$$

$$=\alpha(27-4\alpha^2)$$

$$=\pm\frac{3}{2}(27-4\times\frac{9}{4})$$

$$=\pm 27$$

10. $(x-1)(4x^2+ax-1)^2=16x^5-20x^3+5x-1$ (B)

Let $x=2$, $1.(15+2a)^2=16.2^5-20.2^3+5.2-1=361$

$\therefore 15+2a=\pm 19$

$\therefore 2a=-15\pm 19=4 \text{ or } -34$

$\therefore a=2 \text{ or } -17.$

Question 11 (a)

Criteria
• One for integration and one for substitution.

Answer:

$$\int_0^{\ln 3} \frac{e^x}{1+e^x}dx = \left[\ln(1+e^x)\right]_0^{\ln 3} = \ln(1+e^{\ln 3})-\ln(1+e^0) = \ln 4-\ln 2=\ln 2,$$

Question 11 (b)

Criteria
• One for completing the square and one for simplification.

Answer:

(b) $$\int \frac{-2}{\sqrt{3+2x-x^2}}dx$$

$$=-2\int \frac{1}{\sqrt{-(x^2-2x-3)}}dx$$

$$=-2\int \frac{1}{\sqrt{-(x^2-2x+1-1-3)}}dx$$

$$=-2\int \frac{1}{\sqrt{4-(x-1)^2}}dx \qquad u=x-1, \quad \frac{du}{dx}=1$$

$$=-2\sin^{-1}\frac{(x-1)}{2}+c$$

Question 11 (c)

Criteria
• (i) One each for a, b and c. (ii) One for $\ln(x+2)$ and one for simplification

Answer:

(i) By partial fractions

$$\frac{x^2-11}{(3x-1)(x+2)^2} \equiv \frac{a}{(3x-1)}+\frac{b}{(x+2)}+\frac{c}{(x+2)^2}=\frac{a(x+2)^2+b(3x-1)(x+2)+c(3x-1)}{(3x-1)(x+2)^2}$$

$$x^2-11=a(x+2)^2+b(3x-1)(x+2)+c(3x-1)$$

Let $x=-2$

$$(-2)^2-11=c(3(-2)-1)$$

$c=1$

Let $x=\frac{1}{3}$

$$\left(\frac{1}{3}\right)^2-11=a\left(\frac{1}{3}+2\right)^2$$

$\frac{1}{9}-11=a\left(\frac{7}{3}\right)^2$

$-\frac{98}{9}=\frac{49}{9}a$

$-2=a$

Let $x=0$

$-11=-2(2)^2+b(-1)(2)+1(-1)$

$1=b$

(ii) $\int \frac{x^2-11}{(3x-1)(x+2)^2}\,dx$

$=\int \frac{-2}{3x-1}+\frac{1}{x+2}+\frac{1}{(x+2)^2}\,dx$

$=-\frac{2}{3}\ln(3x-1)+\ln(x+2)+\frac{-1}{x+2}+c$

Question 11 (d)

Criteria
• (i) One for substituting into the distance for either PS or PS' and one each for finding the distances PS and PS' (ii) One for explaining $PS-PS'=-2a$, one for explaining $PS-PS'=+2a$ and one for conclusion

Answer:

(a) (i)

Length of PS is $\sqrt{(a\sec\theta-ae)^2+(b\tan\theta)^2}=\sqrt{a^2(\sec\theta-e)^2+b^2\tan^2\theta}$.

Given $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \Rightarrow b^2=a^2(e^2-1)$.

Length of PS :

$=\sqrt{a^2(\sec\theta-e)^2+a^2(e^2-1)\tan^2\theta}$

$=a\sqrt{\sec^2\theta-2e\sec\theta+e^2+e^2\tan^2\theta-\tan^2\theta}$

$a\sqrt{e^2(1+\tan^2\theta)-2e\sec\theta+(\sec^2\theta-\tan^2\theta)}$

$=a\sqrt{e^2\sec^2\theta-2e\sec\theta+1}$

$=a\sqrt{(e\sec\theta-1)^2}$

$=a(e\sec-1)$.

Length of PS'

$=\sqrt{(a\sec\theta+ae)^2+(b\tan\theta)^2}$

$=\sqrt{a^2(\sec\theta+e)^2+b^2\tan^2\theta}$.

Given $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $b^2=a^2(e^2-1)$.

Length PS' :

$=\sqrt{a^2(\sec\theta+e)^2+a^2(e^2-1)\tan^2\theta}$

$=a\sqrt{\sec^2\theta+2e\sec\theta+e^2+e^2\tan^2\theta-\tan^2\theta}$

$=a\sqrt{e^2(1+\tan^2\theta)+2e\sec\theta+(\sec^2\theta-\tan^2\theta)}$

$=a\sqrt{e^2\sec^2\theta+2e\sec\theta+1}$

$=a\sqrt{(e\sec\theta+1)^2}$

Thus the length of PS' is $a(e\sec\theta+1)$.

(ii) If P lies on the right-hand branch of the hyperbola, then $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$.

For hyperbola $e>1$, $PS=a(e\sec\theta-1)$ and $PS'=a(e\sec\theta+1)$.

Thus $PS-PS'=-2a$.

If P lies on the left-hand branch of the hyperbola, thus $-\pi<\theta<-\frac{\pi}{2}$ or $\frac{\pi}{2}<\theta\le\pi$.

For hyperbola $e>1$, $PS=-a(e\sec\theta-1)$ and $PS'=-a(e\sec\theta+1)$.

Thus $PS-PS'=+2a$. Hence $|PS-PS'|=2a$.

Question 12(a)

Let $P(x)=x^3-3x^2-24x+k=0$

$P'(x)=3x^2-6x-24$

For stationary points $(P'(x)=0)$:

$3x^2-6x-24=0$

$x^2-2x-8=0$

$(x-4)(x+2)=0$

$x=4,\ x=-2$

$x=4,\quad P(4)=4^3-3(4)^2-24(4)+k$

$=k-80$

$x=-2,\quad P(-2)=(-2)^3-3(-2)^2-24(-2)+k$

$=k+28$

$P(-2)\times P(4)>0$ (since one real root)

$(k-80)(k+28)>0$

$k<-28,\ k>80$

4 marks	Correct answer
3 marks	Showing $(k-80)(k+28)>0$
2 marks	Showing $P(4)=k-80$ and $P(-2)=k+28$
1 mark	Correctly finding x-values of stationary points

Question 12 (b)

$$|z|^2 - 7 = 2i(z+2)$$

Let $z = a + ib$

$$|z| = \sqrt{a^2 + b^2}$$

$$|z|^2 = a^2 + b^2$$

$$a^2 + b^2 - 7 = 2i(a + ib + 2)$$

$$= -2b + 2ai + 4i$$

$$a^2 + b^2 + 2b - 2ai = 7 + 4i$$

so $-2a = 4$

$$a = -2$$

so $a^2 + b^2 + 2b = 7$

becomes $b^2 + 2b - 3 = 0$

$$(b+3)(b-1) = 0$$

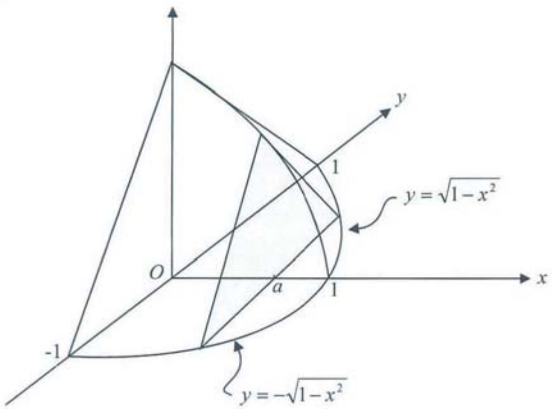
$$b = -3 \text{ or } b = 1$$

$$z = -2 + i, \quad z = -2 - 3i$$

4 marks	Correct answer
3 marks	Obtaining $b = -3$ and $b = 1$
2 marks	Obtaining $a = -2$
1 mark	Obtaining $a^2 + b^2 + 2b - 2ai = 7 + 4i$

Question 12(c)

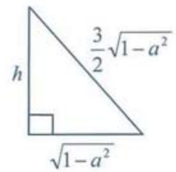
(b) (i)



The base of the shaded isosceles triangle has length $2\sqrt{1-a^2}$.
The two equal side lengths are therefore of length

$$\frac{3}{4} \times 2\sqrt{1-a^2} = \frac{3}{2}\sqrt{1-a^2}$$

The height of the isosceles triangle is given by



$$h = \sqrt{\frac{9}{4}(1-a^2) - (1-a^2)}$$

$$= \sqrt{\frac{5}{4} - \frac{5}{4}a^2}$$

$$= \sqrt{\frac{5}{4}(1-a^2)}$$

So the area of the isosceles triangle is

$$\frac{1}{2} \times \text{base} \times \text{height}$$

$$= \frac{1}{2} \times 2\sqrt{1-a^2} \times \sqrt{\frac{5}{4}(1-a^2)}$$

$$= \sqrt{\frac{5}{4}(1-a^2)^2}$$

$= \frac{\sqrt{5}}{2}(1-a^2)$ as required.

2 marks	Correct derivation
1 mark	Correct method with one mistake OR identifying the base length of the isosceles triangle as $2\sqrt{1-a^2}$

(ii) Now $\delta V = A\,\delta x$ where A = area of isosceles triangle

$$= \frac{\sqrt{5}}{2}(1-x^2)\,\delta x$$

$$V = \lim_{\delta x \to 0} \sum_{x=0}^{1} \frac{\sqrt{5}}{2}(1-x^2)\,\delta x$$

$$= \frac{\sqrt{5}}{2}\int_0^1 (1-x^2)\,dx$$

$$= \frac{\sqrt{5}}{2}\left[x - \frac{x^3}{3}\right]_0^1$$

$$= \frac{\sqrt{5}}{2}\left\{\left(1 - \frac{1}{3}\right) - 0\right\}$$

$$= \frac{\sqrt{5}}{2} \times \frac{2}{3}$$

$= \frac{\sqrt{5}}{3}$ cubic units

2 marks	Correct answer
1 mark	Correct method with one mistake OR obtaining $V = \frac{\sqrt{5}}{2}\int_0^1 (1-x^2)\,dx$

Question 12(d)

Let $P(x) = (2x^2 - 5x + 2)Q(x) + R(x)$

since $\deg D(x) > \deg R(x)$

then $\deg R(x) < 2$

let $R(x) = ax + b$

$P(x) = (2x-1)(x-2)Q(x) + ax + b$

$P\left(\frac{1}{2}\right) = \frac{1}{2}a + b = -2 \qquad -(1)$

$P(2) = 2a + b = 1 \qquad -(2)$

$(2)-(1) \Rightarrow \frac{3a}{2} = 3$

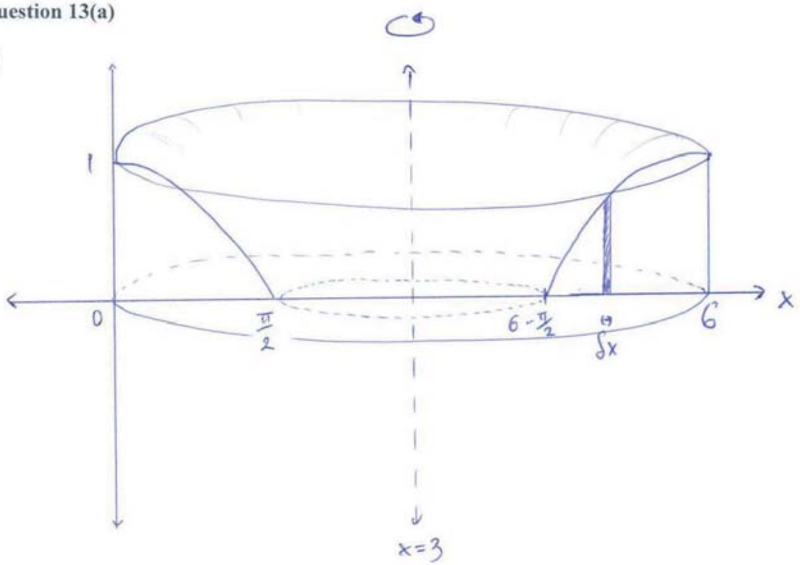
$a = 2,\ b = -3$

so $R(x) = 2x - 3$

3 marks	Correct answer
2 marks	Correctly finding $P\left(\frac{1}{2}\right)$ and $P(2)$
1 mark	Finding $P(x) = (2x-1)(x-2)Q(x) + ax + b$

Question 13(a)

(i)



(ii)

$\delta V = \pi[(3-x)^2 - (3-x-\delta x)^2]y$

$= \pi[2(3-x)\delta x - (\delta x)^2]y$

$\approx 2\pi(3-x)y\delta x$ **1**

$V = 2\pi \int_0^{\frac{\pi}{2}} (3-x)\cos x\, dx$ **1**

$= 2\pi \times [3\sin x]_0^{\frac{\pi}{2}} - 2\pi \int_0^{\frac{\pi}{2}} x\cos x\, dx$

$= 2\pi[3] - 2\pi\left\{[x\sin x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x dx\right\}$ **1**

$= 6\pi - \pi^2 - 2\pi(-1)$

$= (8\pi - \pi^2)\text{u}^3$ **1**

Question 13(b)

(i)

$\angle TDB = \angle BCD$ (The angle between a tangent and a chord is equal to the angle in the alternate segment.) **1**

$\angle BCD = \angle BKT$ (Corresponding angles on parallel lines.) **1**

$\therefore \angle TDB = \angle BKT$

$TBKD$ is a cyclic quadrilateral. (TB subtends equal angles at K and D.) **1**

(ii)

$OB \perp BT$ (A tangent is perpendicular to the radius at the point of contact.) **1**

$\angle OBT = 90°$

Similarly, $\angle ODT = 90°$

$\angle OBT + \angle ODT = 180°$

$TBOD$ is a cyclic quadrilateral. (If opposite angles of a quadrilateral are supplementary, then the quadrilateral is a cyclic quadrilateral.) **1**

(iii)

Join OK.
From (i) and (ii), $TBKOD$ is a cyclic quadrilateral.

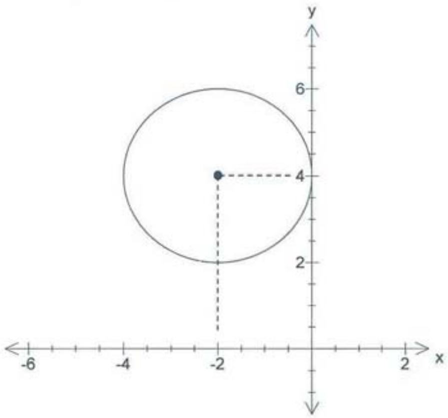
$\angle OKD = \angle OBD$ (Angles in the same segment are equal.) 1
$\angle DKT = \angle DBT$ (Angles in the same segment are equal.)

$\angle OKD + \angle DKT$
$= \angle OBD + \angle DBT$
$= 90°$ (A tangent is perpendicular to the radius at the point of contact.)

$\therefore OK \perp PQ$
$PK = QK$ (A perpendicular from the centre to a chord bisects the chord.) 1

Question 13(c)

(i)
Centre $= (-2,4)$, $r = 2$ 2



(ii)

Distance from centre to the point (1,0)
$\sqrt{(-2-1)^2 + (4-0)^2} = 5$ 1
$\therefore$ Maximum value $5+2 = 7$

Question 14(a)

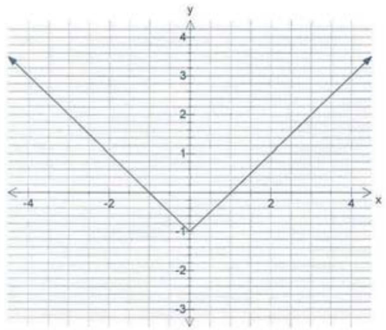
$$\int \frac{2x}{\sqrt{x^4+16}}dx = \ln\left(x^2 + \sqrt{x^4+16}\right) + c$$ 2

Question 14(b)

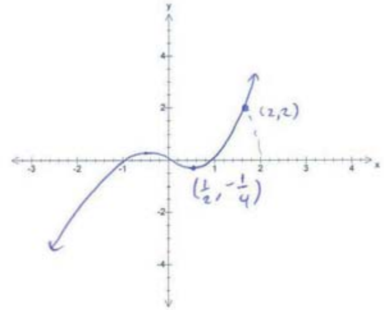
Year 12 Ext 2 Graphing Transformations Answer sheet

DETACH and place in QUESTION 14 booklet

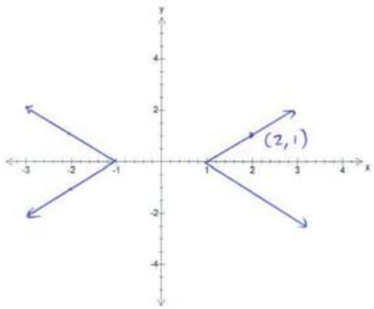
$y = f(x)$



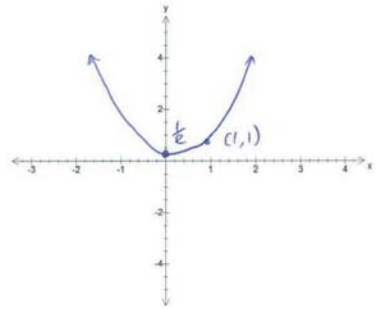
$y = xf(x)$



$|y| = f(x)$



$y = e^{f(x)}$



Question 14(c)

(i) $\int_1^n \ln x\,dx = [x\ln x]_1^n - \int_1^n x.\frac{1}{x}dx$

$= n\ln(n) - 0 - [x]_1^n$

$= n\ln(n) - n + 1$

(ii) $S_u = l_2b_2 + l_3b_3 + \cdots + l_{n-1}b_{n-1}$

$= \ln 2.1 + \ln 3.1 + \cdots + \ln(n-1).1$

$= \ln[2.3.4\ldots(n-1)]$

$= \ln[(n-1)!]$

(iii) $S_a = \ln(n!)$

(iv) $$S_u < \int_1^n \ln x\,dx < S_a$$

$\ln[(n-1)!] < n\ln(n) - n + 1 < \ln(n!)$

$\ln[(n-1)!] < \ln(n^n) - \ln(e^n) + \ln e < \ln(n!)$

$\ln[(n-1)!] < \ln\left(\frac{n^n.e}{e^n}\right) < \ln(n!)$

$\ln[(n-1)!] < \ln(n^n.e^{1-n}) < \ln(n!)$

$(n-1)! < n^n.e^{1-n} < n!$

Question 15(a)

$z = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$

$z^6 = \cos(6\times\frac{\pi}{6}) + i\sin(6\times\frac{\pi}{6})$

$= \cos\pi + i\sin\pi$

$= -1$

2 Marks: Correct answer.

1 Mark: Uses De Moivre's theorem

Question 15(b)

(i) To find the gradient of the tangent.

$xy = 4$

$x\frac{dy}{dx} + y = 0$

$\frac{dy}{dx} = -\frac{y}{x}$

At $P(2t, \frac{2}{t})$

$\frac{dy}{dx} = -\frac{\frac{2}{t}}{2t}$

$= -\frac{1}{t^2}$

Equation of the tangent at P

$y - \frac{2}{t} = -\frac{1}{t^2}(x - 2t)$

$t^2y - 2t = -x + 2t$

$x + t^2y - 4t = 0$

2 Marks: Correct answer.

1 Mark: Correctly calculates the gradient of the tangent to the hyperbola

(ii) Tangent cuts the x axis when $y=0$.

To find point Q substitute $y=0$ into $x+t^2y-4t=0$.

$x+t^2\times 0-4t=0$

$x=4t \qquad \therefore\ Q(4t,0)$

At $Q(4t,0)$ Gradient of the tangent is $-\frac{1}{t^2}$

Gradient of the normal is t^2 ($m_1m_2=-1$)

Equation of the normal at Q

$y-0=t^2(x-4t)$

$y=t^2x-4t^3$

$t^2x-y-4t^3=0$

1 Mark: Correct answer.

(iii) R and S are the points of intersection of normal at Q and the hyperbola.

Solve equations simultaneously

$xy=4 \qquad (1)$

$t^2x-y-4t^3=0 \qquad (2)$

Eqn (1) + $x\times$ Eqn (2)

$t^2x^2-4t^3x=4$

$t^2x^2-4t^3x-4=0$

Sum of the roots is $x_1+x_2=-\frac{-4t^3}{t^2}=4t$.

(Sum of the roots at R and S)

Midpoint of the roots $x=\frac{x_1+x_2}{2}=\frac{4t}{2}=2t$

This is the x coordinate of M.

Substitute $x=2t$ into the equation of the normal at Q.

$t^2\times 2t-y-4t^3=0$ or $y=-2t^3$

Coordinates of M are $(2t,-2t^3)$

2 Marks: Correct answer.

1 Mark: Determines $t^2x^2-4t^3x-4=0$ or makes similar progress.

(iv) Eliminate t from the two parametric equations.

$x=2t \qquad (1)$

$y=-2t^3 \qquad (2)$

Substitute $t=\frac{x}{2}$ from Eqn (1) into Eqn (2).

$y=-2\left(\frac{x}{2}\right)^3$ or $y=-\frac{x^3}{4}$

Locus is $x^3+4y=0$ (Excludes case when $t=0$ (undefined)).

2 Marks: Correct answer.

1 Mark: Finds the equation of the locus or states the correct restriction.

Question 15(c)

(i) Let the roots of $\alpha, \frac{1}{\alpha}, \beta, -\beta$.

$\sum \alpha\beta\chi\delta=e$

$\alpha\cdot\frac{1}{\alpha}\cdot\beta\cdot(-\beta)=e$

$-\beta^2=e \qquad (1)$

$\sum \alpha\beta=c$

$\alpha\cdot\frac{1}{\alpha}+\alpha\beta+\alpha(-\beta)+\frac{\beta}{\alpha}+\frac{(-\beta)}{\alpha}=c$

$1-\beta^2=c \qquad (2)$

Solving Eqns (1) and (2) simultaneously.

$1+e=c$

$c=1+e$

2 Marks: Correct answer

1 Mark: Correctly finds relations between roots and coefficients.

(ii) $\sum \alpha\beta\chi=d$

$\alpha\cdot\frac{1}{\alpha}.\beta+\alpha.\frac{1}{\alpha}.(-\beta)+\frac{1}{\alpha}.\beta.(-\beta)+\alpha.\beta.(-\beta)=d$

$-\beta^2(\frac{1}{\alpha}+\alpha)=d \qquad (3)$

$\sum \alpha=e$

$\alpha+\frac{1}{\alpha}+\beta+(-\beta)=-b$

$\alpha+\frac{1}{\alpha}=-b \qquad (4)$

Solving Eqns (3), (4) and (1) simultaneously.

$eb=d$

$d=be$

2 Marks: Correct answer

1 Mark: Correctly finds relations between roots and coefficients.

Question 15(d)

$e^x > 1 + x$ or $e^x - 1 - x > 0$ Let $f(x) = e^x - 1 - x$	2 Marks: Correct answer.
$f'(x) = e^x - 1$ > 0 when $x > 0$	1 Mark: Shows some understanding.

Hence $f(x)$ is an increasing function for $x > 0$

Absolute minimum for $f(x)$ is 0 when $x = 0$

Therefore $f(x) > 0$

$e^x - 1 - x > 0$

$e^x > 1 + x$

Question 16(a)

Let $P(x) = x^3 - 4x^2 - 15x + 18$

$P(1) = 1 - 4 - 15 + 18$

$= 0$

$\therefore x - 1$ is a factor of $P(x)$.

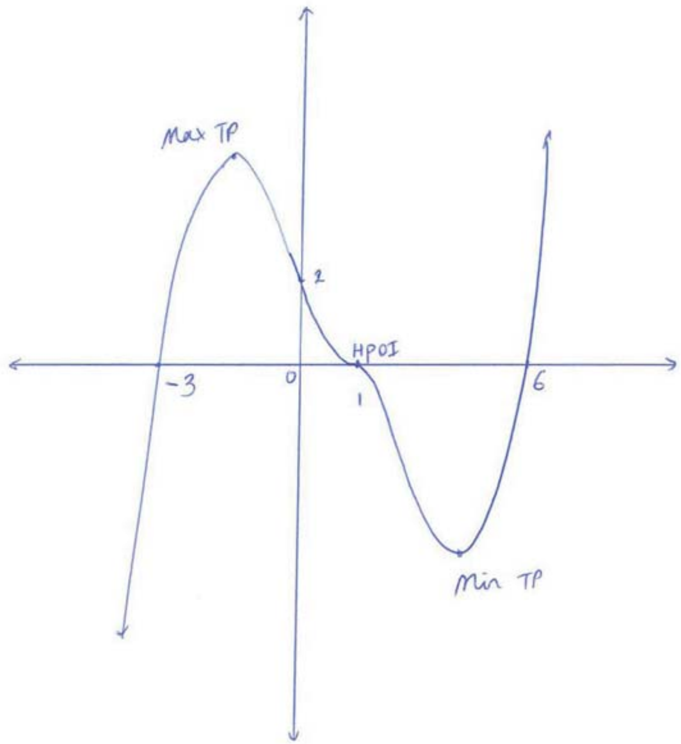
$$\begin{array}{r|rrrr} 1 & 1 & -4 & -15 & 18 \\ & & 1 & -3 & -18 \\ \hline & 1 & -3 & -18 & 0 \end{array}$$

$x^2 - 3x - 18$

$\therefore P(x) = (x-1)(x^2 - 3x - 18)$

$= (x-1)(x+3)(x-6)$

$\therefore y = \frac{1}{9}(x-1)^3(x+3)(x-6)$



Question 16(b)

(i) $f(x)=x^3-3x+1,\quad f(0)=1,\ f(1)=-1$

Hence, there is at least one root between 0 and 1. Note that $f'(x)=3x^2-3$ and $f'(x)=0$ when $x=\pm1$ and $f'(x)<0$ (i.e. monotonic decreasing) in the interval $0<x<1$. Hence, there is exactly one root between 0 and 1.

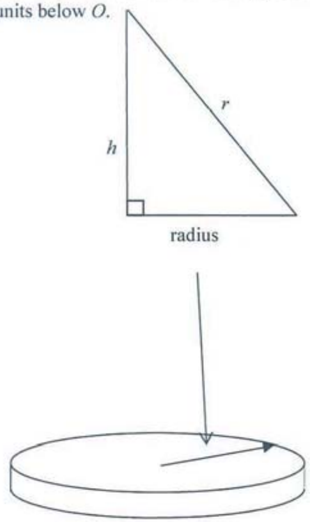
(ii) Let $x_0=0$. Using Newton's method,

$x_1=x_0-\dfrac{f(x_0)}{f'(x_0)}=\dfrac{1}{3}$ and

$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}=\frac{1}{3}-\frac{\left(\frac{1}{3}\right)^3-3\times\left(\frac{1}{3}\right)+1}{3\times\left(\frac{1}{3}\right)^2-3}\approx 0.347$$

(iii)

Consider the volume of the water by considering horizontal discs of thickness □*h*. Consider the disc *h* units below *O*.

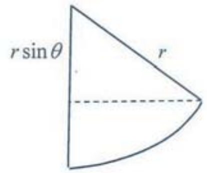


$\text{radius}^2=r^2-h^2$

Let $\delta V=$ Volume of the typical disc

$=\pi\left(r^2-h^2\right).\delta h$

The total volume of water $\approx \sum_{B}^{W}\pi\left(r^2-h^2\right)\delta h$



$=\sum_{h=r\sin\theta}^{h=r}\pi\left(r^2-h^2\right)\delta h$

Exact volume $=\lim_{\delta h\to0}\sum_{h=r\sin\theta}^{h=r}\pi\left(r^2-h^2\right)\delta h$

$=\int_{r\sin\theta}^{r}\pi\left(r^2-h^2\right)dh$

(ii) $V=\int_{r\sin\theta}^{r}\pi\left(r^2-h^2\right)dh$

$=\pi\left[r^2h-\dfrac{h^3}{3}\right]_{r\sin\theta}^{r}$

$=\pi\left[r^3-\dfrac{r^3}{3}-\left(r^3\sin\theta-\dfrac{r^3\sin\theta}{3}\right)\right]$

$=\dfrac{r^3\pi}{3}\left(3-1-3\sin\theta+\sin^3\theta\right)$

$=\dfrac{r^3\pi}{3}\left(2-3\sin\theta+\sin^3\theta\right)$

(iii) Volume of a hemisphere $=\dfrac{2}{3}\pi r^3$

$\therefore\dfrac{r^3\pi}{3}\left(2-3\sin\theta+\sin^3\theta\right)=\dfrac{1}{3}\pi r^3$

$\therefore 2-3\sin\theta+\sin^3\theta=1$

$\sin^3\theta-3\sin\theta+1=0$

Now, in 14 (c), $\alpha^3-3\alpha+1=0\Rightarrow\alpha\approx0.347$

Hence, $\sin\theta\approx0.347\Rightarrow\theta\approx20.3^0$