CRANBROOK SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2001

MATHEMATICS

4 UNIT (Additional)

Time allowed – Three hours

DIRECTIONS TO CANDIDATES

* Attempt all questions.
* ALL questions are of equal value.
* All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
* Standard integrals are printed on the back page.
* Board-approved calculators may be used.
* You may ask for extra Writing Booklets if you need them.

* Submit your work in four 8 page booklets :

(i) QUESTIONS 1 & 2

(ii) QUESTIONS 3 & 4

(iii) QUESTIONS 5 & 6

(iv) QUESTIONS 7 & 8

1. (8 page booklet)

(a) Find (i) $\int \cot x \ \operatorname{cosec}^2 x \ dx$ (ii) $\int \frac{\sec^2 x}{3-\tan x} dx$ **[4 marks]**

(b) Prove that $\int_{5\frac{1}{2}}^{6\frac{1}{2}} \frac{dx}{\sqrt{(x-5)(7-x)}} = \frac{\pi}{3}$, by using the substitution $u = x - 6$. **[3 marks]**

(c) (i) Prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$. **[2 marks]**

(ii) Hence or otherwise evaluate $\int_0^{\pi/2} \frac{\cos^5 x}{\cos^5 x + \sin^5 x} dx$. **[2 marks]**

(d) Evaluate $\int_0^{\pi/4} \frac{dx}{2\sin 2x + \cos x}$ **[4 marks]**

2.

(a) Evaluate $\int_{-\pi/4}^{\pi/4} \frac{x^3}{\cos x} dx$ **[2 marks]**

(b) Find $\int \sin^3 2x \ \cos^2 2x \ dx$ **[3 marks]**

(c) Find $\int \frac{4x-3}{\sqrt{6+2x-3x^2}} \ dx$ **[4 marks]**

(d) If $I_n = \int_0^{\pi/2} \cos^n x \sin^2 x \, dx$ for $n \geq 0$, show that $I_n = \frac{n-1}{n+2} I_{n-2}$ for $n \geq 2$. **[4 marks]**

Hence or otherwise evaluate $\int_0^{\pi/2} \cos^4 x \sin^2 x \, dx$. **[2 marks]**

3. (new 8 page booklet please)

(a) (i) Given $z_1 = 1 - i$ and $z_2 = -1 + \sqrt{3}i$ evaluate $|z_1 z_2|$ and $\arg(z_1 z_2)$

(ii) Find $z_1 z_2$ in cartesian form, and hence show that $\cos\frac{5\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$ **[6 marks]**

(b) If z is a complex number for which $|z| = 1$ show that

(i) $1 \leq |z+2| \leq 3$ and (ii) $-\frac{\pi}{6} \leq \arg(z+2) \leq \frac{\pi}{6}$ **[4 marks]**

(c) (i) Given that $z + \frac{1}{z} = k$, a real number, show that z lies either on the real axis or on the unit circle, centre the origin.

(ii) If z lies on the real axis, show that $|k| \geq 2$; if z lies on the unit circle, show that $|k| \leq 2$. **[5 marks]**

4.

(a) Find integers a and b such that $(x+1)^2$ is a factor of $x^5 + 2x^2 + ax + b$. **[3 marks]**

(b) The equation $z^2 + (1+i)z + k = 0$ has a root $1 - 2i$. Find the other root, and the value of k. **[3 marks]**

(c) Let α, β, γ be the roots (none of which is zero) of $x^3 + 3px + q = 0$.

(i) Obtain the monic equation whose roots are $\frac{\alpha\beta}{\gamma}, \frac{\beta\gamma}{\alpha}, \frac{\gamma\alpha}{\beta}$.

(ii) Deduce that $\gamma = \alpha\beta$ if and only if $(3p-q)^2 + q = 0$ **[9 marks]**

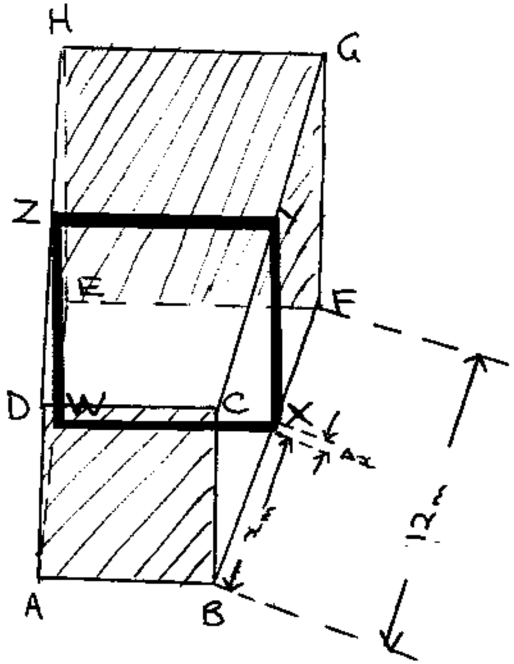
5. (new 8 page booklet please)

(a) The region bounded by the circle $x^2 + y^2 = 4$ and the parabola $y^2 = 3x$ is rotated about the x-axis. By including appropriate diagrams in each case, find the volume of the solid of revolution by using :

(i) circular discs **[5 marks]**

(ii) cylindrical shells. **[5 marks]**

(b) In the solid shown ABCD and EFGH are squares of side 6 m and 10 m respectively. BCGF is a ~~parallelogram~~ trapezium of height 12 m. Cross-sections parallel to the ends are squares. Show that at a distance x m from the base AB the area of the cross-section is $\left(6 + \frac{x}{3}\right)^2$. Hence, by taking slices of thickness Δx find the total volume of the solid. **[5 marks]**



6.

(a) $P\left(3p,\dfrac{3}{p}\right)$ and $Q\left(3q,\dfrac{3}{q}\right)$ are points on the rectangular hyperbola $xy=9$. The equation of chord PQ is $x+pqy=3(p+q)$.

(i) Find the co-ordinates of N, the midpoint of PQ.

(ii) If chord PQ is a tangent to the parabola $y^2=3x$ prove that the locus of N is $3x=-8y^2$.

[5 marks]

(b) A cylinder of constant volume V has its radius increasing at 5% per minute. At what % rate is the height diminishing?

[4 marks]

(c) A cyclist and a jogger journey along two roads OA and OB, which are inclined at 60^0 to one another. The cyclist starts at a point P, 10 km from O along OA and cycles towards O. At the same instant the jogger starts from O and runs away from O along OB. If the cyclist travels at 8 km/h and the jogger runs at 5 km/h find the rate at which the distance between the two is changing after 90 minutes (in km/h), correct to 2 decimal places.

[6 marks]

7. (new 8 page booklet please)

(a) Show that the ellipse $4x^2+9y^2=36$ and the hyperbola $4x^2-y^2=4$ intersect at right angles.

[5 marks]

(b) You are given that the equation of the normal to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1 \quad (a^2>b^2)$ at the point $P(x_1,y_1)$ is $a^2y_1x-b^2x_1y=\left(a^2-b^2\right)x_1y_1$.

(i) This normal meets the major axis of the ellipse at G. S is a focus of the ellipse. Show that $GS=e\times PS$, where e is the eccentricity of the ellipse.

[5 marks]

(ii) The normal at the point $P(5\cos\theta,3\sin\theta)$ on $\dfrac{x^2}{25}+\dfrac{y^2}{9}=1$ cuts the major and minor axes of the ellipse at G and H respectively. Show that as P moves on the ellipse, the mid-point of GH describes another ellipse with the same eccentricity as the first.

[5 marks]

8.

(a) In a certain cricket club there are 15 players available for selection, including 2 Smith brothers, 3 Brown brothers and 10 others. In how many ways may an eleven be selected for a game, if no more than 1 Smith and 2 Browns may be chosen?

[3 marks]

(b) Given that $\sin^{-1}x$, $\cos^{-1}x$ and $\sin^{-1}(1-x)$ are all acute

(i) show that $\sin\left[\sin^{-1}x-\cos^{-1}x\right]=2x^2-1$

(ii) solve the equation $\sin^{-1}x-\cos^{-1}x=\sin^{-1}(1-x)$

[5 marks]

(c) The equation of a curve is $x^2y^2-x^2+y^2=0$.

(i) Show that the numerical value of y is always less than 1.

(ii) Find the equations of the asymptotes.

(iii) Show that $\dfrac{dy}{dx}=\dfrac{y^3}{x^3}$

(iv) Sketch the curve.

[7 marks]

STANDARD INTEGRALS

$$\int x^n\,dx=\frac{1}{n+1}x^{n+1} \qquad (n\neq -1;\ x\neq 0\ if\ n<0)$$

$$\int \frac{1}{x}\,dx=\log_e x \qquad (x>0)$$

$$\int e^{ax}\,dx=\frac{1}{a}e^{ax} \qquad (a\neq 0)$$

$$\int \cos ax\,dx=\frac{1}{a}\sin ax \qquad (a\neq 0)$$

$$\int \sin ax\,dx=-\frac{1}{a}\cos ax \qquad (a\neq 0)$$

$$\int \sec^2 ax\,dx=\frac{1}{a}\tan ax \qquad (a\neq 0)$$

$$\int \sec ax\tan ax\,dx=\frac{1}{a}\sec ax \qquad (a\neq 0)$$

$$\int \frac{1}{a^2+x^2}\,dx=\frac{1}{a}\tan^{-1}\frac{x}{a} \qquad (a\neq 0)$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx=\sin^{-1}\frac{x}{a} \qquad (a>0,\ -a<x<a)$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx=\log_e\left\{x+\sqrt{x^2-a^2}\right\} \qquad (|x|>|a|)$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx=\log_e\left\{x+\sqrt{x^2+a^2}\right\}$$

4UNIT CRANBROOK TRIAL 2001 SOLUTIONS

1 (a) (i) $I = \int \cot x \operatorname{cosec}^2 x \, dx$

let $u = \cot x$

$\therefore \frac{du}{dx} = -\operatorname{cosec}^2 x$

$\therefore I = \int u \cdot -du$

$= \frac{-u^2}{2} + c$

$= \frac{-\cot^2 x}{2} + c$

(ii) $I = \int \frac{\sec^2 x}{3 - \tan x} \, dx$

let $u = 3 - \tan x$

$\frac{du}{dx} = -\sec^2 x$

$\therefore I = \int \frac{-du}{u}$

$= -\ln|u| + c$

$= -\ln|3 - \tan x| + c$

(b) $I = \int_{5\frac{1}{2}}^{6\frac{1}{2}} \frac{dx}{\sqrt{(x-5)(7-x)}}$

let $u = x - 6$ when $x = 5\frac{1}{2}$ $u = -\frac{1}{2}$

$\therefore \frac{du}{dx} = 1$ $x = 6\frac{1}{2}$ $u = \frac{1}{2}$

$\therefore I = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{du}{\sqrt{(u+1)(1-u)}}$

$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{du}{\sqrt{1-u^2}}$

$= 2\int_0^{\frac{1}{2}} \frac{du}{\sqrt{1-u^2}}$

$= 2\left[\sin^{-1} u\right]_0^{\frac{1}{2}}$

$= 2\left[\frac{\pi}{6} - 0\right] = \frac{\pi}{3}.$

(c) (i) TO PROVE: $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

PROOF: LHS $= \int_0^a f(x)\,dx$

let $x = a - u$ when $x = a$ $u = 0$

$\therefore \frac{dx}{du} = -1$ $x = 0$ $u = a$

$\therefore$ LHS $= \int_a^0 f(a-u) \cdot -du$

$= \int_0^a f(a-u) \cdot du$

$= \int_0^a f(a-x)\,dx$ (reverting to the variable x)

$=$ RHS.

(ii) $I = \int_0^{\frac{\pi}{2}} \frac{\cos^5 x \, dx}{\cos^5 x + \sin^5 x}$

$= \int_0^{\frac{\pi}{2}} \frac{\cos^5(\frac{\pi}{2} - x) \, dx}{\cos^5(\frac{\pi}{2} - x) + \sin^5(\frac{\pi}{2} - x)}$

$= \int_0^{\frac{\pi}{2}} \frac{\sin^5 x \, dx}{\sin^5 x + \cos^5 x}$

$\therefore 2I = \int_0^{\frac{\pi}{2}} \frac{\cos^5 x + \sin^5 x}{\sin^5 x + \cos^5 x} \, dx$

$= \int_0^{\frac{\pi}{2}} 1 \, dx$

$= \left[x\right]_0^{\frac{\pi}{2}}$

$= \left[\frac{\pi}{2} - 0\right]$

$= \frac{\pi}{2}$

$\therefore I = \frac{\pi}{4}.$

(d) $I = \int_0^{\frac{\pi}{4}} \frac{dx}{2\sin 2x + \cos x}$

$= \int_0^{\frac{\pi}{4}} \frac{dx}{4\sin x \cos x + \cos x}$

$= \int_0^{\frac{\pi}{4}} \frac{dx}{\cos x(4\sin x + 1)}$

let $t = \tan\frac{x}{2}$, $x \neq \pi$

$\therefore \cos x = \frac{1-t^2}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$

$dx = \frac{2\,dt}{1+t^2}$ when $x = 0$ $t = 0$

$x = \frac{\pi}{4}$ $t = \tan\frac{\pi}{8}$

$\therefore I = \int_0^{\tan\frac{\pi}{8}} \frac{\frac{2\,dt}{1+t^2}}{\left(\frac{1-t^2}{1+t^2}\right)\left(\frac{8t}{1+t^2} + 1\right)}$

$= \int_0^{\tan\frac{\pi}{8}} \frac{2(1+t^2)\,dt}{(1-t^2)(t^2+8t+1)}$

By partial fractions:

$\frac{2(1+t^2)}{(1-t)(1+t)(t^2+8t+1)} = \frac{A}{1-t} + \frac{B}{1+t} + \frac{Ct+D}{t^2+8t+1}$

$\therefore 2 + 2t^2 = A(1+t)(t^2+8t+1)$
$+ B(1-t)(t^2+8t+1)$
$+ (Ct+D)(1-t^2)$

let $t = -1$ $\therefore 4 = -12B$ $\therefore B = -\frac{1}{3}$

$t = 1$ $\therefore 4 = 12A$ $\therefore A = \frac{1}{3}$

$t = 0$ $\therefore 2 = A + B + D$ $\therefore D = 2$

$t = 2$ $\therefore 10 = 63A - 21B - 6C - 3D$

$\therefore C = 2$

$\therefore I = \int_0^{\tan\frac{\pi}{8}} \frac{\frac{1}{3}}{1-t} - \frac{\frac{1}{3}}{1+t} + \frac{2t+2}{t^2+8t+1} \, dt$

$= \int_0^{\tan\frac{\pi}{8}} \frac{\frac{1}{3}}{1-t} - \frac{\frac{1}{3}}{1+t} + \frac{(2t+8)-6}{t^2+8t+1} \, dt$

$= \int^{\tan\frac{\pi}{8}} \frac{1}{3} \quad -\frac{1}{3} \quad +\frac{2t+8}{} \quad \frac{-6}{} \, dt$

$= \left[-\frac{1}{3}\ln|1-t| - \frac{1}{3}\ln|1+t| + \ln|t^2+8t+1| - \frac{6}{2\sqrt{15}} \ln\left|\frac{t+4-\sqrt{15}}{t+4+\sqrt{15}}\right|\right]_0^{\tan\frac{\pi}{8}}$

$= \left[-\frac{1}{3}\ln|1-\tan\frac{\pi}{8}| - \frac{1}{3}\ln|1+\tan\frac{\pi}{8}| + \ln|\tan^2\frac{\pi}{8} + 8\tan\frac{\pi}{8} + 1| - \frac{3}{\sqrt{15}}\ln\left|\frac{\tan\frac{\pi}{8}+4-\sqrt{15}}{\tan\frac{\pi}{8}+4+\sqrt{15}}\right| + \frac{3}{\sqrt{15}}\ln\left|\frac{4-\sqrt{15}}{4+\sqrt{15}}\right|\right]$

$= \ln(\tan^2\frac{\pi}{8} + 8\tan\frac{\pi}{8} + 1) + \frac{3}{\sqrt{15}}\ln\left|\frac{4-\sqrt{15}}{4+\sqrt{15}}\right|$
$-\frac{1}{3}\ln|1-\tan^2\frac{\pi}{8}|.$

2 (a) $I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x^3}{\cos x} \, dx$

Let $f(x) = x^3$, $g(x) = \cos x$

Now as $f(x)$ is odd and $g(x)$ is even then $\frac{f(x)}{g(x)}$ is an odd function. (odd fn $\div$ even fn is odd fn.)

$\therefore I = 0$ (the integration of an odd function about symmetrical limits is zero.)

(b) $I = \int \sin^3 2x \cos^2 2x \, dx$

$= \int \sin 2x(1 - \cos^2 2x)\cos^2 2x \, d$

let $u = \cos 2x$

$\therefore \frac{du}{dx} = -2\sin 2x$

$$I = -\frac{1}{2}\int (1-u^2)u^2\,du$$
$$= -\frac{1}{2}\int u^2 - u^4\,du$$
$$= -\frac{1}{2}\left[\frac{\cos^3 2x}{3} - \frac{\cos^5 2x}{5}\right] + c$$

(c) $$I = \int \frac{4x-3}{\sqrt{6+2x-3x^2}}\,dx$$
$$= \int \frac{-\frac{2}{3}(-6x+2) - \frac{5}{3}}{\sqrt{6+2x-3x^2}}\,dx$$
$$= -\frac{2}{3}\int \frac{-6x+2}{\sqrt{6+2x-3x^2}}\,dx - \frac{5}{3}\int \frac{dx}{\sqrt{-3(x^2-\frac{2}{3}x-2)}}$$

let $u = \sqrt{6+2x-3x^2}$

$$\therefore \frac{du}{dx} = \frac{\frac{1}{2}(-6x+2)}{\sqrt{6+2x-3x^2}}$$
$$= -\frac{2}{3}\cdot 2\int \frac{\frac{1}{2}(-6x+2)\,dx}{\sqrt{6+2x-3x^2}} - \frac{5}{3}\int \frac{dx}{\sqrt{-3[(x-\frac{1}{3})^2 - \frac{19}{9}]}}$$
$$= -\frac{4}{3}\sqrt{6+2x-3x^2} - \frac{5}{3}\int \frac{dx}{\sqrt{3}\sqrt{(\frac{\sqrt{19}}{3})^2 - (x-\frac{1}{3})^2}}$$
$$= -\frac{4}{3}\sqrt{6+2x-3x^2} \quad \frac{-5}{3\sqrt{3}}\sin^{-1}\left(\frac{x-\frac{1}{3}}{\sqrt{19}/3}\right) + c$$
$$= -\frac{4}{3}\sqrt{6+2x-3x^2} \quad \frac{-5}{3\sqrt{3}}\sin^{-1}\left(\frac{3x-1}{\sqrt{19}}\right) + c$$

(d) $$I_n = \int_0^{\frac{\pi}{2}} \cos^n x \sin^2 x\,dx, \; n \geq 0$$
$$= \int_0^{\frac{\pi}{2}} \cos^n x(1-\cos^2 x)\,dx$$
$$= \int_0^{\frac{\pi}{2}} \cos^n x - \cos^{n+2} x\,dx$$
$$= U_n - U_{n+2}$$

(where $U_n = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$)

Now for $U_n = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$

$$\therefore U_n = \int_0^{\frac{\pi}{2}} \cos^{n-1}x \cos x\,dx$$

let $u = \cos^{n-1}x$, $dv = \cos x\,dx$

$\therefore \frac{du}{dx} = (n-1)\cos^{n-2}x \cdot -\sin x \qquad v = \sin x$

$$\therefore U_n = \left[\cos^{n-1}x \sin x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2}x \sin^2 x\,dx$$
$$= 0 + (n-1)\,I_{n-2}$$

$\therefore U_n = (n-1)\,I_{n-2}$

Now as $I_n = U_n - U_{n+2}$

$$\therefore I_n = (n-1)I_{n-2} - (n+2-1)I_{(n+2-2)}$$
$$= (n-1)I_{n-2} - (n+1)I_n$$
$$\therefore I_n(1+n+1) = (n-1)I_{n-2}$$
$$\therefore I_n = \frac{n-1}{n+2}\,I_{n-2}$$

Now $$I_4 = \int_0^{\frac{\pi}{2}} \cos^4 x \sin^2 x\,dx$$
$$= \frac{4-1}{4+2}\,I_2 \quad \text{(using the above result)}$$
$$= \frac{3}{6}\cdot\frac{1}{4}\cdot I_0$$
$$= \frac{1}{8}\int_0^{\frac{\pi}{2}} \cos^0 x \sin^2 x\,dx$$
$$= \frac{1}{8}\int_0^{\frac{\pi}{2}} \frac{1}{2}[1-\cos 2x]\,dx$$
$$= \frac{1}{16}\left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$$
$$= \frac{1}{16}\left[\left(\frac{\pi}{2}-0\right) - 0\right]$$
$$= \frac{\pi}{32}.$$

15

3 (a) (i) $z_1 = 1-i \qquad |z_1| = \sqrt{2} \qquad \arg z_1 = -\frac{\pi}{4}$

$z_2 = -1+\sqrt{3}i \qquad |z_2| = 2 \qquad \arg z_2 = \frac{2\pi}{3}$

$\therefore |z_1 z_2| = \underline{2\sqrt{2}}$ 2

$\arg(z_1 z_2) = \underline{\frac{5\pi}{12}}$ 2

(ii) $$z_1 z_2 = (1-i)(-1+\sqrt{3}i)$$
$$= \underline{(\sqrt{3}-1) + i(\sqrt{3}+1)}$$ 1

$$\therefore (\sqrt{3}-1) + i(\sqrt{3}+1) = 2\sqrt{2}\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$$

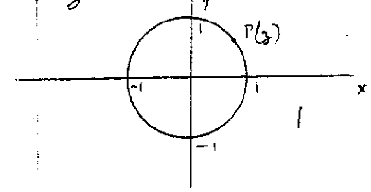
Equate real parts

$$\therefore \sqrt{3}-1 = 2\sqrt{2}\cos\frac{5\pi}{12}$$

ie $$\underline{\cos\frac{5\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}}$$ 1 5

(b) If $|z| = 1$

z lies on unit circle



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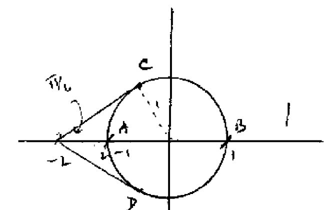

1

A gives min. value of $|z+2|$
B gives max. value of $|z+2|$ $\} \quad \underline{1 \leq |z+2| \leq 3}$ 1

C gives max. value of $\arg(z+2)$
D gives min value of $\arg(z+2)$ $\} \quad \underline{-\frac{\pi}{6} \leq \arg(z+2) \leq \frac{\pi}{6}}$ 1

(c) (i) If $z = x+iy$

$$x+iy + \frac{1}{x+iy} = x+iy + \frac{x-iy}{x^2+y^2} = \frac{x(x^2+y^2+1) + iy(x^2+y^2-1)}{x^2+y^2}$$

$\therefore$ If $z + \frac{1}{z}$ real $\qquad y(x^2+y^2-1) = 0$

$\therefore \underline{y=0}$ (real axis) or $\underline{x^2+y^2=1}$ (unit circle, centre the origin) 3

(ii) If $y = 0 \qquad z + \frac{1}{z} = \frac{x(x^2+1)}{x^2} = \frac{x^2+1}{x}$ and $\left|z+\frac{1}{z}\right| - 2 = \frac{x^2+1-2|x|}{|x|}$

$$= \frac{(|x|-1)^2}{|x|} \geq 0$$

$$\therefore \left|z+\frac{1}{z}\right| \geq 2$$

If $x^2+y^2=1 \qquad \left|z+\frac{1}{z}\right| = \frac{|2x|}{\cancel{1}} \leq 2$ since $\underline{|x| \leq 1}$

(a) $P(x) = x^5 + 2x^2 + ax + b$

$P'(x) = 5x^4 + 4x + a$ 1

$P(-1) = P'(-1) = 0$ $\therefore$ $1 - a + b = 0$ 1

$1 + a = 0$

$\therefore$ $\underline{\underline{a = -1, \; b = -2}}$ 1

(b) One root $= 1 - 2i$

Sum of roots $= -1 - i$ 1 $\therefore$ Other root $: -1 - i - 1 + 2i$

$= \underline{\underline{-2 + i}}$ 1

And product of roots

$b = (1-2i)(-2+i)$

$= -2 + 4i + i + 2 = \underline{\underline{5i}}$ 1

(c) $x^3 + 3px + q = 0$ $\therefore \alpha + \beta + \gamma = 0$ $\alpha\beta + \beta\gamma + \gamma\alpha = 3p$ $\alpha\beta\gamma = -q$ 1

Then $\dfrac{\alpha\beta}{\gamma} + \dfrac{\beta\gamma}{\alpha} + \dfrac{\gamma\alpha}{\beta} = \dfrac{\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2}{\alpha\beta\gamma}$

$= \dfrac{(\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)}{\alpha\beta\gamma} = \underline{\underline{-\dfrac{9p^2}{q}}}$ 2

$\dfrac{\alpha\beta}{\cancel{\gamma}} \cdot \dfrac{\beta\cancel{\gamma}}{\alpha} + \dfrac{\beta\gamma}{\cancel{\alpha}} \cdot \dfrac{\gamma\cancel{\alpha}}{\beta} + \dfrac{\gamma\alpha}{\cancel{\beta}} \cdot \dfrac{\alpha\cancel{\beta}}{\gamma} = \alpha^2 + \beta^2 + \gamma^2$

$= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) = \underline{\underline{-6p}}$ 2

$\dfrac{\alpha\beta}{\cancel{\gamma}} \cdot \dfrac{\beta\cancel{\gamma}}{\cancel{\alpha}} \cdot \dfrac{\gamma\alpha}{\beta} = \alpha\beta\gamma = \underline{\underline{-q}}$ 1

$\therefore$ Monic equation is $\underline{\underline{x^3 + \dfrac{9p^2}{q}x^2 - 6px + q = 0}}$ 1

If $\gamma = \alpha\beta$, one root of this equation is 1

$\therefore$ $1 + \dfrac{9p^2}{q} - 6p + q = 0$

i.e. $q + 9p^2 - 6pq + q^2 = 0$

i.e. $\underline{\underline{(3p - q)^2 + q = 0}}$ 2

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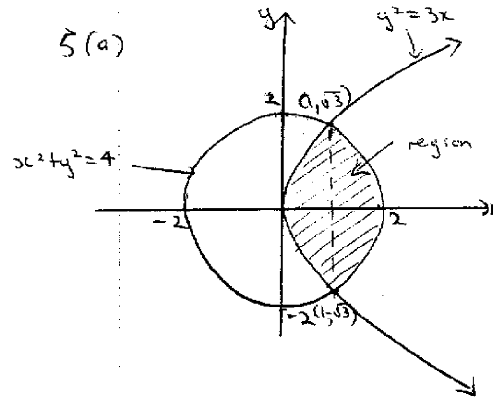
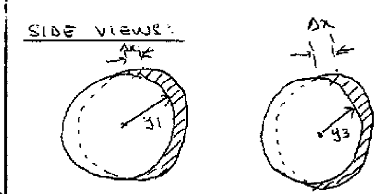
5 (a)





Now areas of the cross-sectional slices are: $A_1(x) = \pi y_1^2$

$= \pi(3x)$

$A_2(x) = \pi y_3^2$

$= \pi(4 - x^2)$

<u>For points of intersection:</u>

$y^2 = 3x$ — ①

$x^2 + y^2 = 4$ — ②

sub $y^2 = 3x$ into ②

$\therefore x^2 + 3x - 4 = 0$

$(x+4)(x-1) = 0$

$\therefore x = -4$ or 1

when $x = -4$ y does not exist

$x = 1$ $y = \pm\sqrt{3}$

$\therefore$ pts of intersection are $(1, \sqrt{3})$ and $(1, -\sqrt{3})$

Now volume, ΔV, of each slice

$= A(x)\,\Delta x$

$\therefore$ Total volume $= \lim\limits_{\Delta x \to 0} \left\{ \left[\sum\limits_{x=0}^{1} A_1(x) + \sum\limits_{x=1}^{2} A_2(x) \right] \Delta x \right\}$

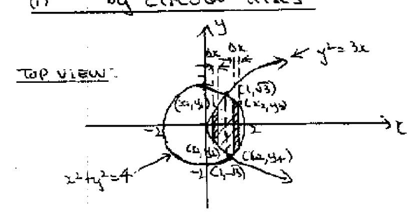
$= \pi \int_0^1 3x\,dx + \pi \int_1^2 4 - x^2\,dx$

$= \pi \left[\dfrac{3x^2}{2} \right]_0^1 + \pi \left[4x - \dfrac{x^3}{3} \right]_1^2$

$= \pi \left[\dfrac{3}{2} - 0 \right] + \pi \left[\left(8 - \dfrac{8}{3}\right) - \left(4 - \dfrac{1}{3}\right) \right]$

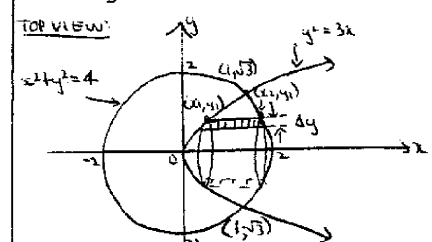
$= \dfrac{19\pi}{6}$ units^3

(i) <u>by circular discs</u>



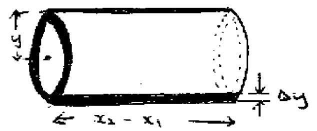
Take 2 slices of thickness Δx through the region as shown.

(ii) <u>by cylindrical shells.</u>



Take a slice of thickness Δy perpendicular to the y-axis

SIDE VIEW:



When the slice is rotated about the x-axis it generates a thin cylindrical shell of area $2\pi rh$.

Now area of shell, $A(y) = 2\pi y(x_2 - x_1)$

$$= 2\pi y\left(\sqrt{4-y^2} - \frac{y^2}{3}\right)$$

Now volume, ΔV, of each shell $= A(y)\Delta y$

$\therefore$ Total volume $= \lim_{\Delta y \to 0} \sum_{y=0}^{\sqrt3} 2\pi y\left(\sqrt{4-y^2} - \frac{y^2}{3}\right)$

$$= 2\pi \int_0^{\sqrt3} y\left(\sqrt{4-y^2} - \frac{y^2}{3}\right) dy$$

$$= 2\pi \int_0^{\sqrt3} y(4-y^2)^{\frac12} dy$$

$$- 2\pi \int_0^{\sqrt3} \frac{y^3}{3} dy$$

let $u = (4-y^2)^{3/2}$

$\therefore \frac{du}{dy} = \frac32(4-y^2)^{\frac12}\cdot -2y$

$= -3y(4-y^2)^{\frac12}$

$\therefore$ Total volume $= -\frac{2\pi}{3}\int_0^{\sqrt3} -3y(4-y^2)^{\frac12} dy$

$$-\frac{2\pi}{3}\int_0^{\sqrt3} y^3 dy$$

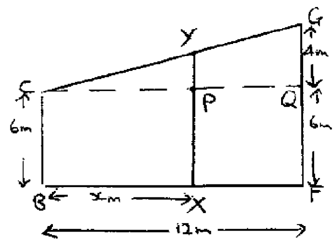
$$= -\frac{2\pi}{3}\left[(4-y^2)^{3/2}\right]_0^{\sqrt3} - \frac{2\pi}{3}\left[\frac{y^4}{4}\right]_0^{\sqrt3}$$

$$= -\frac{2\pi}{3}[1-8] - \frac{2\pi}{3}\left[\frac94 - 0\right]$$

$$= \frac{14\pi}{3} - \frac{3\pi}{2}$$

$$= \frac{19\pi}{6} \text{ units}^3$$

(b)



As $BC \parallel XY \parallel FG$

$\angle CPY = \angle CQG$ (corr. ∠s bounded by ∥ lines are equal)

similarly $\angle CYP = \angle CGQ$

$\angle C$ is common.

$\therefore \triangle CYP \,|||\, \triangle CGQ$ (Δs are equiangular)

$\therefore \frac{PY}{4} = \frac{x}{12}$ (corr. sides of similar Δs are in the same ratio)

$\therefore PY = \frac{x}{3}$

$\therefore XY = 6 + \frac{x}{3}$

$\Rightarrow$ at a distance x m from the base AB the area of the cross-section is $\left(6+\frac{x}{3}\right)^2$ m².

Now take slices of thickness Δx.

$\therefore$ volume, ΔV, of each slice $= A(x)\Delta x$

$\therefore$ Total volume $= \lim_{\Delta x \to 0} \sum_{x=0}^{12} A(x)\Delta x$

$$= \int_0^{12} \left(6+\frac{x}{3}\right)^2 dx$$

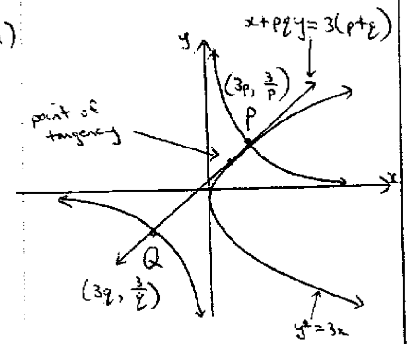
$$= \int_0^{12} 36 + 4x + \frac{x^2}{9}\, dx$$

$$= \left[36x + 2x^2 + \frac{x^3}{27}\right]_0^{12}$$

$$= \left[\left(432 + 288 + \frac{1728}{27}\right) - 0\right]$$

$$= 784 \text{ m}^3$$

6 (a)



(i) $N = \left(\frac{3p+3q}{2}, \frac{\frac3p + \frac3q}{2}\right)$

$$= \left(\frac{3(p+q)}{2}, \frac{3(p+q)}{2pq}\right)$$

(ii) $y^2 = 3x$

$\therefore 2y\frac{dy}{dx} = 3$

$\therefore \frac{dy}{dx} = \frac{3}{2y}$

Let parametric eq'ns of $y^2 = 3x$ be $y = t$, $x = \frac{t^2}{3}$

$\therefore$ At point of tangency $\frac{dy}{dx} = \frac{3}{2t}$

$\therefore$ Eq'n of tangent is:

$y - t = \frac{3}{2t}\left(x - \frac{t^2}{3}\right)$

$\therefore 2ty - 2t^2 = 3x - t^2$

$\therefore 3x - 2ty = -t^2$

$\therefore -\frac{2t}{3}y + x = -\frac{t^2}{3}$

But PQ is this tangent as well.

$\therefore -\frac{2t}{3} = pq$, $-\frac{t^2}{3} = 3(p+q)$

$\therefore \frac{3(p+q)}{2} = \frac{-t^2}{6}$

Now as $N = \left(\frac{3(p+q)}{2}, \frac{3(p+q)}{2pq}\right)$

$\therefore N = \left(\frac{-t^2}{6}, \frac{-t^2}{6}\cdot\frac{-3}{2t}\right)$

$= \left(\frac{-t^2}{6}, \frac{t}{4}\right)$

$\therefore x = \frac{-t^2}{6}$, $y = \frac{t}{4}$

$\therefore t = 4y$

$\therefore x = \frac{-(16y^2)}{6}$

$\therefore 6x = -16y^2$

$\therefore 3x = -8y^2$ is the locus of N.

(see over for alternative solution to (a)(ii))

(b) GIVEN: V is constant, $\frac{dr}{dt} = 0.05r$

TO FIND: $\frac{dh}{dt}$ as a %.

SOLUTION: As $V = \pi r^2 h$

$\therefore h = \frac{V}{\pi r^2} = \frac{V}{\pi}r^{-2}$

$\therefore \frac{dh}{dr} = \frac{-2V}{\pi}r^{-3}$

$= \frac{-2V}{\pi r^3}$

Now $\frac{dh}{dt} = \frac{dh}{dr} \cdot \frac{dr}{dt}$

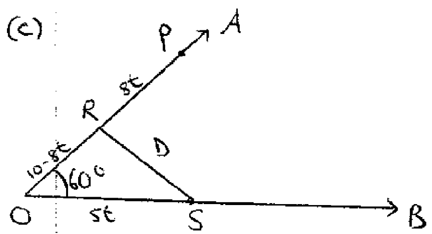
$\therefore \frac{dh}{dt} = \frac{-2V}{\pi r^3} \times 0\cdot05 r$

$= \frac{-2\pi r^2 h}{\pi r^3} \times 0\cdot05 r$

$= -0\cdot1 h$

$\Rightarrow$ height is diminishing at 10% per minute.

(c)



After t hours the cyclist has travelled $8t$ km from P to a point R and the jogger has travelled $5t$ km from O to a point S. Let $RS = D$.

Now by the cosine rule:

$D^2 = (10-8t)^2 + (5t)^2 - 2\times(10-8t)(5t)\cos 60°$

$= 100 - 160t + 64t^2 + 25t^2 - 50t + 40t^2$

$= 129t^2 - 210t + 100$

$\therefore 2D\frac{dD}{dt} = 258t - 210$

$\therefore \frac{dD}{dt} = \frac{129t - 105}{D}$

When $t = 1\frac{1}{2}$ [90 mins $= 1\frac{1}{2}$ hrs]

$D^2 = 129(1\frac{1}{2})^2 - 210(1\frac{1}{2}) + 100$

$= 75\cdot25$

$\therefore D = \sqrt{75\cdot25}$

$\therefore \frac{dD}{dt} = \frac{129(1\cdot5) - 105}{\sqrt{75\cdot25}}$

$= 10\cdot20211039\ldots$

$= 10\cdot20$ (2 d.p.)

$\Rightarrow$ distance between the cyclist and jogger is changing at approx $10\cdot20$ km/h after 90 mins.

(a)(ii) Alternative solution

As $x + pqy = 3(p+q)$ is a tangent to $y^2 = 3x$ then the quadratic equation formed by substituting $x = \frac{y^2}{3}$ into the equation of this tangent will have a discriminant of zero.

ie quadratic formed is: $\frac{y^2}{3} + pqy = 3(p+q)$

$\therefore y^2 + 3pqy - 9(p+q) = 0$

Now $\Delta = (3pq)^2 - 4.1.(-9(p+q))$

$= 9p^2q^2 + 36(p+q)$

But as $\Delta = 0$ $\therefore p^2q^2 = -4(p+q)$ — (1)

Now from the coords of N:

$y = \frac{3}{2}\left(\frac{p+q}{pq}\right)$

$\therefore y^2 = \frac{9}{4}\left(\frac{(p+q)^2}{p^2q^2}\right)$

$= \frac{9}{4}\left(\frac{(p+q)^2}{-4(p+q)}\right)$ (sub. (1))

$= \frac{-9}{16}(p+q)$ — (2)

But also from the coords of N $x = \frac{3(p+q)}{2}$

$\therefore p+q = \frac{2x}{3}$ sub into (2)

$\therefore y^2 = \frac{-9}{16}\left(\frac{2x}{3}\right)$

$= -\frac{3x}{8}$

$\Rightarrow$ locus of N is: $3x = -8y^2$.

7. (a) $4x^2 + 9y^2 = 36$

$8x + 18y\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = -\frac{4x}{9y}$

$4x^2 - y^2 = 4$

$8x - 2y\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = \frac{4x}{y}$

Solve simultaneously

$4x^2 + 9y^2 = 36$

$4x^2 - y^2 = 4$

$10y^2 = 32$

$y^2 = \frac{16}{5}$ $\quad y = \pm\frac{4}{\sqrt5}$

$4x^2 = 4 + \frac{16}{5}$

$x^2 = \frac{9}{5}$ $\quad x = \pm\frac{3}{\sqrt5}$

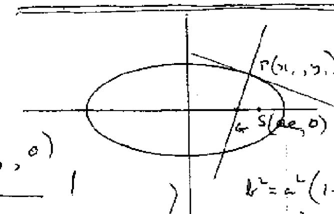
2

If x, y of same sign gradients are $-\frac{4}{9}\cdot\frac{3}{\sqrt5}\cdot\frac{\sqrt5}{4} = -\frac{1}{3}$ and $\frac{3}{\sqrt5}\cdot\frac{\sqrt5}{4}\cdot 4 = 3$

..... opposite signs .. .. $\frac{1}{3}$ and -3

$\therefore$ Ellipse and Hyperbola intersect at right angle

(b) $a^2y_1x - b^2x_1y = (a^2 - b^2)x_1y_1$



For G cut $y = 0$

$\therefore$ G is $\left(\frac{a^2-b^2}{a^2}x_1, 0\right) = (e^2x_1, 0)$

$b^2 = a^2(1-e^2)$

$= a^2 - a^2e^2$

$a^2e^2 = a^2 - b^2$

$\therefore GS = |e(a - ex_1)|$

$PS = \sqrt{(x_1 - ae)^2 + y_1^2}$

$= \sqrt{x_1^2 - 2aex_1 + a^2e^2 + \frac{a^2b^2 - b^2x_1^2}{a^2}}$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$

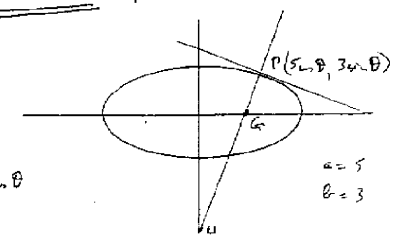
$= \frac{b^2(a^2 - x^2)}{a^2}$

$= \frac{1}{a}\sqrt{x_1^2(a^2-b^2) - 2a^3ex_1 + a^2(a^2e^2 + b^2)}$

$= \frac{1}{a}\sqrt{a^2e^2x_1^2 - 2a^3ex_1 + a^4}$ $= \frac{1}{a}\cdot a\sqrt{e^2x_1^2 - 2aex_1 + a^2}$

$= |a - ex_1|$

$\therefore GS = e \times PS$



(c) Normal is

$75\sin\theta . x - 45\cos\theta . y = 16 \times 15\sin\theta\cos\theta$

ie. $5\sin\theta . x - 3\cos\theta . y = 16\sin\theta\cos\theta$

$a = 5$

$b = 3$

$\therefore$ G is $\left(\frac{16\cos\theta}{5}, 0\right)$ | H is $\left(0, -\frac{16\sin\theta}{3}\right)$

$\therefore$ Midpoint of GH $= \left(\frac{8\cos\theta}{5}, -\frac{8}{3}\sin\theta\right)$ |

$\therefore$ satisfies
$$\begin{cases} x = \frac{8\cos\theta}{5} & \cos\theta = \frac{5x}{8} \\ y = -\frac{8}{3}\sin\theta & \sin\theta = -\frac{3y}{8} \end{cases}$$

ie. $\frac{25x^2}{64} + \frac{9y^2}{64} = 1$

ie. $\frac{x^2}{64/25} + \frac{y^2}{64/9} = 1$ |

ie. lies on ellipse
with e given by $\frac{64}{25} = \frac{64}{9}(1-e^2)$

$\frac{9}{25} = 1 - e^2$

$e^2 = 1 - \frac{9}{25} = \frac{16}{25}$

$\therefore$ $e = 4/5$ |

And original eccentricity given by
$9 = 25(1-e^2)$

$\frac{9}{25} = 1 - e^2$ $\therefore$ $e = \frac{4}{5}$

ie. Same eccentricity |

8. (a)

Smiths	Browns	Others	No. of ways
0	0	11	$0 = 0$
0	1	10	${}^2C_0 \times {}^3C_1 \times {}^{10}C_{10} = 3$
0	2	9	${}^2C_0 \times {}^3C_2 \times {}^{10}C_9 = 30$
1	0	10	${}^2C_1 \times {}^3C_0 \times {}^{10}C_{10} = 2$
1	1	9	${}^2C_1 \times {}^3C_1 \times {}^{10}C_9 = 60$
1	2	8	${}^2C_1 \times {}^3C_2 \times {}^{10}C_8 = 270$

Total no of ways $= 365$ |

(b) Let $\sin^{-1}x = \alpha$ $\cos^{-1}x = \beta$

$\therefore$ $x = \sin\alpha$ $x = \cos\beta$

(i) $\sin(\sin^{-1}x - \cos^{-1}x) = \sin(\alpha - \beta)$

$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= x \cdot x - \sqrt{1-x^2} \cdot \sqrt{1-x^2}$

$= x^2 - (1-x^2) = 2x^2 - 1$ 2

(ii) If $\sin^{-1}x - \cos^{-1}x = \sin^{-1}(1-x)$

$\sin(\sin^{-1}x - \cos^{-1}x) = 1 - x$

$\therefore$ $2x^2 - 1 = 1 - x$

$\therefore$ $2x^2 + x - 2 = 0$

$x = \frac{-1 \pm \sqrt{17}}{4}$ 2

But $\sin^{-1}x$ acute $\therefore$ x +ve

$\therefore$ $x = \frac{-1+\sqrt{17}}{4}$ |

(c) $x^2y^2 - x^2 + y^2 = 0$

(i) $y^2(x^2+1) = x^2$ |

$\therefore$ $y^2 = \frac{x^2}{x^2+1}$

$\therefore$ $0 \le y^2 \le 1$

$\therefore$ $0 \le |y| \le 1$ |

(ii) As $x \to \pm\infty$ $y^2 \to 1^-$ $\therefore y \to 1^-$ or $y \to -1^+$

$\therefore$ Asymptotes are $y = \pm 1$ 1

(iii) $x^2 \cdot 2y\frac{dy}{dx} + y^2 \cdot 2x - 2x + 2y\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx}(2x^2y + 2y) = 2x - 2xy^2$

$\therefore \frac{dy}{dx} = \frac{\cancel{2}x(1-y^2)}{\cancel{2}y(x^2+1)}$ 2

But $y^2(x^2+1) = x^2$ $\therefore y(x^2+1) = x^2/y$

and $x^2(1-y^2) = y^2$ $\therefore x(1-y^2) = y^2/x$

$\therefore \frac{dy}{dx} = \frac{y^2}{x} \div \frac{x^2}{y}$

$= \frac{y^3}{x^3}$ 1

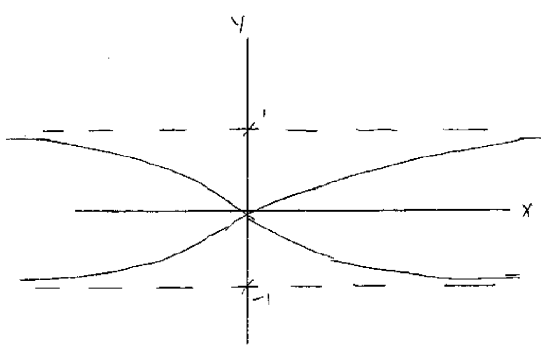
(iv)

Note

Even in x and y

$\therefore$ Symmetry about both axes

As $x \to \pm\infty$ $\frac{dy}{dx} \to 0$



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