CRANBROOK SCHOOL

YEAR 12 MATHEMATICS – EXTENSION 2

Term 1 2002 **Time : 3 h / SKB and MJB**

All questions are of equal value.
All necessary working should be shown in every question.
Full marks may not be awarded if work is careless or badly arranged.
Approved silent calculators may be used.
Standard Integrals appear at the end of the paper.

Submit your work for Question 1 in a 4 page booklet.
Submit your work for Question 2 in a 4 page booklet.
Submit your work for Questions 3 and 4 in the same 8 page booklet.
Submit your work for Questions 5 and 6 in the same 8 page booklet.
Submit your work for Questions 7 and 8 in the same 8 page booklet.

1. (15 Marks) (Begin a new 4 page booklet.) skb

(a) The complex number z is given by $z = \sqrt{3} + \dfrac{1+i}{1-i}$.

Find: (i) $\mathrm{Re}(z)$
(ii) $\mathrm{Im}(z)$
(iii) $|z|$
(iv) $\arg z$ 4

(b) If $z = \cos\theta + i\sin\theta$, show that $\dfrac{1}{1+z} = \dfrac{1}{2}(1 - i\tan\dfrac{\theta}{2})$ 4

(c) (i) On an Argand Diagram, shade in the region for which $0 \le |z| \le 2$ and $1 \le \mathrm{Im}\, z \le 2$. 2

(ii) What is the complex number with the largest argument which satisfies the inequalities of (i)? 1

(d) If $z_1 = 1 + \sqrt{3}\, i$ and $z_2 = \sqrt{3} - i$ show that:

(i) $\dfrac{(z_1)^{10}}{(z_2)^8} = -2 + 2\sqrt{3}\, i$ 2

(ii) $\left(\dfrac{z_1}{z_2}\right)^{79} = -i$ 2

2. **(15 marks) (Begin a 4 page booklet.)** **mjb**

(a) Prove that if $P(z) = z^4 + 2z^2 + 1$ has two double roots find these roots and factorise the polynomial $P(z)$ over the complex number field. **3**

(b) It is known that $P(x) = x^4 + 2x^3 + x^2 - 1$ has a zero, $x = \dfrac{-1 + i\sqrt{3}}{2}$.

Find all the other zeros of $P(x)$. **3**

(c) Find the sixth-degree polynomial $P(x)$ and the constant A such that $x^4(1-x)^4 \equiv (1+x^2)P(x) + A$. **3**

(d) **(i)** If $P(x) = x^3 - 9x^2 + 24x + c$ for some real number c, find the values of x for which $P'(x) = 0$. Hence find the two values of c for which the equation $P(x) = 0$ has a repeated root. **3**

(ii) Sketch the graphs of $y = P(x)$ for these values of c. Hence write down the values of c for which the equation $P(x) = 0$ has three distinct real roots. **3**

3. **(15 marks) (Begin a 8 page booklet for Questions 3 and 4.)** **skb**

(a) If $P(x_1, y_1)$ is any point on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and $S(ae, 0)$ and $S'(-ae, 0)$ are the foci prove that $|PS'| - |PS| = 2a$. **3**

(b) For the ellipse $x^2 + 4y^2 = 100$:

(i) find the eccentricity and the co-ordinates of the foci; **2**

(ii) find the equation of the tangent at $P(8,3)$ in general form. **2**

(iii) If the normal at P meets the major axis at $G(6,0)$ and the perpendicular from the centre O to the tangent at P meets that tangent at K, prove that PG.OK is equal to the square of the semi minor-axis. [Include a labelled diagram with your answer.] **4**

(c) The condition for the line $L: \ y = mx + c$ to touch the ellipse $E: \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is given by $c^2 = b^2 + a^2m^2$.

(i) From a variable point $T(X,Y)$ on L, tangents are drawn to E. Show that $(Y - mX)^2 = b^2 + a^2m^2$. **1**

(ii) If these tangents are at right angles to one another, prove that T lies on the circle $x^2 + y^2 = a^2 + b^2$. **3**

4. (15 Marks) **skb**

(a) Find : $\displaystyle\int \frac{dx}{\sqrt{6 - x - x^2}}$ **3**

(b) Evaluate in exact form : $\displaystyle\int_1^e x^3 \log_e x \, dx$ **3**

(c) (i) By using the substitution $t = a - x$, prove that:
$\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$ **2**

(ii) **Hence** prove that : $\displaystyle\int_0^3 x(3-x)^{11}\,dx = \frac{3^{12}}{52}.$ **3**

(d) Find : $\displaystyle\int \cos^3\theta \sqrt{1 - \sin\theta}\; d\theta$ **4**

5. (15 marks) (Begin a new 8 page booklet for Questions 5 and 6.) **skb**

(a) (i) Find real numbers a and b such that :
$$\frac{1}{(2t-1)(t+2)} = \frac{a}{2t-1} + \frac{b}{t+2}$$ **2**

(ii) By using the substitution $t = \tan\dfrac{\theta}{2}$ evaluate :
$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{3\sin\theta - 4\cos\theta}$$ **6**

(b) Evaluate : $\displaystyle\int_{-1}^{1} \frac{\tan^{-1} x}{1 + \sin^2 x}\,dx$ **2**

(c) It is given that if $I_n = \int \cos^{n-1}x \sin nx \, dx$ and $n \geq 1$ then :

$I_n = \frac{1}{2n-1}((n-1)I_{n-1} - \cos^{n-1} x \cos nx)$.

Use this reduction formula to show that:

$\int_0^{\frac{\pi}{4}} \cos^2 x \sin 3x \, dx = \frac{1}{60}(28 - \sqrt{2})$. **5**

6. (15 Marks) **skb**

(a) A solid is built on the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ such that cross-sectional slices perpendicular to the $y-$axis are equilateral triangles with one side lying in the base of the solid. Find the exact volume of this solid. **4**

(b) The region bounded by the curve $y = x^2 + 1$ and the line $y = 3 - x$ is rotated about the x-axis. By considering slices perpendicular to the $x-$axis , prove that the volume generated is $\frac{117\pi}{5}$ units^3. **5**

(c) **(i)** The circle $(x-3)^2 + y^2 = 4$ is rotated about the $y-$axis. By using the method of cylindrical shells show that the volume, V is given by : $V = 4\pi \int_1^5 x\sqrt{4-(x-3)^2} \, dx$ **3**

(ii) Hence show that the volume is $24\pi^2$ units^3. **3**

7. (15 marks) (Begin a new 8 page booklet for Questions 7 and 8.) **skb**

(a) Solve for x: $\frac{x+1}{(x-1)(x+2)} \geq 3$ **5**

(b) P, Q and R are the vertices of an equilateral triangle. $P = (-3, 2)$ and $Q = (3, -2)$ and OR is the perpendicular bisector of PQ where O is the origin.

(i) If m is the gradient of RQ show that the acute angle between PQ and RQ is given by $\tan 60^0 = \left| \frac{m + \frac{2}{3}}{1 - \frac{2m}{3}} \right|$ **1**

(ii) By taking the positive absolute value case show that

$$m = \frac{3\sqrt{3}-2}{3+2\sqrt{3}}$$ 1

(iii) **Hence** or **otherwise** find the possible co-ordinates of R. 3

(c) Consider a pack of 50 playing cards which consists of 5 colours {Yellow, Green, Blue, Indigo and Violet} containing cards numbered from 1 to 10 inclusive, respectively. A joker is added to the pack. The joker can stand for any card and when there are equal numbers of different cards it takes the value of the higher card. Otherwise, the joker stands for the card which is occurring most often.
e.g. two 4's, two 6's and a joker = two 4's and three 6's;
two 4's, one 6, one 8 and a joker = three 4's, one 6 and one 8.

If five cards are dealt to a player, determine the probability (leaving your answer as a fraction in its simplest form) that the player has received :

(i) four 10's. 2

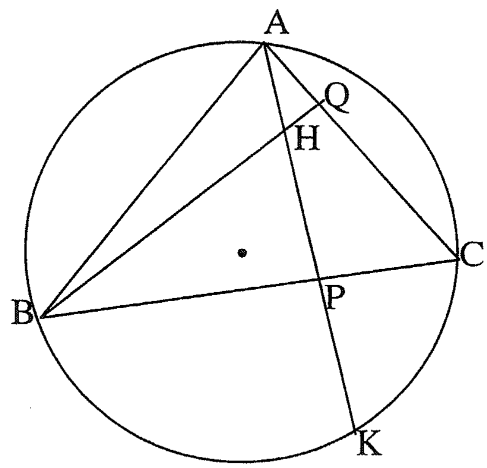
(ii) any three of one number and any two of another number e.g. three 10's and two 8's. 3

8. (15 Marks) **skb**

(a) (i) On a particular July day with a strong off shore easterly wind blowing and a large sea swell, the motion of a buoy situated off Bondi Beach can be considered to be simple harmonic. If the motion of the buoy is given by : $\ddot{x} = -n^2(x-b)$, where $x = b$ is the centre of motion and 'n' is a positive constant show that $x = b + a\cos nt$ (where 'a' is the amplitude) satisfies the buoy's motion. 2

(ii) The crest height of the buoy is 25m above sea level and the trough height is 20m above sea level. At one instant the buoy is at a crest position and 15 seconds later it is at a trough position. If initially the buoy is at a crest position, how long after this **on the first two occasions**, will the buoy be 24m above sea level? Leave your answers in seconds correct to 2 decimal places. 5

(b)



ABC is a triangle inscribed in a circle as shown. AP and BQ are altitudes meeting at H. AP produced cuts the circle at K. Prove that HP = KP. **5**

(c) Find the general solutions of $\tan 3\theta - \cot 5\theta = 0$ **3**

END OF EXAMINATION.

HMX2 2002 TRIAL SOLUTIONS.

(a) $z = \sqrt{3} + \frac{1+i}{1-i} \times \frac{1+i}{1+i}$

$= \sqrt{3} + \frac{1+2i-1}{2}$

$= \sqrt{3} + i$

(i) $Re(z) = \sqrt{3}$ ✓

(ii) $Im(z) = 1$ ✓

(iii) $|z| = \sqrt{(\sqrt{3})^2 + 1^2} = 2$ ✓

(iv) $\arg z = \frac{\pi}{6}$ ✓

(b) $LHS = \frac{1}{1+z}$

$= \frac{1}{1+\cos\theta + i\sin\theta} \times \frac{1+\cos\theta - i\sin\theta}{1+\cos\theta - i\sin\theta}$ ✓

$= \frac{1+\cos\theta - i\sin\theta}{(1+\cos\theta)^2 + \sin^2\theta}$

$= \frac{1+\cos\theta - i\sin\theta}{1+2\cos\theta + \cos^2\theta + \sin^2\theta}$

$= \frac{1+\cos\theta - i\sin\theta}{2(1+\cos\theta)}$ ✓

$= \frac{1}{2}\left[1 - \frac{i\sin\theta}{1+\cos\theta}\right]$

$= \frac{1}{2}\left[1 - \frac{i\left(\frac{2t}{1+t^2}\right)}{1+\frac{1-t^2}{1+t^2}}\right]$ if $t = \tan\frac{\theta}{2}$ ✓

$= \frac{1}{2}\left[1 - \frac{i\,2t}{1+t^2+1-t^2}\right]$

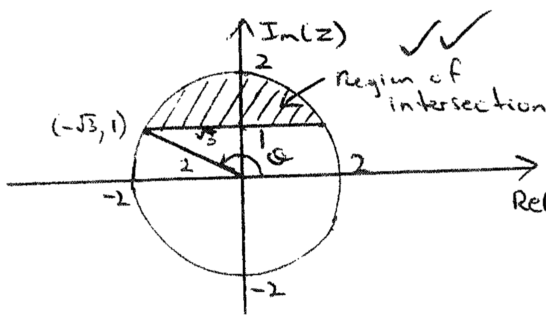
$= \frac{1}{2}\left[1 - \frac{i\,2t}{2}\right]$

$= \frac{1}{2}[1 - it]$

$= \frac{1}{2}\left[1 - i\tan\frac{\theta}{2}\right]$ ✓

$= RHS.$

(c) (i) ✓✓



(d) $z_1 = 1+\sqrt{3}i \quad \therefore \arg z_1 = \frac{\pi}{3}, |z_1| = 2$

$z_2 = \sqrt{3} - i \quad \therefore \arg z_2 = -\frac{\pi}{6}, |z_2| = 2$

(i) $\frac{(z_1)^{10}}{(z_2)^8} = \frac{\left(2\,\text{cis}\frac{\pi}{3}\right)^{10}}{\left(2\,\text{cis}\left(-\frac{\pi}{6}\right)\right)^8} = \frac{2^{10}\,\text{cis}\frac{10\pi}{3}}{2^8\,\text{cis}\left(-\frac{4\pi}{3}\right)}$ ✓

$\therefore \frac{(z_1)^{10}}{(z_2)^8} = 2^2\,\text{cis}\left(\frac{14\pi}{3}\right) = 2^2\,\text{cis}\left(\frac{2\pi}{3}\right)$

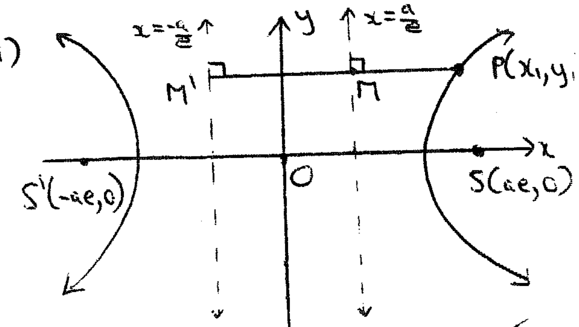
$\therefore \frac{(z_1)^{10}}{(z_2)^8} = 4\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -2+2\sqrt{3}i$ ✓

(ii) $\left(\frac{z_1}{z_2}\right)^{79} = \left(\frac{2\,\text{cis}\frac{\pi}{3}}{2\,\text{cis}\left(-\frac{\pi}{6}\right)}\right)^{79} = \left(\text{cis}\frac{\pi}{2}\right)^{79}$ ✓

$\therefore \left(\frac{z_1}{z_2}\right)^{79} = i^{79} = (i^4)^{19} \cdot i^3 = 1^{19} \cdot -i$

$= -i$ ✓

3. (a)



✓

Now from the definition of a hyperbola:

$\frac{|PS'|}{|PM'|} = e = \frac{|PS|}{|PM|}$

$\therefore |PS'| - |PS| = e\,|PM' - PM|$ ✓

$= e\left|x_1 + \frac{a}{e} - \left(x_1 - \frac{a}{e}\right)\right|$

$= e\left|\frac{2a}{e}\right|$

$= 2a.$ ✓

(b) $x^2 + 4y^2 = 100$

$\therefore \frac{x^2}{100} + \frac{y^2}{25} = 1 \quad \Rightarrow a = 10, b = 5$

(i) For eccentricity: $b^2 = a^2(1-e^2)$ ✓

$\therefore 25 = 100(1-e^2) \quad \therefore e^2 = \frac{3}{4} \quad \therefore e = \frac{\sqrt{3}}{2}$

$(0 < e < 1)$

Foci $(\pm ae, 0) = (\pm 5\sqrt{3}, 0)$ ✓

a i) $P(z) = z^4 + 2z^2 + 1$ has double roots it shares two roots with $P'(z)$. 1

$P'(z) = 4z^3 + 4z$ has single roots.

$P'(z) = 0 \Rightarrow 4z(z^2+1) = 0 \quad \therefore z = 0, \; z = \pm i \qquad P(0) = 1 \neq 0$ not root.

$P(i) = i^4 + 2i^2 + 1 = 1 - 2 + 1 = 0 \quad \therefore z = i$ is a double root. 1

$P(-i) = (-i)^4 + 2(-i)^2 + 1 = 0 \quad \therefore z = -i$ is the other one.

$\therefore P(z) = (z-i)^2(z+i)^2$ over C. 1

b) If $\frac{-1+i\sqrt{3}}{2}$ is a zero of $P(x)$ then so is $\frac{-1-i\sqrt{3}}{2}$ by part (a) above.

Let $z_1 = \frac{-1+i\sqrt{3}}{2}$, $z_2 = \frac{-1-i\sqrt{3}}{2}$ then

$x^2 - (z_1+z_2)x + z_1 z_2 = 0$

i.e. $x^2 + x + 1 = 0$

$$\begin{array}{r} x^2 + x - 1 \\ x^2+x+1 \,\big)\, \overline{x^4 + 2x^3 + x^2 + 0x - 1} \\ \underline{x^4 + x^3 + x^2} \\ x^3 + 0x^2 + 0x \\ \underline{x^3 + x^2 + x} \\ -x^2 - x - 1 \\ \underline{-x^2 - x - 1} \end{array}$$

$\therefore P(x) = (x^2+x+1)(x^2+x-1)$

For $x^2 + x - 1 = 0$, $x = \frac{1 \pm \sqrt{5}}{2}$

Thus there are 4 zeros of $P(x)$

$\frac{-1 \pm i\sqrt{3}}{2}$, $\frac{1 \pm \sqrt{5}}{2}$

3

c) $x^4(1-x)^4 = x^4(1 - 4x + 6x^2 - 4x^3 + x^4)$

$= x^4 - 4x^5 + 6x^6 - 4x^7 + x^8$

Let $P(x) = ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g$

$\therefore (1+x^2)P(x) + A = ax^8 + bx^7 + (a+c)x^6 + (b+d)x^5 + (c+e)x^4 + (d+f)x^3 + (e+g)x^2 + fx + g + A$

Equate coefficients: $\therefore a = 1, \; b = -4$

$a + c = 6 \Rightarrow c = 5$, $b + d = -4 \Rightarrow d = 0$

$c + e = 1 \Rightarrow e = -4$, $d + f = 0 \Rightarrow f = 0$, $e + g = 0 \Rightarrow g = 4$, $f = 0$

$A + g = 0 \Rightarrow A = -4$

$\therefore P(x) = x^6 - 4x^5 + 5x^4 - 4x^2 + 4$ and $A = -4$. 3

d i) $P(x) = x^3 - 9x^2 + 24x + c$

$P'(x) = 3x^2 - 18x + 24$

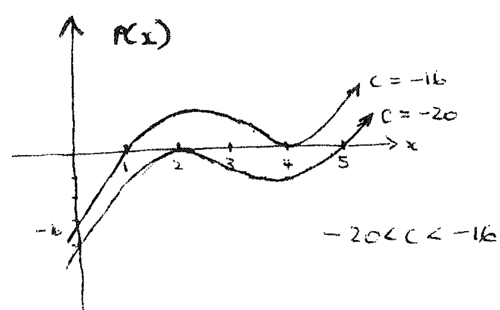
$= 3(x-4)(x-2) = 0$

$\therefore x = 4, 2$

For repeated roots:

$P(2) = 20 + c = 0 \quad \therefore c = -20$

$P(4) = 16 + c = 0 \quad c = -16$. 3



$-20 < c < -16$

d ii) $c = -20 \qquad P(x) = (x-2)^2(x-5)$

$\therefore 2x + 8y\frac{dy}{dx} = 0 \qquad \therefore \frac{dy}{dx} = \frac{-x}{4y}$

At $P(8,3)$ $\frac{dy}{dx} = \frac{-8}{12} = -\frac{2}{3} = m_{tangent}$ ✓

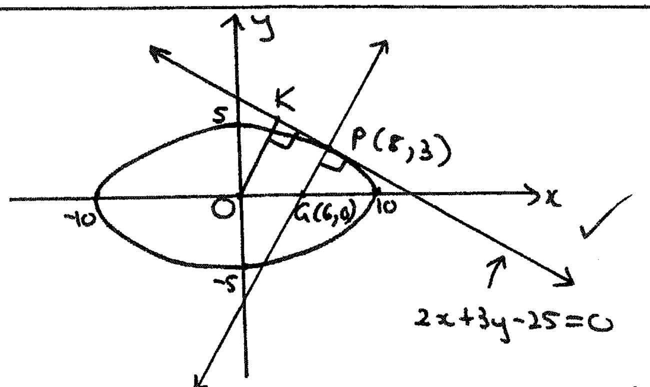
$\therefore$ Eqn of req'd tangent is:

$y - 3 = \frac{-2}{3}(x-8)$

$\therefore 3y - 9 = -2x + 16$

ie $2x + 3y - 25 = 0$ ✓

iii)



✓

Now $PG = \sqrt{(8-6)^2 + (3-0)^2} = \sqrt{13}$ units ✓

$OK = \frac{|2\times0 + 3\times0 - 25|}{\sqrt{2^2+3^2}} = \frac{25}{\sqrt{13}}$ units ✓

$\therefore PG.OK = \sqrt{13} \times \frac{25}{\sqrt{13}} = 25 = 5^2$ ✓

ie $PG.OK$ = square of semi-minor axis.

(c) If $L: y = mx + c$ touches

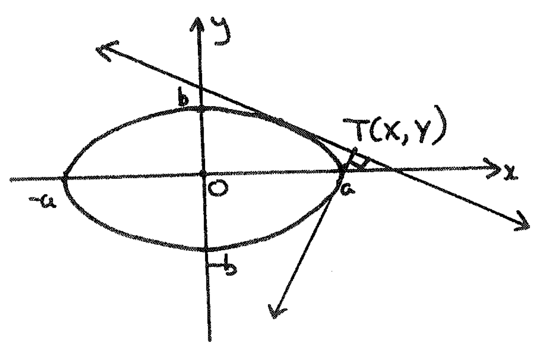
$E: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ then $c^2 = b^2 + a^2m^2$ —①

(i) If $T(X,Y)$ lies on L $\therefore Y = mX + c$

$\therefore c = Y - mX$ sub into ①

$\therefore (Y - mX)^2 = b^2 + a^2m^2$ ✓

(ii)



Now $Y^2 - 2mXY + m^2X^2 = b^2 + a^2m^2$

$\therefore m^2(X^2 - a^2) - 2mXY + (Y^2 - b^2) = 0$ —②

Now if the tangents drawn from T are at right angles $\Rightarrow$ product of gradients $= -1$. ✓

Letting m_1, m_2 be the roots of ② and also the gradients

$\Rightarrow m_1 m_2 = \frac{Y^2 - b^2}{X^2 - a^2} = -1$ ✓

$\therefore Y^2 - b^2 = -X^2 + a^2$

$\therefore X^2 + Y^2 = a^2 + b^2$

ie $T(X,Y)$ lies on the circle $x^2 + y^2 = a^2 + b^2$. ✓

4(a) $I = \int \frac{dx}{\sqrt{6 - x - x^2}}$

$= \int \frac{dx}{\sqrt{-[x^2 + x - 6]}}$ ✓

$= \int \frac{dx}{\sqrt{-[(x+\frac{1}{2})^2 - \frac{25}{4}]}}$

$= \int \frac{dx}{\sqrt{(\frac{5}{2})^2 - (x+\frac{1}{2})^2}}$ ✓

$= \sin^{-1}\left(\frac{x+\frac{1}{2}}{5/2}\right) + c$

$= \sin^{-1}\left(\frac{2x+1}{5}\right) + c$ ✓

(b) $I = \int_1^e x^3 \log_e x \, dx$

let $u = \log_e x$ $\quad \therefore \frac{du}{dx} = \frac{1}{x}$ ✓

$dv = x^3\,dx \qquad v = \frac{x^4}{4}$

$\therefore I = \left[\frac{x^4}{4}\log_e x\right]_1^e - \frac{1}{4}\int_1^e x^3 . dx$

$= \left[\frac{e^4}{4}.1 - 0\right] - \frac{1}{4}\left[\frac{x^4}{4}\right]_1^e$ ✓

$= \frac{e^4}{4} - \frac{1}{16}\left[e^4 - 1\right]$

$= \frac{1}{20}\left[3e^4 + 1\right]$ ✓

(c) (i) $LHS = \int_0 f(x)\,dx$

let $t = a - x \quad \therefore \frac{dt}{dx} = -1$

when $x = a \quad t = 0$

$x = 0 \quad t = a$

$$\therefore LHS = \int_a^0 f(a-t).-dt$$

$$= \int_0^a f(a-t)\,dt$$

$$= \int_0^a f(a-x)\,dx$$

(reverting to the variable x)

$= RHS.$

(ii) $$I = \int_0^3 x(3-x)^{11}\,dx$$

$$= \int_0^3 (3-x)(3-(3-x))^{11}\,dx$$

$$= \int_0^3 (3-x)x^{11}\,dx$$

$$= \int_0^3 3x^{11} - x^{12}\,dx$$

$$= \left[\frac{3x^{12}}{12} - \frac{x^{13}}{13}\right]_0^3$$

$$= \left[\frac{\cancel{3}^1 . 3^{12}}{\cancel{12}_4} - \frac{3^{13}}{13} - 0\right]$$

$$= \left[\frac{13.3^{12} - 4.3^{13}}{52}\right]$$

$$= \frac{3^{12}(13-12)}{52}$$

$$= \frac{3^{12}}{52}.$$

(d) $$I = \int \cos^3\theta\sqrt{1-\sin\theta}\,d\theta$$

$$= \int \cos\theta(1-\sin^2\theta)\sqrt{1-\sin\theta}\,d\theta$$

$$= \int \cos\theta\,(1-\sin\theta)^{3/2}(1+\sin\theta)\,d\theta$$

let $u = 1-\sin\theta$, $\sin\theta = 1-u$

$\therefore \frac{du}{d\theta} = -\cos\theta$

$$\therefore I = \int u^{3/2}(2-u).-du$$

$$\therefore I = \frac{2u^{7/2}}{7} - \frac{2.2u^{5/2}}{5} + c$$

$$= \frac{2}{7}(1-\sin\theta)^{7/2} - \frac{4}{5}(1-\sin\theta)^{5/2} + c$$

5 (a) (i) $$\frac{1}{(2t-1)(t+2)} = \frac{a}{2t-1} + \frac{b}{t+2}$$

$$= \frac{a(t+2)+b(2t-1)}{(2t-1)(t+2)}$$

$\therefore 1 = a(t+2) + b(2t-1)$

let $t = -2 \quad \therefore 1 = -5b \quad \therefore b = -\frac{1}{5}$

$t = \frac{1}{2} \quad \therefore 1 = \frac{5a}{2} \quad \therefore a = \frac{2}{5}$

(ii) $$I = \int_0^{\frac{\pi}{2}} \frac{d\theta}{3\sin\theta - 4\cos\theta}$$

$\boxed{\text{let } t = \tan\frac{\theta}{2} \quad \text{when } \theta = 0 \; t = 0 \quad \theta = \frac{\pi}{2} \; t = 1}$

$$= \int_0^1 \frac{\frac{2dt}{1+t^2}}{3\left(\frac{2t}{1+t^2}\right) - 4\left(\frac{1-t^2}{1+t^2}\right)}$$

$$= \int_0^1 \frac{2dt}{6t - 4 + 4t^2}$$

$$= \int_0^1 \frac{dt}{2t^2 + 3t - 2}$$

$$= \int_0^1 \frac{dt}{(2t-1)(t+2)}$$

$$= \left[\frac{1}{5}\ln|2t-1| - \frac{1}{5}\ln|t+2|\right]_0^1$$

$$= \left[\left(\frac{1}{5}\ln 1 - \frac{1}{5}\ln 3\right) - \left(\frac{1}{5}\ln|-1| - \frac{1}{5}\ln 2\right)\right]$$

$$= \frac{1}{5}\ln\frac{2}{3}$$

(b) $$I = \int_{-1}^1 \frac{\tan^{-1}x}{1+\sin^2 x}\,dx$$

let $$f(x) = \frac{\tan^{-1}x}{1+\sin^2 x}$$

$$\therefore f(-x) = \frac{\tan^{-1}(-x)}{1+\sin^2(-x)}$$

$$= \frac{-\tan^{-1}x}{1+\sin^2 x}$$

$$= -f(x)$$

$\therefore f(x)$ is an odd function

$I_n = \frac{1}{2n-1}(\ldots)$ $, n \geq 1$

Now $I_3 = \int_0^{\frac{\pi}{4}} \cos^2 x \sin 3x \, dx$

$= \frac{1}{5}\left(2I_2 - [\cos^2 x \cos 3x]_0^{\frac{\pi}{4}}\right)$

$= \frac{2}{5} I_2 - \frac{1}{5}\left[\left(\frac{1}{2}\cdot\frac{-1}{\sqrt{2}}\right) - (1.1)\right]$

$= \frac{2}{5}\left[\frac{1}{3}\left(I_1 - [\cos x \cos 2x]_0^{\frac{\pi}{4}}\right)\right] + \frac{1}{10\sqrt{2}} + \frac{1}{5}$

$= \frac{2}{15}\left[1\left(0 - [\cos x]_0^{\frac{\pi}{4}}\right)\right] - \frac{2}{15}\left[\left(\frac{1}{\sqrt{2}}\cdot 0\right) - (1.1)\right] + \frac{1}{10\sqrt{2}} + \frac{1}{5}$

$= \frac{-2}{15}\left[\frac{1}{\sqrt{2}} - 1\right] - 0 + \frac{2}{15} + \frac{1}{10\sqrt{2}} + \frac{1}{5}$

$= \frac{-2}{15\sqrt{2}} + \frac{2}{15} - 0 + \frac{2}{15} + \frac{1}{10\sqrt{2}} + \frac{1}{5}$

$= \frac{-8 + 8\sqrt{2} + 8\sqrt{2} + 6 + 12\sqrt{2}}{60\sqrt{2}}$

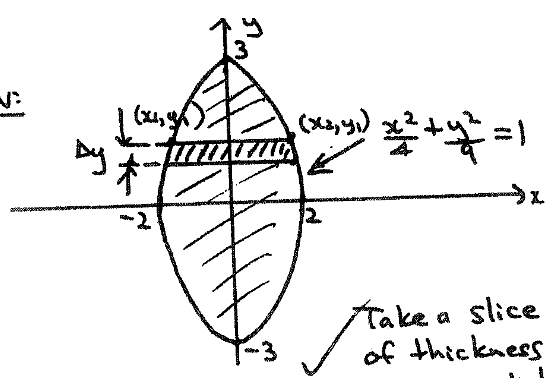
$= \frac{28\sqrt{2} - 2}{60\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{56 - 2\sqrt{2}}{120}$

$= \frac{1}{60}\left(28 - \sqrt{2}\right)$

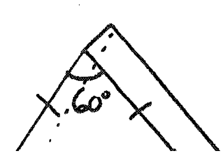
(a)

TOP VIEW:



Take a slice of thickness Δy perpendicular to the y-axis as shown.

SIDE VIEW:



Each slice is an equilateral Δ of thickness Δy.

Area of cross-sectional slice,

$A(y) = \frac{1}{2} \cdot 2x \cdot 2x \sin 60°$

$= 2x^2 \cdot \frac{\sqrt{3}}{2}$

$= \sqrt{3} x^2$

$= \sqrt{3}\left[4\left[1 - \frac{y^2}{9}\right]\right]$

$= 4\sqrt{3}\left[1 - \frac{y^2}{9}\right]$

Now volume, ΔV, of each slice $= A(y)\Delta y$

$\therefore$ Total volume $= \lim_{\Delta y \to 0} \sum_{y=-3}^{3} A(y)\Delta y$

$= 4\sqrt{3}\int_{-3}^{3} \left(1 - \frac{y^2}{9}\right) dy$

$= 8\sqrt{3}\left[y - \frac{y^3}{27}\right]_0^3$

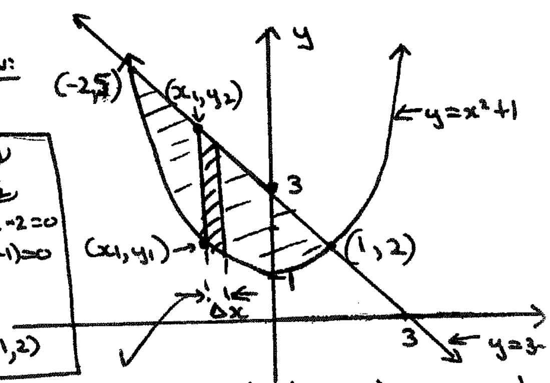
$= 8\sqrt{3}\left[(3 - 1) - 0\right]$

$= 16\sqrt{3}$ units^3

(b)

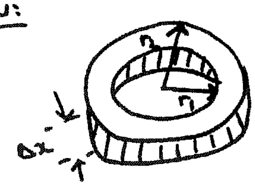
TOP VIEW:

$y = x^2 + 1$ —(1)
$y = 3 - x$ —(2)
(1)−(2): $x^2 + x - 2 = 0$
$(x+2)(x-1) = 0$
$\therefore x = -2, 1$
$\therefore$ pts of int. are $(-2,5), (1,2)$



Take a slice of thickness Δx perp. to the x-axis.

SIDE VIEW:



Each slice represents a hollow disc as shown.

$r_2 = y_2, \quad r_1 = y_1$

Now area of cross-sectional slice,

$A(x) = \pi\left(y_2^2 - y_1^2\right)$

$= \pi\left((3-x)^2 - (x^2+1)^2\right)$

$= \pi\left(9 - 6x + x^2 - x^4 - 2x^2 - 1\right)$

$= \pi\left(8 - 6x - x^2 - x^4\right)$

Now volume, ΔV, of each slice $= A(x)\Delta x$

$\therefore$ Total volume $= \lim \sum^{1} A(x)\Delta x$

$$\therefore \text{Total volume} = \pi \int_{-2}^{1} (8 - 6x - x^2 - x^4)\,dx$$

$$= \pi \left[8x - 3x^2 - \frac{x^3}{3} - \frac{x^5}{5} \right]_{-2}^{1}$$

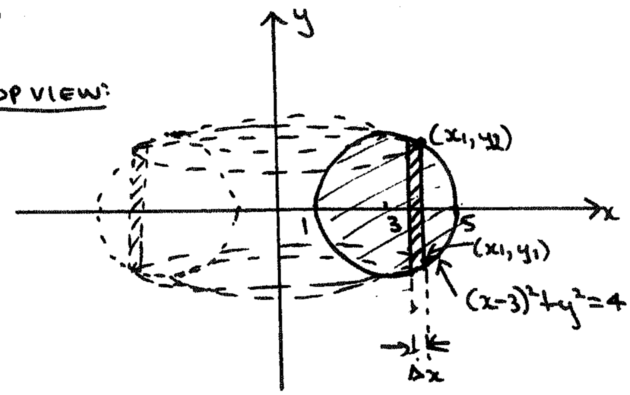
$$= \pi \left[\left(8 - 3 - \tfrac{1}{3} - \tfrac{1}{5}\right) - \left(-16 - 12 + \tfrac{8}{3} + \tfrac{32}{5}\right) \right]$$

$$= \pi \left[23\tfrac{2}{5} \right]$$

$$= \frac{117\pi}{5} \text{ units}^3$$

(i)

TOP VIEW:



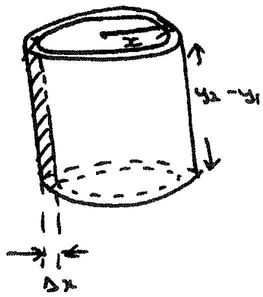
Take a slice of thickness Δx perpendicular to the x-axis inside the circle. Each slice when rotated about the y-axis generates a thin cylindrical shell of area $2\pi r h$.

SIDE VIEW:

$y_2 = \sqrt{4-(x-3)^2}$

$y_1 = -\sqrt{4-(x-3)^2}$

$y_2 - y_1 = 2\sqrt{4-(x-3)^2}$



Now area of each cylindrical shell,

$$A(x) = 2\pi x \cdot 2\sqrt{4-(x-3)^2}$$

$$= 4\pi x \sqrt{4-(x-3)^2}$$

Now volume, ΔV, of each shell $= A(x)\,\Delta x$

$$\therefore \text{Total volume}, V = \lim_{\Delta x \to 0} \sum_{x=1}^{5} A(x)\,\Delta x$$

$$= 4\pi \int_1^5 x\sqrt{4-(x-3)^2}\,dx$$

(ii) $\text{Volume} = 4\pi \int_1^5 x\sqrt{4-(x-3)^2}\,dx$

Let $u = x-3$ when $x=5$ $u=2$

$\therefore \frac{du}{dx} = 1$ $x=1$ $u=-2$

$$\therefore \text{Volume} = 4\pi \int_{-2}^{2} (u+3)\sqrt{4-u^2}\,du$$

$$= 4\pi \underbrace{\int_{-2}^{2} u\sqrt{4-u^2}\,du} + 12\pi \underbrace{\int_{-2}^{2} \sqrt{4-u^2}}$$

Let $f(u) = u\sqrt{4-u^2}$

$\therefore f(-u) = -u\sqrt{4-u^2}$

$= -f(u)$

$\therefore f(u)$ is odd.

$\Rightarrow$ This integral is zero as the integration of an odd function about symmetrical limits is zero.

This represents the area of a semi-circle of radius 2 units.

$\boxed{\text{OR let } x = 3 + 2\sin \ldots \text{ etc}}$

$$\therefore \text{Volume} = 12\pi \cdot \tfrac{1}{2} \cdot \pi \cdot 2^2$$

$$= 24\pi^2 \text{ units}^3.$$

7 (a) $\dfrac{x+1}{(x-1)(x+2)} \geq 3$, $x \neq 1, x \neq -2$.

bs. $\times (x-1)^2(x+2)^2$

$\therefore (x+1)(x-1)(x+2) \geq 3(x-1)^2(x+2)^2$

$\therefore 0 \geq (x-1)(x+2)\left[3(x-1)(x+2) - (x+1)\right]$

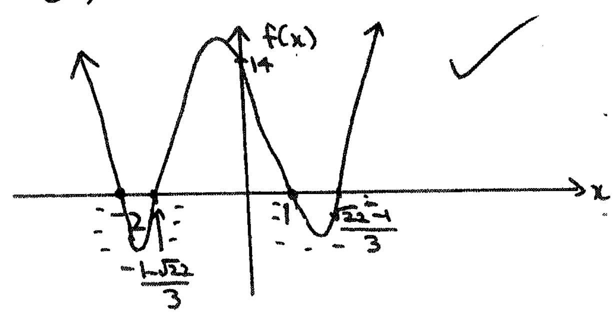
$\therefore 0 \geq (x-1)(x+2)\left[3(x^2+x-2) - x - 1\right]$

$\therefore 0 \geq (x-1)(x+2)(3x^2+2x-7)$

$\therefore 0 \geq (x-1)(x+2)\left[3\left(x^2+\tfrac{2}{3}x-\tfrac{7}{3}\right)\right]$

$\therefore 0 \geq (x-1)(x+2)\left[3\left(\left(x+\tfrac{1}{3}\right)^2 - \tfrac{22}{9}\right)\right]$

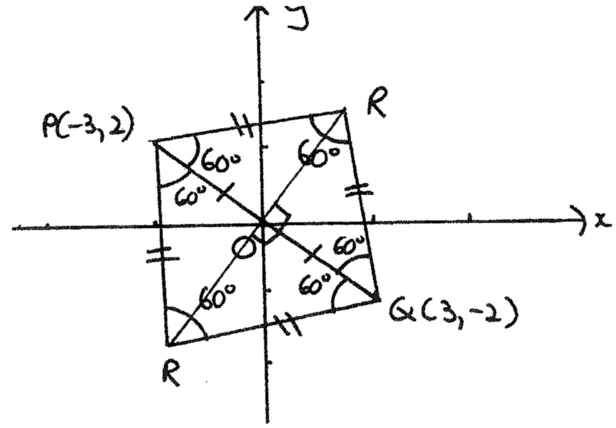
$\therefore 0 \geq 3(x-1)(x+2)\left(x+\tfrac{1}{3}-\tfrac{\sqrt{22}}{3}\right)\left(x+\tfrac{1}{3}+\tfrac{\sqrt{22}}{3}\right)$



$\Rightarrow -2 < x \leq \dfrac{-1-\sqrt{22}}{3}$

$1 < x \leq \dfrac{\sqrt{22}-1}{3}$

)



(i) If $m = m_{RQ}$ and $m_{PQ} = \frac{-2-2}{3--3}$
$= -\frac{2}{3}$

$\therefore$ acute angle between PQ and RQ is given by : $\tan 60° = \left|\dfrac{m - (-\frac{2}{3})}{1 + m.(-\frac{2}{3})}\right|$

$= \left|\dfrac{m + \frac{2}{3}}{1 - \frac{2m}{3}}\right|$

(ii) Now taking the positive absolute value case: $\sqrt{3} = \dfrac{m + \frac{2}{3}}{1 - \frac{2m}{3}}$

$\therefore \sqrt{3} = \dfrac{3m+2}{3-2m}$

$\therefore 3\sqrt{3} - 2\sqrt{3}m = 3m + 2$

$\therefore m(3 + 2\sqrt{3}) = 3\sqrt{3} - 2$

$\therefore m = \dfrac{3\sqrt{3}-2}{3+2\sqrt{3}}$

(iii) Now eqn of RQ is given by:

$y + 2 = \dfrac{3\sqrt{3}-2}{3+2\sqrt{3}}(x-3)$ —— ①

$m_{OR} = \frac{3}{2}$ (as OR ⊥ PQ)

$\therefore$ eqn of OR is: $y = \frac{3}{2}x$ —— ②

Now OR and RQ meet at R.

sub ② into ①: $\frac{3}{2}x + 2 = \dfrac{3\sqrt{3}-2}{3+2\sqrt{3}}(x-3)$

$\therefore (3x+4)(3+2\sqrt{3}) = (6\sqrt{3}-4)(x-3)$

$\therefore 13x = -26\sqrt{3}$ $\therefore x = -2\sqrt{3}$

when $x = -2\sqrt{3}$ $y = \frac{3}{2}(-2\sqrt{3}) = -3\sqrt{3}$

$\therefore R = (-2\sqrt{3}, -3\sqrt{3})$

Now due to symmetry of figure

$R = (2\sqrt{3}, 3\sqrt{3})$ as well.

$\Rightarrow$ possible coordinates of R are : $(\pm 2\sqrt{3}, \pm 3\sqrt{}$

(c)

(i) P(4 10's) = P(4 10's without joker) + P(4 10's with joker)

$= \dfrac{{}^5C_4 \times {}^{45}C_1 + {}^5C_3 \times {}^1C_1 \times {}^{4}}{{}^{51}C_5}$

$= \dfrac{225 + 450}{2\,349\,060}$

$= \dfrac{675}{2\,349\,060}$

$= \dfrac{45}{156604}$

(ii) P(3 of one number and 2 of another number)

= P(3 of one number and 2 of another number without joker) + P(2 of one number and 2 of another number and joker)

$= \dfrac{{}^5C_3 \times {}^5C_2 \times {}^{10}C_2 \times 2! + {}^5C_2 \times {}^5C_2 \times {}^1C_1 \times {}^{1}}{{}^{51}C_5}$

$= \dfrac{9000 + 4500}{2\,349\,060}$

$= \dfrac{225}{39151}$

<u>N.B.</u> The case of 3 of one number, 1 of another number and joker does not apply as this would result in 4 of one number and 1 of the other number under the rule

(a) (i) $\ddot{x} = -n^2(x-b)$ —①

$x = b + a\cos nt$ —②

sub ② into ①: LHS $= \ddot{x}$

$= \frac{d}{dt}(\dot{x})$

$= \frac{d}{dt}(-an\sin nt)$

$= -an^2\cos nt$

$= -an^2\left(\frac{x-b}{a}\right)$

$= -n^2(x-b)$

$=$ RHS

$\therefore x = b + a\cos nt$ satisfies the buoy's motion.

(ii) $b = \frac{25+20}{2} = \frac{45}{2}$; $a = \frac{25-20}{2} = \frac{5}{2}$

$T = 30 = \frac{2\pi}{n}$ $\quad \therefore n = \frac{\pi}{15}$

$\therefore x = \frac{45}{2} + \frac{5}{2}\cos\frac{\pi t}{15}$

When $x = 24$, $t = ?$

$\therefore 24 = \frac{45}{2} + \frac{5}{2}\cos\frac{\pi t}{15}$

$\therefore \frac{3}{5} = \cos\frac{\pi t}{15}$ $\quad \therefore \frac{\pi t}{15} = \cos^{-1}\left(\frac{3}{5}\right)$ (require 1st, 4th quads)

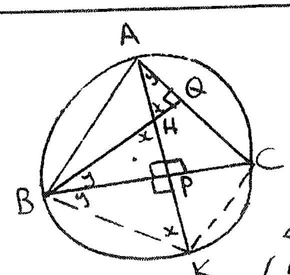
$\therefore t = \frac{15}{\pi}\cos^{-1}\left(\frac{3}{5}\right)$ and $\left[2\pi - \cos^{-1}\left(\frac{3}{5}\right)\right]\frac{15}{\pi}$

for first two occasions $x = 24$

$\therefore t = 4.43$ and 25.57 (2dp)

ie. The first two occasions that the buoy is 24m above sea level after being in a crest position will occur approx. 4.43 seconds and 25.57 seconds afterwards.

(b)



TO PROVE: $HP = KP$

PROOF: Join B to K

Join K to C

$\angle KBC = \angle KAC = y$ (say)

(∠s at circum. of circle subtended from a common

In Δs AHQ and BPK:

$\angle HAQ = \angle KBP = y$

$\angle AQH = \angle BPK = 90°$ (Altitude property)

$\therefore \angle AHQ = \angle BKP = x$ (say)

[∠ sum of respective Δs = 180°]

$\angle BHP = x$ (vert-opp. ∠s are equal)

[Now Δs HQA and HPB are similar

(2∠'s of one Δ = 2∠'s of the other Δ)

$\Rightarrow \angle HBP = y$] ← not necessary

Now $\Delta HPB \equiv \Delta KPB$ (AAS Test)

($\angle HPB = \angle KPB = 90°$;

$\angle PHB = \angle PKB = x$;

BP is common)

$\therefore HP = KP$ (corr. sides of congruent Δs are equal).

(c) $\tan 3\theta - \cot 5\theta = 0$

$\therefore \tan 3\theta = \cot 5\theta$

$= \tan\left(\frac{\pi}{2} - 5\theta\right)$

$\therefore 3\theta = n\pi + \left(\frac{\pi}{2} - 5\theta\right)$

$\therefore 8\theta = n\pi + \frac{\pi}{2}$

$= \frac{\pi}{2}(2n+1)$

$\therefore \theta = \frac{\pi}{16}(2n+1)$,

where n is any integer.