CRANBROOK SCHOOL

YEAR 12 MATHEMATICS – EXTENSION 2

Term 3 2004 **Time : 3 h / SKB**

All questions are of equal value.
All necessary working should be shown in every question.
Full marks may not be awarded if work is careless or badly arranged.
Approved silent calculators may be used.
Standard Integrals appear at the end of the paper.

Submit your work for Questions 1 and 2 in the same 8 page booklet.
Submit your work for Questions 3 and 4 in the same 8 page booklet.
Submit your work for Questions 5 and 6 in the same 8 page booklet.
Submit your work for Questions 7 and 8 in the same 8 page booklet.

1. **(15 Marks) (Begin a 8 page booklet for Questions 1 and 2.)**

(a) The complex number z is given by $z = \dfrac{1+\sqrt{3}i}{1-i}$.

Find: **(i)** $|z|$ 2

(ii) $\arg z$ 2

(iii) $\bar{z}$ in mod-arg form. 1

(b) Find the Cartesian equation of the locus represented by

$\left|\dfrac{z-4}{z+3i}\right| = 1$ where z is a complex number. 2

(c) If z is a complex number then on an Argand Diagram, shade in the region for which $|z+i|<1$ and $-\dfrac{2\pi}{3} \le \arg z \le -\dfrac{\pi}{3}$. 3

(d) **(i)** Prove that $\tan^{-1}(a) - \tan^{-1}(b) = \tan^{-1}\left(\dfrac{a-b}{1+ab}\right)$. 2

(ii) Hence or otherwise find the locus of z where z is a complex number if $\arg\left(\dfrac{z+i}{z-2}\right) = \dfrac{\pi}{2}$. 3

2. (15 marks)

(a) The polynomial $P(x) = 8x^4 + 12x^3 - 30x^2 + 17x - 3$ has a root of multiplicity 3. Factorise $P(x)$ and find all of its roots. **4**

(b) The roots of $x^3 + 6x^2 + 5x - 8 = 0$ are α, β and γ. Find the monic cubic polynomial whose roots are α^2, β^2 and γ^2. **3**

(c) If α, β and γ are the roots of $2x^3 - 4x^2 - 3x - 1 = 0$ find the value of $\alpha^3 + \beta^3 + \gamma^3$. **4**

(d) If $x = 1 + i$ is a root of $P(x) = 3x^3 - 7x^2 + 8x - 2$ find all the roots of $P(x)$. **3**

3. (15 marks) (Begin a 8 page booklet for Questions 3 and 4.)

(a) If $P(x_1, y_1)$ is any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $S(ae, 0)$ and $S'(-ae, 0)$ are the foci prove that $|PS'| + |PS| = 2a$.

(b) For the hyperbola $5x^2 - 4y^2 = 20$:

(i) find the eccentricity and the co-ordinates of the foci; **2**

(ii) find the equations of the directrices and asymptotes. **2**

(iii) If the point $P(2\sec\theta, \sqrt{5}\tan\theta)$ lies on this hyperbola prove that the equation of the tangent to the hyperbola at P is $\frac{x\sec\theta}{2} - \frac{y\tan\theta}{\sqrt{5}} = 1$. **3**

(iv) If this tangent cuts the asymptotes at L and M prove that LP = PM. **3**

(v) Prove that the area of triangle OLM is independent of the position of P, where O is the origin. **2**

4. (15 Marks)

(a) **(i)** Find the roots of $z^3 = 1$ expressing them in the form $a + ib$. **2**

(ii) Show these roots on an Argand Diagram as $1, \omega$ and ω^2. **1**

(iii) Hence or otherwise show that:
$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$ **3**

(b) **(i)** The ellipse E, is given in terms of the complex number z by: $|z+3| + |z-3| = 10$. Find the Cartesian equation of E. **3**

(ii) Prove that the area enclosed by E is 20π units^2. **3**

(c) If $w = \frac{1+z}{1-z}$ and $|z| = 1$ where w and z are complex numbers, determine the locus of w. **3**

5. (15 marks) (Begin a new 8 page booklet for Questions 5 and 6.)

(a) Find : $\int \frac{dx}{x^2 - x - 2}$ **3**

(b) Evaluate in exact form : $\int_0^1 x \tan^{-1} x \, dx$ **4**

(c) **(i)** Prove that $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{x(\pi - 2x)} dx = \frac{2\ln 2}{\pi}$ **3**

(ii) By using the substitution $t = a + b - x$, prove that:
$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$. **2**

(iii) Hence prove that : $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin^2 x}{x(\pi - 2x)} dx = \frac{\ln 2}{\pi}$. **3**

6. (15 marks)

(a) By using the substitution $t = \tan\frac{\theta}{2}$ evaluate :

$$\int_0^{\frac{\pi}{2}} \frac{4\,d\theta}{3+5\cos\theta}$$ 4

(b) Evaluate : $\int_{-1}^{1} \frac{\sin^{-1} x}{1+\tan^2 x}\,dx$ 2

(c) Prove that $\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx = \frac{2\pi - 3\sqrt{3}}{6}$. 3

(d) If $I_n = \int_0^{\frac{\pi}{2}} \sin^n 2x \cos^2 2x\,dx$ and $n \ge 1$ prove that $I_n = \frac{n-1}{n+2} I_{n-2}$ for $n \ge 2$. 4

Hence use this reduction formula to evaluate

$\int_0^{\frac{\pi}{2}} \sin^4 2x \cos^2 2x\,dx$. 2

7. (15 marks) (Begin a new 8 page booklet for Questions 7 and 8.)

(a) A solid is built on the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ such that cross-sectional slices perpendicular to the x-axis are parabolic segments with axes of symmetry passing through the x-axis and heights half the length of their bases. Find the exact volume of this solid. 6

(b) The region bounded by the curve $xy = 4$ and the line $y = 5 - x$ is rotated about the y-axis.

(i) By considering slices perpendicular to the y-axis to produce thin hollow discs, prove that the volume generated is 9π units^3. 5

(ii) Confirm your answer in (b) (i) by taking slices parallel to the y-axis to generate thin cylindrical shells. 4

8. (15 Marks)

(a) Harry, the poker-faced hooker, acting on instructions from his astute half back Drapes, propels an ellipsoid 2m above the ground from the sideline into a lineout. The ball passes 2.3m above the head of a 1.7m prop standing 15m from the sideline and is caught 3.8m above the ground by a 1.98m lock, assisted in jumping, 20m from the sideline. If the ball is thrown at a speed of v m/s at an angle of θ to the horizontal and assuming that there is no air resistance and that the acceleration due to gravity is $10\,\text{m/s}^2$:

(i) show that the Cartesian equation of motion of the ball is given by $y = -\frac{5x^2}{v^2\cos^2\theta} + x\tan\theta + 2$; 2

(ii) find the angle at which Harry should propel the ball, to the nearest minute; 3

(iii) find the speed at which the ball should be thrown into the lineout, to the nearest m/s. 1

(b) A light aircraft flying horizontally at a speed of 50 m/s is observed on a bearing of 330^0T at an elevation of 45^0. After two minutes it is observed on a bearing of 030^0T at an elevation of 30^0.

(i) Determine the altitude of the aircraft to the nearest m. 3

(ii) Find the direction of the flight of the plane to the nearest degree T. 2

(c) Prove by mathematical induction that for each positive integer n, there are unique integers x_n and y_n such that $(1+\sqrt{2})^n = x_n + y_n\sqrt{2}$. 4