CRANBROOK

MATHEMATICS EXTENSION 2

2007

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

General Instructions

- Reading time – 5 minutes
- Writing time – 3 hours
- All eight questions should be attempted
- Total marks available - 120
- All questions are worth 15 marks
- An approved calculator may be used
- All relevant working should be shown for each question
- Answer each question in a separate 8 page booklet.

Question 1 (15 marks) Marked by SKB **Marks**

(a) Find $\int_1^2 3\sqrt{x-1}\,dx$ **2**

(b) Using the substitution $u = e^x - 1$ or otherwise, **2**

find $\int \frac{2e^x}{e^{2x} - 2e^x + 1}dx$

(c) Use integration by parts to evaluate $\int_0^{\frac{\pi}{4}} x\cos 4x\,dx$ **3**

(d) Use the substitution $t = \tan\frac{\theta}{2}$ to find $\int \frac{\sin\theta}{1+\cos\theta}d\theta$ **4**

(e) (i) Find the real numbers a, b and c such that **2**

$$\frac{x^3 + 5x^2 + x + 2}{x^2(x^2+1)} \equiv \frac{x+a}{x^2} + \frac{bx+c}{x^2+1}$$

(ii) Find $\int \frac{x^3 + 5x^2 + x + 2}{x^2(x^2+1)}dx$ **2**

Question 2 (15 marks) Marked by SKB **Marks**

(a) Evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\sin^3 x + \cos^3 x}\, dx$. **2**

(b) Evaluate $\int_0^1 \sqrt{1+x^2}\, dx$. **4**

(c) Find $\int \frac{3x - 4}{\sqrt{4 + 5x - 3x^2}}\, dx$ **4**

(d) If $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \sin^2 x\, dx$ for $n \geq 0$,show that **5**

$I_n = \frac{n-1}{n+2} I_{n-2}$ for $n \geq 2$. Hence or otherwise

evaluate $\int_0^{\frac{\pi}{2}} \cos^4 x \sin^2 x$ dx.

Question 3 (15 marks) Marked by CJL **Marks**

(a) Let $z = 3 - 4i$ and $\omega = 2 - i$

(i) Find $\dfrac{1}{z}$ in the form $x + yi$ **1**

(ii) Show that $\operatorname{Im} z + \overline{\omega} + z\omega = -10i$ **2**

(b) If ai is a solution to the equation **2**

$$z^2 + (1-i)z + (2-2i) = 0$$

find the real value of a.

(c) Let $u = 1 - i$

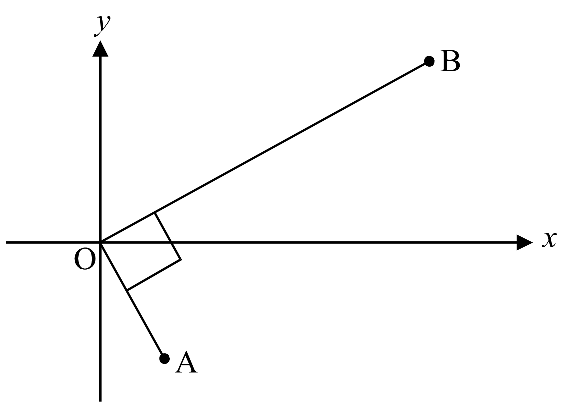
(i) Find $|u|$ and $\arg u$ **2**

(ii) Hence find u^{12}. Express your answer in the form $x + yi$. **2**

(d) Sketch the region on an Argand diagram where the inequalities **3**

$|z - 2 + i| \le |z + 2 - i|$ and $\operatorname{Im} z \ge 0$ both hold.

(e)



The point A on the Argand diagram above corresponds to the complex number z.
Triangle ABO is a right-angled triangle where $OB = 2OA$.

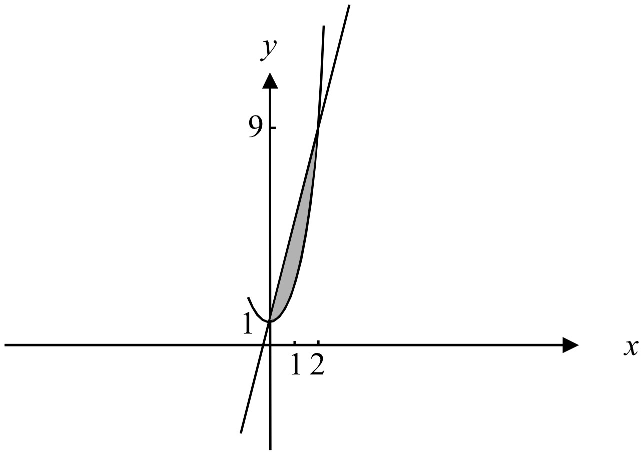
(i) Show that point B corresponds to the complex number $2iz$. **1**

(ii) The point C corresponds to the complex number v and C is situated so that $OACB$ is a rectangle.
Given that $z = x + yi$, $x, y \in R$, find $\overline{v}$ in terms of x and y. **2**

Question 4 (15 marks) Marked by SKB **Marks**

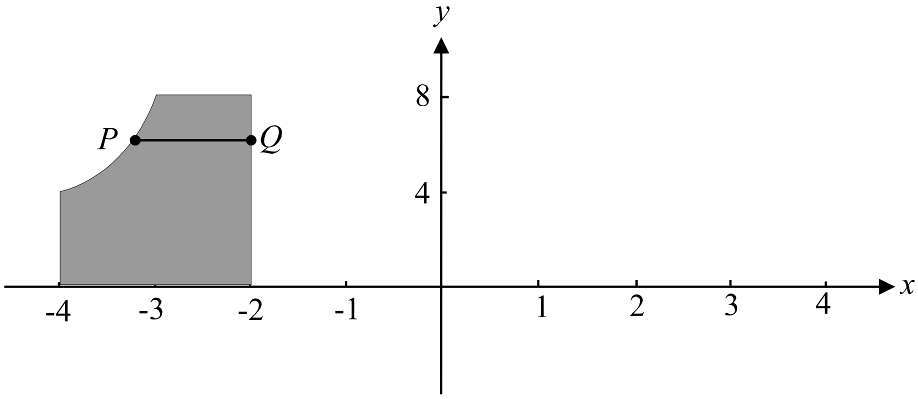
(a) The base of a certain solid S lies on the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$. The cross-section of this solid by planes perpendicular to the $x-$axis are equilateral triangles. By including appropriate views of slices to this solid calculate its volume in exact form. **5**

(b)



The shaded area shown in the diagram above is the area between the graph of $y=4x+1$ and the graph of $y=2x^2+1$. This shaded area is rotated about the y axis to form a solid.
Use the method of cylindrical shells to find the volume of the solid. **4**

(c)



The shaded area is bounded by the lines $x=-4, x=-2, y=8$, by the curve $y=\frac{-8}{x+2}$ and by the x-axis.
The region is rotated about the line $x = 3$ until it reaches its original position thus forming a solid. The horizontal line segment PQ forms an annulus as a result of this rotation.

(i) Show that the area of this annulus at height y where $y \geq 4$, is equal to

$$16\pi\left(\frac{4}{y^2}+\frac{5}{y}\right)$$ **2**

(ii) Hence find the volume of the solid. **4**

Question 5 (15 marks) Marked by CJL **Marks**

(a) Let $f(x)=\cos^{-1} x$ for $-1 \le x \le 1$ and $g(x)=\sin^{-1} x$ for $-1 \le x \le 1$.

(i) Sketch $f(x)$ and $g(x)$ on the same set of axes. **1**

(ii) By differentiating, evaluate $f(x)+g(x)$ **1**

(iii) Hence evaluate $\int_{-1}^{1}\left(f(x)+g(x)\right)dx$ **1**

(b) The ellipse E has the equation $x^2+\frac{y^2}{4}=1$.

(i) Find the eccentricity and the foci of E. **2**

(ii) Find the length of the major and minor axes of E. **1**

(iii) Write down the equations of the directrices of E. **1**

(iv) Sketch E. **1**

(c) (i) The polynomial equation $p(x)=0$ has a root α of multiplicity 3. **2**
Show that α is a root of $p'(x)=0$ and is of multiplicity 2.

(ii) The polynomial $q(x)=x^6+ax^5+bx^4-x^2-2x-1$ has a **2**
quadratic factor of x^2+2x+1. Find *a* and *b*.

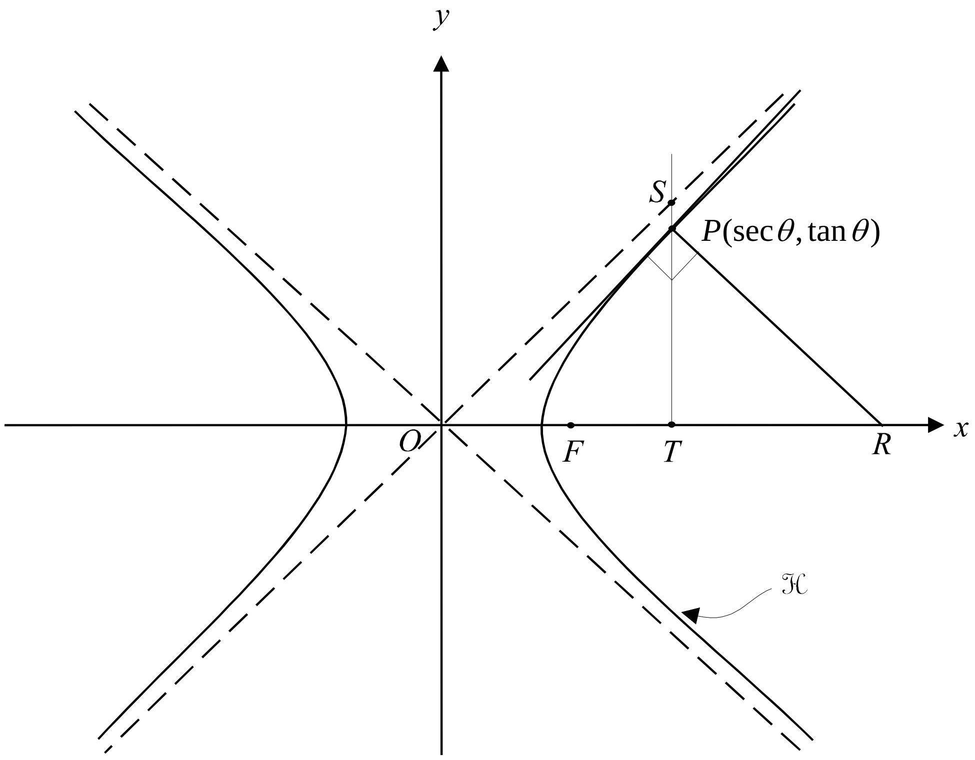
(i) Consider the polynomial **3**

$$r'(x)=1+x+\frac{x^2}{2!}+...+\frac{x^n}{n!} \quad \text{where} \quad r(0)=1.$$

Show that $r(x)$ has no double roots.

Question 6 (15 marks) Marked by CJL **Marks**

(a)



The point $P(\sec\theta, \tan\theta)$ lies on the hyperbola H with equation $x^2 - y^2 = 1$. A vertical line through P intersects with an asymptote at S and with the x-axis at T as shown. A normal to H at P intersects the x-axis at R. The point F is a foci of H.

(i) Show that the equation of the normal to H at the point P is $y = -\sin\theta\, x + 2\tan\theta$. **2**

(ii) Show that $RS = \sqrt{2}RT$. **3**

(iii) Find the coordinates of the point U which lies on SR such that TU is parallel to the asymptote on which S lies. **2**

(iv) For what values of θ will FU be the perpendicular bisector of SR? **2**

(b) Let $\omega = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$.

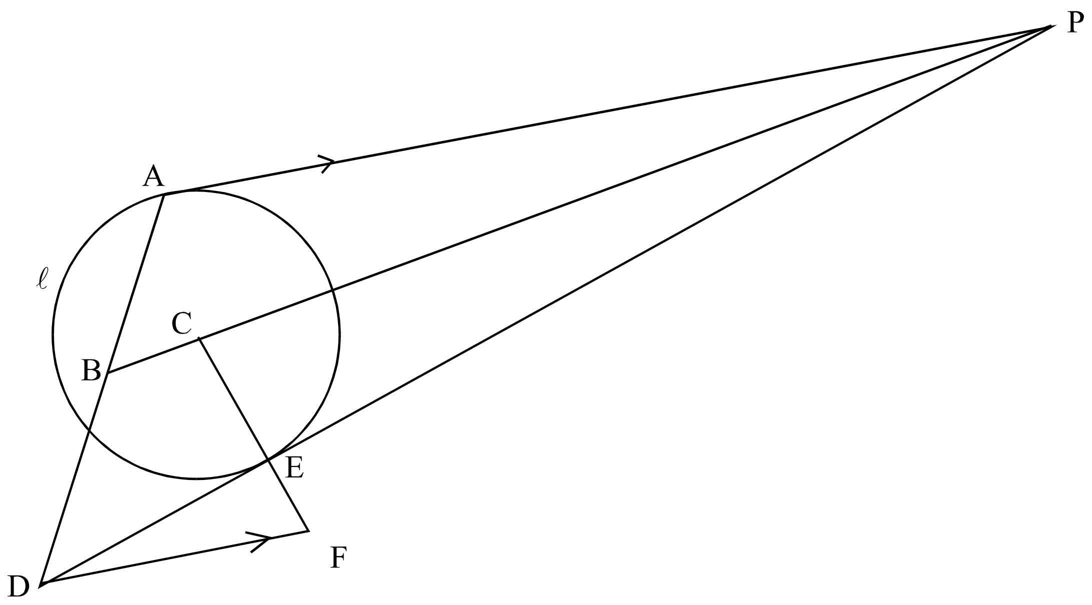
(i) Write down in terms of ω, and the positive integer k, all the solutions of the equation $z^{10} - 1 = 0$. **2**

(ii) Prove that $\omega + \omega^2 + \omega^3 + ... + \omega^{10} = 0$. **2**

(iii) The quadratic equation $x^2 + bx + c = 0$, where b and c are real, has the root $\omega + \omega^4$. Find the other root in terms of ω. **2**

Question 7 (15 marks) **Marked by CJL** **Marks**

(a)



In the diagram, ℓ is a circle with exterior point *P*. Tangents from *P* are drawn to meet ℓ at points *A* and *E*. The point *C* is the centre of ℓ. The line *BP* passes through *C*. The line *AD* passes through *B*. The line *CF* passes through *E*. *AP* is parallel to *DF*.

(i) Show that *ACEP* is a cyclic quadrilateral. **1**

(ii) Use a double angle formula to show that $DE = \dfrac{DF(EP^2 - CE^2)}{CP^2}$ **2**

(iii) Use the sine rule to show that $\dfrac{AB}{BD} = \dfrac{AP}{DP}$ **2**

Marks

(b) (i) Draw the graph of $y = \ln(x+1)$ **1**

(ii) Hence explain why **1**

$$\int_0^n \ln(x+1)dx < n\ln(n+1), \qquad n = 1,2,3...$$

(iii) Use integration by parts to show that **3**

$$\int_0^n \ln(x+1)dx = \ln(n+1)^{n+1} - n$$

(iv) Hence deduce that $\ln(n+1) < n$ **1**

(v) Show that $\sum_{k=1}^{n} \frac{1}{2}\ln(k+1) = \frac{1}{2}\ln(n+1)!$ **1**

(vi) Use the results from parts (iii) and (v) together with your graph **3**

to deduce that $n! < \left(\frac{n+1}{e}\right)^{2n}(n+1)$

Question 8 (15 marks) **Marked by SKB** **Marks**

(a) (i) For all real, positive numbers *a* and *b*, where *a* > *b* show that **4**

(α) $a+b>2\sqrt{ab}$

(β) $b^2-a^2<2\sqrt{ab}(b-a)$

(ii) Hence deduce that *a* > *c* given that *c* is a positive real number and **4**

$$\sqrt{a}(b-a)+\sqrt{c}(c-b)>\frac{c^2-a^2}{2\sqrt{b}}$$

(b) If $h(n)=n^4+6n^2+9$

(i) show that $h(n+2)-h(n)=8(n+1)(n^2+2n+5)$ **3**

(ii) hence prove by mathematical induction that $h(n)$ is divisible by 8 if n is an odd positive integer. **4**

END OF EXAM

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \; x \neq 0, \; \text{if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$