CRANBROOK

MATHEMATICS EXTENSION 2

2008

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

General Instructions

- Reading time – 5 minutes
- Writing time – 3 hours
- All eight questions should be attempted
- Total marks available - 120
- All questions are worth 15 marks
- An approved calculator may be used
- All relevant working should be shown for each question
- Answer each question in a separate 8 page booklet.
- Standard integrals sheet at back of examination.

Question 1 (15 marks) Marked by SKB **Marks**

(a) Find $\int x \tan x^2 \, dx$. **2**

(b) Use the substitution $u = \sqrt{x}$ to evaluate $\int_4^9 \frac{x}{\sqrt{x}(1+x)} dx$. **3**

(c) Use the completion of squares method to find $\int \frac{-2}{\sqrt{3+2x-x^2}} dx$. **2**

(d) (i) Find the real numbers a, b and c such that **2**

$$\frac{2x^2+2x+5}{(x^2+2)(1-x)} \equiv \frac{ax+b}{x^2+2} + \frac{c}{1-x}.$$

(ii) Hence find $\int \frac{2x^2+2x+5}{(x^2+2)(1-x)} dx$. **2**

(e) Use integration by parts to evaluate $\int_1^5 \frac{\ln x}{\sqrt{x}} dx$. **4**

Question 2 (15 marks) Marked by SKB **Marks**

(a) Evaluate $\int_{-1}^{1} \frac{\tan^{-1} x}{1+x^4} dx$ **2**

(b) Evaluate $\int_{0}^{2} \sqrt{4-x^2}\, dx$ **4**

(c) By using the result $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$

evaluate $\int_0^{2\pi} \frac{x \cos x}{1+\sin^2 x} dx$ **3**

(d) Let $I_n = \int_0^1 x(x^2-1)^n \, dx$ for $n = 0,1,2,...$

(i) Use integration by parts to show that **3**

$$I_n = \frac{-n}{n+1} I_{n-1} \text{ for } n \ge 1.$$

(ii) Hence or otherwise show that **2**

$$I_n = \frac{(-1)^n}{2(n+1)} \text{ for } n \ge 0.$$

(iii) Explain why $I_{2n} > I_{2n+1}$ for $n \ge 0$ **1**

Question 3 (15 marks) Marked by JSH **Marks**

(a) Let $z = 3 - i$ and $w = 2 + 4i$.
Find the following in the form $x + yi$.

(i) $z\overline{w}$ **1**

(ii) $\dfrac{z}{w}$ **1**

(b) (i) Express $1 + i$ in modulus-argument form. **2**

(ii) Hence, find the values of n, for which **3**

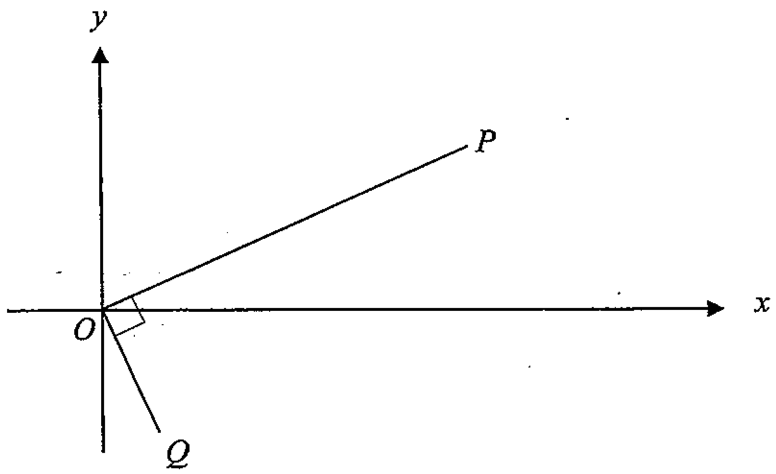
$$(1+i)^n + (1-i)^n = 0$$

where n is a positive integer.

(c) Sketch the region in the Argand diagram where the inequalities **2**

$$|z-1| \le 1 \text{ and } \frac{\pi}{4} \le \arg(z-1) \le \frac{\pi}{2} \text{ both hold.}$$

(d)



In the Argand diagram above, point P corresponds to the complex number z. **1**
The triangle OPQ is a right-angled triangle and $OP = 3OQ$.
What is the complex number that corresponds to point Q?

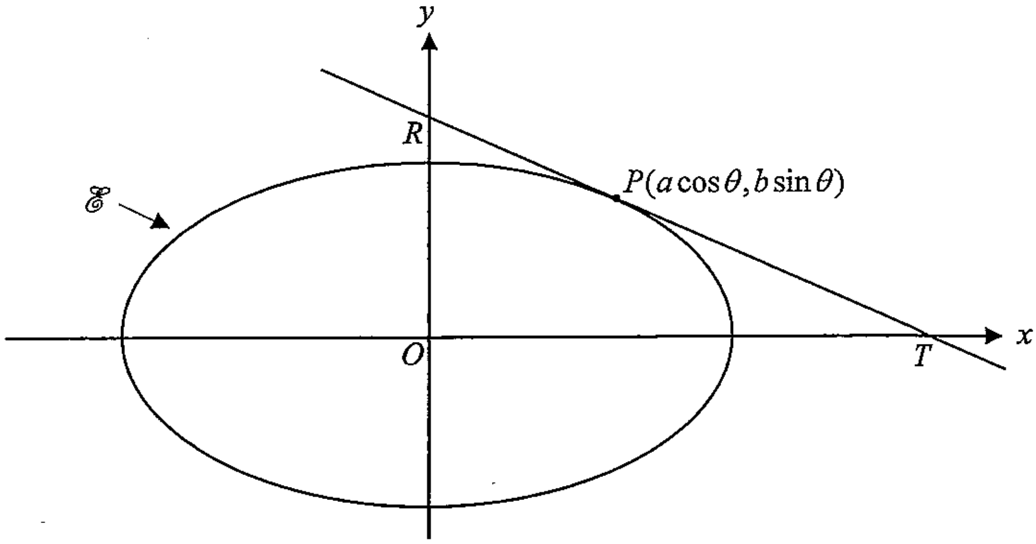
(e) (i) Find all the solutions to the equation $z^6 = 1$ in the form $x + yi$. **2**

(ii) If ω is a non-real solution to the equation $z^6 = 1$, show that $\omega^4 + \omega^2 = -1$. **2**

(iii) By choosing one particular value of ω, explain with the aid of a diagram why $\omega^4 + \omega^2 = -1$. **1**

Question 4 (15 marks) Marked by JSH **Marks**

(a)



The ellipse E with equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ shown in the diagram above, has a tangent at the point $P(a\cos\theta, b\sin\theta)$. The tangent cuts the x-axis at T and the y-axis at R.

(i) Show that the equation of the tangent at the point P is **2**

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1.$$

(ii) If T is the point of intersection between the tangent at point P and one of the directrices of the ellipse, show that $\cos\theta = e$. **3**

(iii) Hence find the angle that the focal chord through P makes with the x-axis. **1**

(iv) Using similar triangles or otherwise, show that $RP = e^2 RT$. **3**

(b) $P(ct, \frac{c}{t}), t \neq 1$ lies on the hyperbola $xy = c^2$. The tangent and normal at P meet the line $y = x$ at T and N respectively. If O is the origin show that $OT.ON = 4c^2$. Include a labelled diagram with your answer. **6**

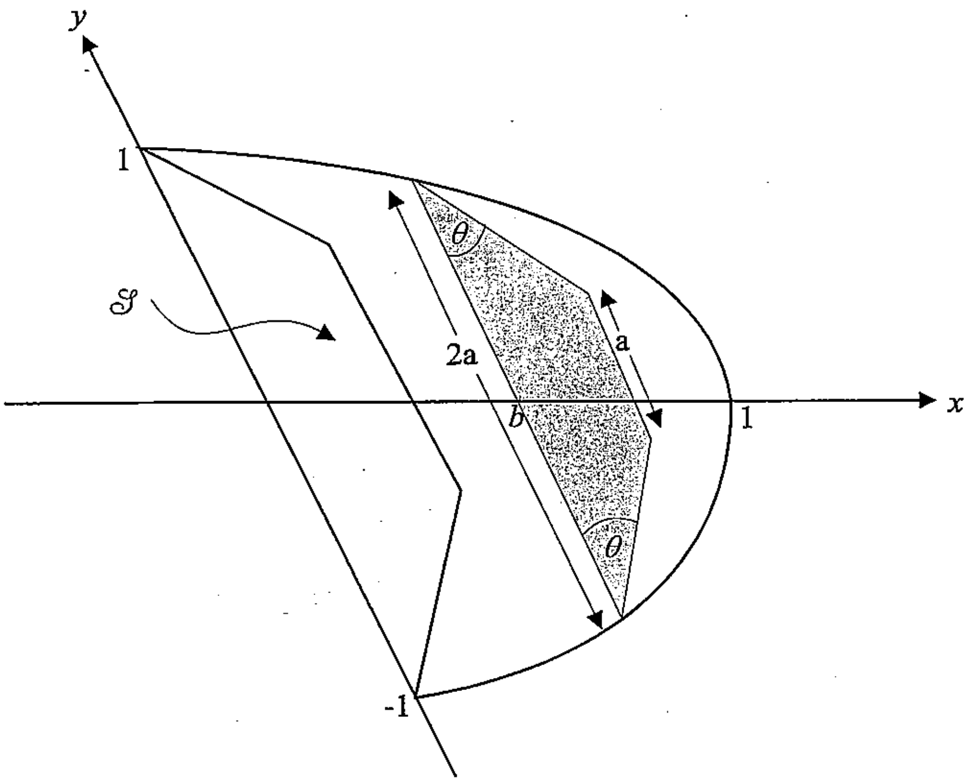
Question 5 (15 marks) Marked by SKB **Marks**

(a) The region bounded by the curve $x = y^2$ and the line $x = 4$ is rotated about the line $y = 2$. Find the volume generated when:

(i) Slices of thickness Δx are taken perpendicular to the $x-$axis in this region to create hollow cylindrical discs. 4

(ii) Slices of thickness Δy are taken perpendicular to the $y-$axis in this region to create thin cylindrical shells. 4

(b)



A solid S has a semi-circular base in the x-y plane with its diameter along the y-axis.
Each cross-section of the solid running perpendicular to the x-y plane is a regular trapezium with its base sidelength twice that of its parallel sidelength. The angle between the base sidelength and the sides of the trapezium is θ.
A typical cross-section taken at $x = b$ is shown in the diagram.

(i) Show that if $\theta = 45^\circ$, the area of the trapezium at $x = b$ is $\dfrac{3a^2}{4}$. 1

(ii) Find the volume of the solid S when $\theta = 45^\circ$. **2**

(iii) Find the volume of the solid, $\mathscr{D}$, generated when the semi-circle is rotated through an angle of 90° about the y-axis. **1**

(iv) Find the values of θ for which the volume of S found in part (ii) is greater than the volume of $\mathscr{D}$ **3**

Question 6 (15 marks) Marked by JSH **Marks**

(a) If $1-i$ is a zero of $P(x)=x^3+ax^2+bx+6$, where $a,b \in$ Real

(i) Evaluate a and b **4**

(ii) Hence fully factorise $P(x)$ over the complex field. **1**

(b) (i) Use De Moivre's theorem to express $\tan 5\theta$ in terms of powers of $\tan\theta$. **3**

(ii) Hence show $x^5-5x^4-10x^3+10x^2+5x-1=0$ has roots $1,\ \tan\frac{\pi}{20},\ \tan\frac{9\pi}{20}, -\tan\frac{3\pi}{20}$ and $-\tan\frac{7\pi}{20}$. **2**

(iii) By solving $x^5-5x^4-10x^3+10x^2+5x-1=0$ another way, show that **5**

$$\tan\frac{9\pi}{20}+\tan\frac{\pi}{20}=2+2\sqrt{5} \text{ and } \tan\frac{7\pi}{20}+\tan\frac{3\pi}{20}=2\sqrt{5}-2.$$

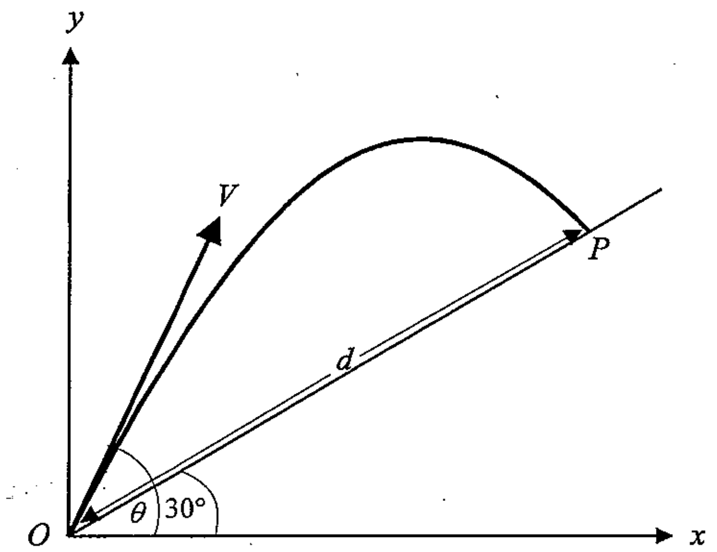
Question 7 (15 marks) Marked by JSH **Marks**

(a) AB is a chord of a circle. X is a point on AB produced. XT is a tangent from X to the circle.

(i) Prove that ΔXAT is similar to ΔXTB. 2

(ii) Deduce that $XT^2 = XA.XB$ 2

(b)



The diagram above shows the path of a particle which has been projected from point 0 at an angle of θ to the horizontal. The speed at which the particle was projected was $\sqrt{g}$ m/sec where g is the acceleration due to gravity. The particle lands at point P which lies on a plane inclined at an angle of 30° to the horizontal. The base of this inclined plane is at O and point P lies d metres from O. The position of the particle at time t seconds is given by

$$x = \sqrt{g}\, t \cos\theta$$

$$\text{and } y = \sqrt{g}\, t \sin\theta - \frac{1}{2} g t^2$$

(i) Show that the path of trajectory of the particle is given by 1

$$y = x \tan\theta - \frac{x^2 \sec^2\theta}{2}.$$

(ii) If there is only one path of trajectory for the particle to land at point P, find θ for that path. 4

Marks

(c) Find the general solutions to the equation **6**

$$\cos 4\theta + \cos 2\theta = \sqrt{2}\cos^2\theta + \frac{1}{\sqrt{2}}\sin 2\theta .$$

Question 8 (15 marks) Marked by SKB **Marks**

(a) (i) If a > 0, b > 0 and c > 0, show that $a^2 + b^2 \geq 2ab$ and hence deduce that $a^2 + b^2 + c^2 \geq ab + bc + ca$. **2**

(ii) If $a + b + c = 9$, show that $ab + bc + ca \leq 27$ and **3**

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{27}{abc}.$$

(b) If $U_1 = 1, U_2 = 5$ and $U_n = 5U_{n-1} - 6U_{n-2}$ for $n \geq 3$, prove by mathematical induction that $U_n = 3^n - 2^n$ for $n \geq 1$. **5**

(c) The lines $y = 0$, $3x - 4y + 3 = 0$ and $3x + 4y - 15 = 0$ are the sides of a triangle. Find the co-ordinates of the centre of the circle inscribed in the triangle. Hence or otherwise write down the equation of the circle. **5**

STANDARD INTEGRALS

$\int x^n \, dx$	$= \frac{1}{n+1} x^{n+1}, \quad n \neq -1;\ x \neq 0,\ \text{if } n < 0$
$\int \frac{1}{x} \, dx$	$= \ln x, \quad x > 0$
$\int e^{ax} \, dx$	$= \frac{1}{a} e^{ax}, \ a \neq 0$
$\int \cos ax \, dx$	$= \frac{1}{a} \sin ax, \ a \neq 0$
$\int \sin ax \, dx$	$= -\frac{1}{a} \cos ax, \ a \neq 0$
$\int \sec^2 ax \, dx$	$= \frac{1}{a} \tan ax, \ \ a \neq 0$
$\int \sec ax \tan ax dx$	$= \frac{1}{a} \sec ax, \ \ a \neq 0$
$\int \frac{1}{a^2 + x^2} \, dx$	$= \frac{1}{a} \tan^{-1} \frac{x}{a}, \ a \neq 0$
$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx$	$= \sin^{-1} \frac{x}{a}, a > 0, \ -a < x < a$
$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx$	$= \ln\left(x + \sqrt{x^2 - a^2}\right), \ x > a > 0$
$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx$	$= \ln\left(x + \sqrt{x^2 + a^2}\right)$

NOTE : $\ln x = \log_e x, \quad x > 0$

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SOLUTIONS
2008

Question 1 (15 marks)

(a) $\int x\tan x^2\,dx = \frac{1}{2}\int \tan u\,du$ where $u = x^2$

(1 mark) $\frac{du}{dx} = 2x$

$= \frac{1}{2}\int \frac{\sin u}{\cos u}\,du$

$= -\frac{1}{2}\ln\left(\cos x^2\right) + c$

(1 mark)

(b) $\int_4^9 \frac{x}{\sqrt{x}(1+x)}\,dx$

$u = \sqrt{x},\ \frac{du}{dx} = \frac{1}{2\sqrt{x}} \quad \therefore 2du = \frac{dx}{\sqrt{x}}$

$x = 9,\ u = 3$

$x = 4,\ u = 2$

$= \int_2^3 \frac{u^2}{\left(1+u^2\right)} 2\frac{du}{dx}\,dx$

$= 2\int_2^3 \frac{u^2}{1+u^2}\,du$

(1 mark) for terminals

$= 2\int_2^3 \left(1 - \frac{1}{1+u^2}\right)du$ **(1 mark)** for function

$= 2\left[u - \tan^{-1} u\right]_2^3$

$= 2\left\{\left(3 - \tan^{-1} 3\right) - \left(2 - \tan^{-1} 2\right)\right\}$

$= 2 - 2\tan^{-1} 3 + 2\tan^{-1} 2$

(1 mark)

(c) $$\int \frac{-2}{\sqrt{3+2x-x^2}}\,dx$$

$$= -2\int \frac{1}{\sqrt{-\left(x^2-2x-3\right)}}\,dx$$

$$= -2\int \frac{1}{\sqrt{-\left(x^2-2x+1-1-3\right)}}\,dx$$

$$= -2\int \frac{1}{\sqrt{4-(x-1)^2}}\,dx$$ **(1 mark)**

$u = x-1$

$\frac{du}{dx} = 1$

$$= -2\sin^{-1}\frac{(x-1)}{2}+c$$ **(1 mark)**

(d) (i) $$\frac{2x^2+2x+5}{\left(x^2+2\right)(1-x)} \equiv \frac{ax+b}{x^2+2}+\frac{c}{1-x}$$

$$\equiv \frac{(ax+b)(1-x)+c\left(x^2+2\right)}{\left(x^2+2\right)(1-x)}$$

True iff $2x^2+2x+5 \equiv (ax+b)(1-x)+c\left(x^2+2\right)$ **(1 mark)**

Put $x=1$, $9=3c$ $c=3$

Put $x=0$, $5=b+2c$ $b=-1$

Put $x=-1$, $5=(-a-1)2+3\times 3$ $a=1$

So, $a=1$, $b=-1$, $c=3$

(1 mark)

(ii) Hence $$\int \frac{2x^2+2x+5}{\left(x^2+2\right)(1-x)}\,dx$$

$$= \int\left(\frac{x-1}{x^2+2}+\frac{3}{1-x}\right)dx$$

$$= \int \frac{x}{x^2+2}\,dx - \int \frac{1}{x^2+2}\,dx + \int \frac{3}{1-x}\,dx$$ **(1 mark)**

$$= \frac{1}{2}\ln\left|x^2+2\right| - \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} - 3\ln|1-x| + c$$

(1 mark)

$$\left(or = \ln\left|\frac{\sqrt{x^2+2}}{(1-x)^3}\right| - \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + c\right)$$

(e) $\int_1^5 \frac{\ln x}{\sqrt{x}}dx = \left[2\sqrt{x}\ln x\right]_1^5 - \int_1^5 2\sqrt{x}.\frac{1}{x}dx$ Let $u = \ln x$ $dv = \frac{1}{\sqrt{x}}dx$

$\therefore \frac{du}{dx} = \frac{1}{x}$ $v = 2\sqrt{x}$

(1 mark) – first function **(1 mark)** – second function
(1 mark) – correct positioning of terminals

$$= \left(2\sqrt{5}\ln 5 - 2\ln 1\right) - \int_1^5 2x^{-\frac{1}{2}}dx$$

$$= 2\sqrt{5}\ln 5 - 2\left[2\sqrt{x}\right]_1^5$$

$$= 2\sqrt{5}\ln 5 - 4\sqrt{5} + 4$$

(1 mark)

Question 2 (15 marks)

(a) $\mathrm{I} = \int_{-1}^{1} \frac{\tan^{-1} x}{1+x^4}\,dx$ Let $f(x) = \frac{\tan^{-1} x}{1+x^4}$

$\therefore f(-x) = \frac{\tan^{-1}(-x)}{1+(-x)^4} = -\frac{\tan^{-1} x}{1+x^4} = -f(x)$

$\therefore f(x)$ is an odd function **(1 mark)**

$\therefore$ I = 0, as the integration of an odd function about symmetrical limits is zero. **(1 mark)**

(b) $\mathrm{I} = \int_0^1 \sqrt{4-x^2}\,dx$ Let $x = 2\sin\theta$ $\therefore \frac{dx}{d\theta} = 2\cos\theta$

When $x = 0, \theta = 0$ and when $x = 1, x = \frac{\pi}{6}$

$\therefore \mathrm{I} = \int_0^{\frac{\pi}{6}} \sqrt{4-4\sin^2\theta}\;\; 2\cos\theta\;\, d\theta$ **(1 mark)**

$= 4\int_0^{\frac{\pi}{6}} \cos^2\theta\;\, d\theta$ **(1 mark)**

$= 2\int_0^{\frac{\pi}{6}} 1+\cos 2\theta\;\, d\theta$, using $\cos 2\theta = 2\cos^2\theta - 1$ and $\cos^2\theta = \frac{1}{2}[1+\cos 2\theta]$

$= 2[\theta + \frac{\sin 2\theta}{2}]_0^{\frac{\pi}{6}}$ **(1 mark)**

$= 2[\frac{\pi}{6} + \frac{\sqrt{3}}{4}]$ or $\frac{2\pi + 3\sqrt{3}}{6}$ *or* $\frac{\pi}{3} + \frac{\sqrt{3}}{2}$ **(1 mark)**

(c) $I=\int_0^{2\pi}\frac{x\cos x}{1+\sin^2 x}\,dx$

$=\int_0^{2\pi}\frac{(2\pi-x)\cos(2\pi-x)}{1+\sin^2(2\pi-x)}\,dx$, using the given result **(1 mark)**

$=\int_0^{2\pi}\frac{2\pi\cos x}{1+\sin^2 x}\,dx-\int_0^{2\pi}\frac{x\cos x}{1+\sin^2 x}\,dx$

$\therefore 2I=\int_0^{2\pi}\frac{2\pi\cos x}{1+\sin^2 x}\,dx$

$\therefore I=\pi\int_0^{2\pi}\frac{\cos x}{1+\sin^2 x}\,dx$ **(1 mark)** Let $u=\sin x, \therefore \frac{du}{dx}=\cos x$

When $x=0, u=0$ and when $x=2\pi, u=0$

$\therefore I=\pi\int_0^{0}\frac{du}{1+u^2}=0$, as the integration about the same limits is zero. **(1 mark)**

(d) (i) $I_n=\int_0^1 x\left(x^2-1\right)^n dx \quad n=0,1,2,\ldots$

$=\left[\frac{x^2}{2}\left(x^2-1\right)^n\right]_0^1-\int_0^1\frac{x^2}{2}\times n\left(x^2-1\right)^{n-1}\times 2x\,dx$ **(1 mark)**

$=\frac{1}{2}\times 0-0-n\int_0^1 x^3\left(x^2-1\right)^{n-1}dx$

$=-n\int_0^1\frac{x^3\left(x^2-1\right)^n}{x^2-1}\,dx$

$=-n\int_0^1\frac{[x(x^2-1)+x]\left(x^2-1\right)^n}{x^2-1}\,dx$

$=-n\int_0^1\left\{x\left(x^2-1\right)^n+\frac{x}{x^2-1}\left(x^2-1\right)^n\right\}dx$ **(1 mark)**

$I_n=-n\int_0^1 x\left(x^2-1\right)^n dx-n\int_0^1 x\left(x^2-1\right)^{n-1}dx$

$I_n=-nI_n-nI_{n-1}$

$(1+n)I_n=-nI_{n-1}$

So $I_n=\frac{-n}{n+1}I_{n-1}$ for $n\geq 1$ as required. **(1 mark)**

(ii) Method 1 – "Hence"

$$I_n = \frac{-n}{n+1} I_{n-1} \qquad \text{for } n \geq 1$$

$$= \frac{-n}{n+1}\frac{-n+1}{n}\frac{-n+2}{n-1}\cdots\frac{-3}{4}\frac{-2}{3}\frac{-1}{2} I_0$$

Now $I_0 = \int_0^1 x\left(x^2-1\right)^0 dx$

$$= \left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{2}$$ **(1 mark)**

So $I_n = \frac{-n}{n+1}\frac{-n+1}{n}\frac{-n+2}{n-1}\cdots\frac{-3}{4}\frac{-2}{3}\frac{-1}{2}\frac{1}{2}$ for $n \geq 0$

$$= (-1)^n \frac{n}{n+1}\frac{n-1}{n}\frac{n-2}{n-1}\cdots\frac{3}{4}\frac{2}{3}\frac{1}{2}\frac{1}{2}$$

$$= (-1)^n \frac{1}{n+1}.\frac{1}{2}$$ (The other terms cancel.) **(1 mark)**

So $I_n = \frac{(-1)^n}{2(n+1)}$, $n \geq 0$ as required.

OR

$$I_n = \frac{-n}{n+1} I_{n-1} \qquad \text{for } n \geq 1$$

$$I_n = \frac{-n}{n+1} \int_0^1 x\left(x^2-1\right)^{n-1} dx$$

$$= \frac{-n}{n+1}\frac{1}{2}\int_0^1 2x\left(x^2-1\right)^{n-1} dx$$

$$= \frac{-n}{n+1}\frac{1}{2}\left[\frac{(x^2-1)^n}{n}\right]_0^1$$ **(2 marks)**

$$= \frac{-n}{n+1}\frac{1}{2}\left[0-\frac{(-1)^n}{n}\right]$$

So $I_n = \frac{(-1)^n}{2(n+1)}$, $n \geq 0$ as required.

(ii) Method 2 – "Otherwise"

$$I_n = \int_0^1 x(x^2-1)^n\,dx \quad \text{for } n = 0,1,2,\ldots$$
$$= \frac{1}{2}\int_0^1 2x(x^2-1)^n\,dx$$
$$= \frac{1}{2}\left[\frac{(x^2-1)^{n+1}}{n+1}\right]_0^1 \qquad \textbf{(1 mark)}$$
$$= \frac{1}{2(n+1)}\left(0-(-1)^{n+1}\right)$$
$$= \frac{-1(-1)^{n+1}}{2(n+1)}$$
$$= \frac{(-1)^n}{2(n+1)} \text{ for } n \geq 0 \text{ as required} \qquad \textbf{(1 mark)}$$

(iii) $I_0 = \frac{1}{2},\ I_1 = \frac{-1}{4},\ I_2 = \frac{1}{6},\ I_3 = \frac{-1}{8},\ I_4 = \frac{1}{10},\ I_5 = \frac{-1}{12}$

Clearly $I_{2n} > 0$ and $I_{2n+1} < 0$

So $I_{2n} > I_{2n+1}$

Alternatively, from (ii), $I_{2n} = \frac{(-1)^{2n}}{2(2n+1)}$

$$= \frac{1}{2(2n+1)}$$
$$> 0$$
$$I_{2n+1} = \frac{(-1)^{2n+1}}{2((2n+1)+1)}$$
$$= \frac{(-1)}{4(n+1)}$$
$$< 0$$

So $I_{2n} > I_{2n+1}$.

(1 mark)

Question 3 (15 marks)

(a) (i) $z\overline{w} = (3-i)(2-4i)$

$= 6-12i-2i-4$ **(1 mark)**

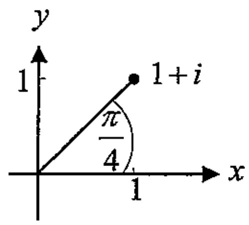
$= 2-14i$

(ii)
$$\frac{z}{w} = \frac{3-i}{2+4i} \times \frac{2-4i}{2-4i}$$
$$= \frac{2-14i}{20}$$
$$= \frac{1}{10} - \frac{7}{10}i$$

(1 mark)

(b) (i) From the diagram,

$1 + i = \sqrt{2} cis \frac{\pi}{4}$



(1 mark) for correct modulus
(1 mark) for correct argument

(ii) $(1+i)^n + (1-i)^n = 0$

Hence

$$\left(\sqrt{2}cis\frac{\pi}{4}\right)^n + \left(\sqrt{2}cis\frac{-\pi}{4}\right)^n = 0$$ **(1 mark)**

$$\left(\sqrt{2}\right)^n\left(\cos\frac{\pi n}{4} + i\sin\frac{\pi n}{4}\right) + \left(\sqrt{2}\right)^n\left(\cos\frac{\pi n}{4} - i\sin\frac{\pi n}{4}\right) = 0$$

$$\cos\frac{\pi n}{4} = 0$$ **(1 mark)**

$$\frac{\pi n}{4} = \frac{(2k+1)\pi}{2}$$

where $k = 0,1,2,...$

$$n = 2(2k+1)$$

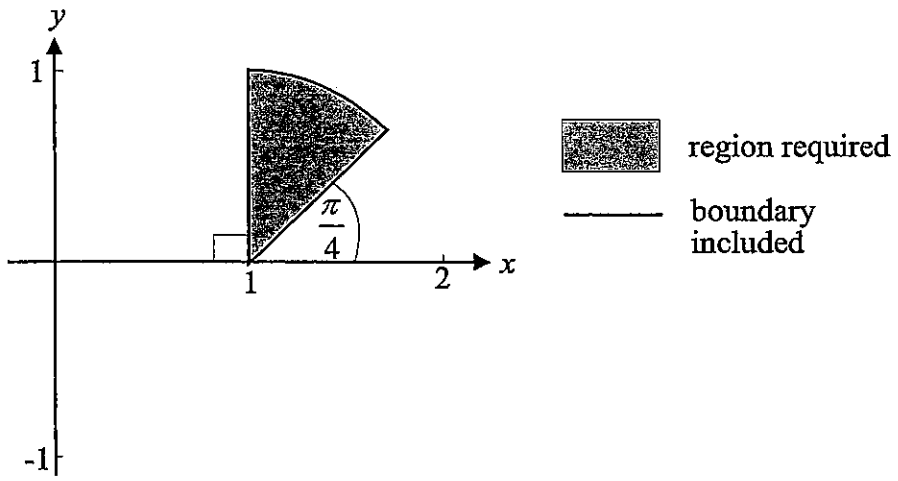
(1 mark)

(c) The inequality $|z-1| \le 1$ corresponds to a disc with centre at $(1,0)$ and radius 1.

The inequality $\frac{\pi}{4} \le \arg(z-1) \le \frac{\pi}{2}$ corresponds to a wedge with vertex $(1,0)$.

(1 mark)

The region where both these inequalities hold is shown in the diagram below.



(1 mark)

(d) Point P corresponds to the complex number z. Point Q is obtained by rotating point P **clockwise** through an angle of $\frac{\pi}{2}$ and reducing it by a factor of $\frac{1}{3}$.

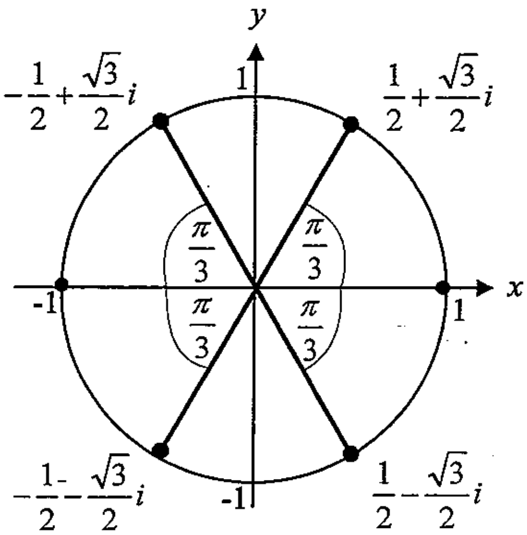
So, point Q corresponds to the complex number $\frac{-i}{3}z$.

(1 mark)

(e) (i) $z^6 = 1$

We are looking for the sixth roots of unity. We know that one root is 1 and another is -1. The 6 roots of unity are evenly spaced around the circumference of a circle of radius 1 unit.

So, the other four must be $cis\frac{\pi}{3}$, $cis\frac{2\pi}{3}$, $cis\frac{4\pi}{3}$ and $cis\frac{5\pi}{3}$.



(1 mark)

So, the six roots are

$\pm 1, +\frac{1}{2} \pm \frac{\sqrt{3}}{2}i, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. **(1 mark)**

(ii)

$$z^6 - 1 = 0$$

$$(z^3 - 1)(z^3 + 1) = 0$$

$$(z-1)(z^2+z+1)(z+1)(z^2-z+1) = 0$$ **(1 mark)**

The two real roots of the equation are revealed by the factors $(z-1)$ and $(z+1)$. The four non-real roots are revealed by the factors $(z^2+z+1)$and (z^2-z+1).

So $(\omega^2+\omega+1)(\omega^2-\omega+1) = 0$

$$\omega^4 + \omega^2 + 1 = 0$$

So $\omega^4 + \omega^2 = -1$ as required. **(1 mark)**

(iii) Let $\omega = cis\left(\frac{\pi}{3}\right)$

Now, $\omega^4 = cis\frac{4\pi}{3}$ (De Moivre)

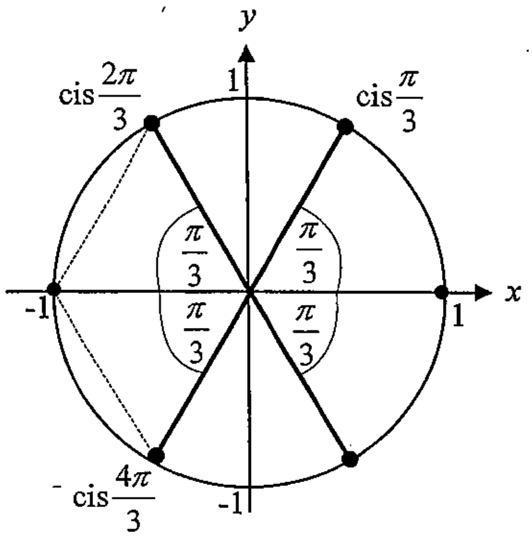
$\omega^2 = cis\frac{2\pi}{3}$ (De Moivre)

$\text{cis}\frac{4\pi}{3} + cis\frac{2\pi}{3} = -1$ by adding

the two complex numbers $cis\frac{4\pi}{3}$ and $cis\frac{2\pi}{3}$.

(Note, any of the four possible values of ω could have been chosen here to illustrate that $\omega^4 + \omega^2 = -1$.)

(1 mark)



Question 4 (15 marks)

(a) (i) P is the point $(a\cos\theta, b\sin\theta)$

$x = a\cos\theta \qquad y = b\sin\theta$

$\frac{dx}{d\theta} = -a\sin\theta \qquad \frac{dy}{d\theta} = b\cos\theta$

$\frac{dy}{dx} = -\frac{b\cos\theta}{a\sin\theta}$ **(1 mark)**

Equation of tangent is

$$y - b\sin\theta = -\frac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$$

$$ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$$

$$bx\cos\theta + ay\sin\theta = ab(\sin^2\theta + \cos^2\theta)$$

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$ **(1 mark)**

as required.

(ii) If T is the point of intersection between the tangent found in part (i) and one of the directrices of the ellipse, then T has the coordinates $\left(\frac{a}{e},0\right)$. **(1 mark)**

From (i), the gradient of the tangent at P is $\frac{-b}{a}\frac{\cos\theta}{\sin\theta}$. So using the coordinates of points P and T we have

$$\frac{b\sin\theta-0}{a\cos\theta-\frac{a}{e}}=\frac{-b}{a}\frac{\cos\theta}{\sin\theta}$$ **(1 mark)**

$$\frac{eb\sin\theta}{ae\cos\theta-a}=\frac{-b}{a}\frac{\cos\theta}{\sin\theta}$$

$$be\sin\theta\times a\sin\theta=-b\cos\theta(ae\cos\theta-a)$$

$$abe\sin^2\theta=-abe\cos^2\theta+ab\cos\theta$$

$$abe(\sin^2\theta+\cos^2\theta)=ab\cos\theta.$$

$$\cos\theta=e$$ as required.

(1 mark)

(iii) Since $\cos\theta=e$, the x-coordinate of P which is $a\cos\theta=ae$. So the focal chord through P makes an angle of 90° with the x-axis.

(1 mark)

(iv) The equation of the tangent through P is

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$

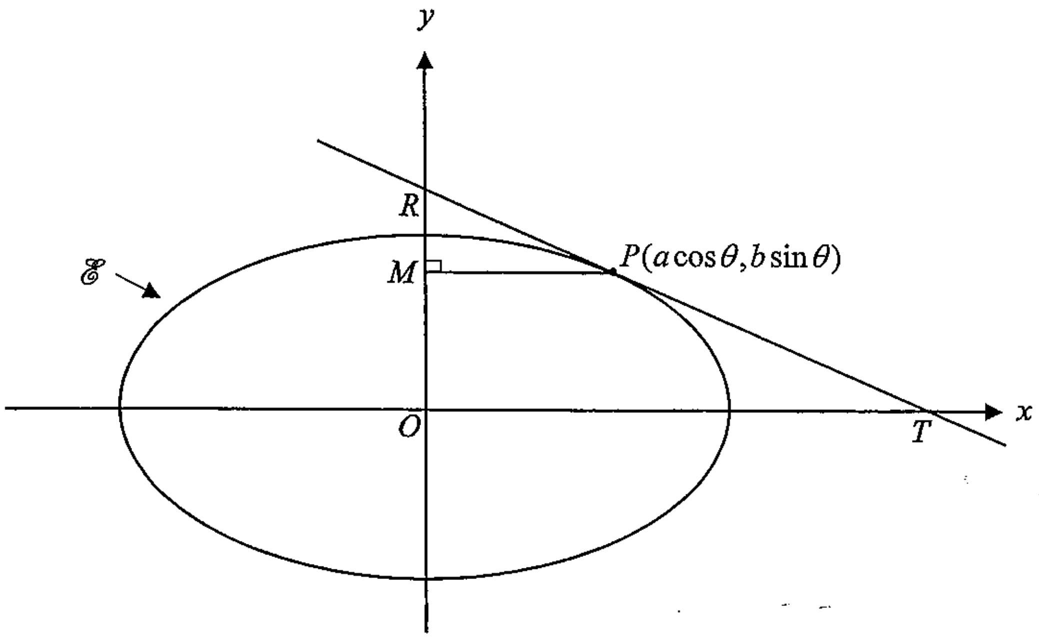
when $x=0$,

$$y\sin\theta=b$$

$$y=\frac{b}{\sin\theta}$$

R is the point $\left(0,\frac{b}{\sin\theta}\right)$

(1 mark)



Let M be the point on the y-axis such that PM is perpendicular to the y-axis.

M is the point $(0, b\sin\theta)$

Now since ΔROT is similar to ΔRMP,

$$\frac{RP}{RT} = \frac{RM}{RO} \qquad \textbf{(1 mark)}$$

$$= \frac{\frac{b}{\sin\theta} - b\sin\theta}{\frac{b}{\sin\theta}}$$

$$= \frac{b - b\sin^2\theta}{\sin\theta} \times \frac{\sin\theta}{b}$$

$$= 1 - \sin^2\theta$$

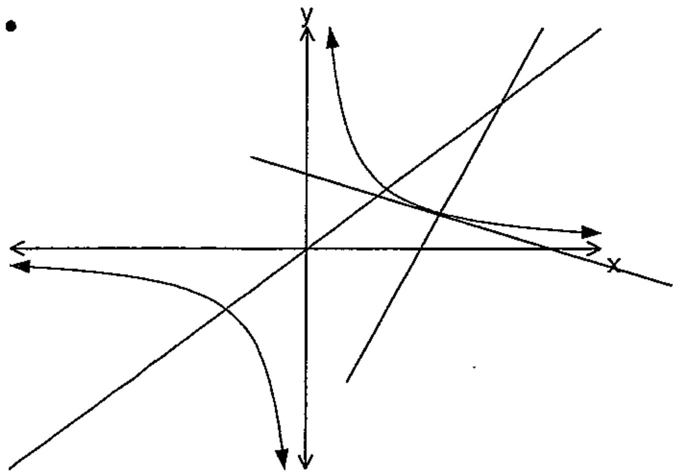
$$= \cos^2\theta$$

$$= e^2 \qquad \text{(from part (ii))}$$

So $\quad RP = e^2 RT \quad$ as required.

(1 mark)

(b) **(1 mark)**



: x Intercept (0, 0)

$xy = c^2 \quad \therefore y = c^2x^{-1}$ and $\dfrac{dy}{dx} = -\dfrac{c^2}{x^2}$

At $P(ct, \dfrac{c}{t})$ $\dfrac{dy}{dx} = \dfrac{-c^2}{c^2t^2} = \dfrac{-1}{t^2} = m_{\text{tangent}}$; $m_{\text{normal}} = t^2$ **(1 mark)**

<u>Now the equation of the tangent at P is:</u>

$y - \dfrac{c}{t} = \dfrac{-1}{t^2}(x - ct)$

$\therefore t^2y - ct = -x + ct$

$\therefore x + t^2y = 2ct$

At T $y = x \quad \therefore x + t^2x = 2ct \quad \therefore x = \dfrac{2ct}{1+t^2}$

$\Rightarrow T = (\dfrac{2ct}{1+t^2}, \dfrac{2ct}{1+t^2})$ **(1 mark)**

<u>Now the equation of the normal at P is:</u>

$y - \dfrac{c}{t} = t^2(x - ct)$

$\therefore t^3x - ty = c(t^4 - 1)$

At N $y = x \quad \therefore t^3x - tx = c(t^4 - 1) \qquad \therefore x = \dfrac{c(t^4-1)}{t(t^2-1)} = \dfrac{c(t^2+1)}{t}$

$\Rightarrow N = (\dfrac{c(t^2+1)}{t}, \dfrac{c(t^2+1)}{t})$ **(1 mark)**

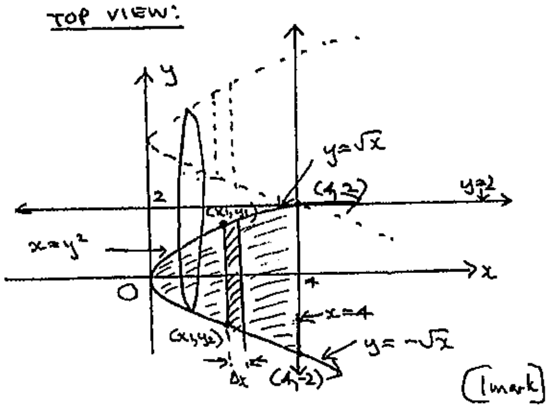
Now $OT.ON = \sqrt{(\dfrac{2ct}{1+t^2})^2 + (\dfrac{2ct}{1+t^2})^2}.\sqrt{(\dfrac{c(t^2+1)}{t})^2 + (\dfrac{c(t^2+1)}{t})^2}$

$= \sqrt{2}(\dfrac{2ct}{1+t^2}).\sqrt{2}(\dfrac{c(t^2+1)}{t})$ **(2 marks)**

$= 4c^2$

5 (a)

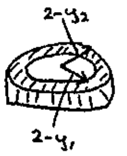
TOP VIEW:



(1 mark)

Take slice of thickness $\Delta x \perp$ to x-axis

SIDE VIEW:



$y_1 = \sqrt{x}$, $R_2 = 2 - y_2$

$y_2 = -\sqrt{x}$, $R_1 = 2 - y_1$

Area of cross-sectional slice, $A(x) = \pi(R_2^2 - R_1^2)$

$\therefore A(x) = \pi\left((2 - -\sqrt{x})^2 - (2-\sqrt{x})^2\right)$

$= \pi\left(4 + 4\sqrt{x} + x - (4 - 4\sqrt{x} + x)\right)$ (1 mark)

$= 8\pi\sqrt{x}$

Volume, ΔV, of each slice $= A(x)\Delta x$

Now total volume $= \lim\limits_{\Delta x \to 0} \sum\limits_{x=0}^{4} A(x)\,\Delta x$

$= 8\pi \int_0^4 x^{\frac{1}{2}}\,dx$ (1 mark)

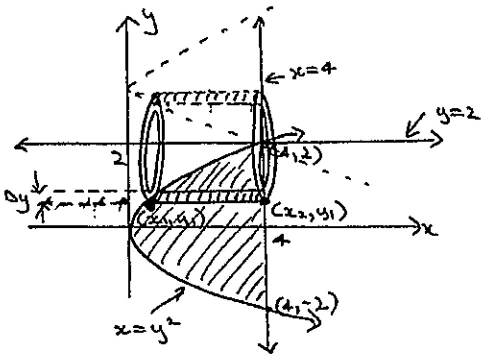
$= 8\pi\left[\dfrac{x^{3/2}}{3/2}\right]_0^4$

$= \dfrac{16\pi}{3}\left[4^{3/2} - 0\right]$

$= \dfrac{128\pi}{3}$ units^3 (1 mark)

(b)

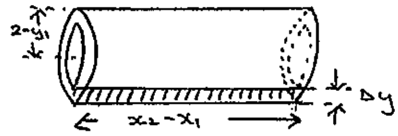
TOP VIEW:



Take slice of thickness $\Delta y \perp$ to y-axis.

(1 mark)

SIDE VIEW:



Each slice when rotated about the line $y=2$ generates a thin cylindrical shell of area $A(y) = 2\pi r h$.

Volume, ΔV, of each slice $= A(y)\,\Delta y$

Now total volume $= \lim\limits_{\Delta y \to 0} \sum\limits_{y=-2}^{2} A(y)\,\Delta y$

where $A(y) = 2\pi(2-y)(4-y^2)$ (1 mark)

$\therefore$ Total volume $= 2\pi\int_{-2}^{2} 8 - 4y - 2y^2 + y^3\,dy$ (1 mark)

$= 2\pi\left[8y - 2y^2 - \dfrac{2y^3}{3} + \dfrac{y^4}{4}\right]_{-2}^{2}$

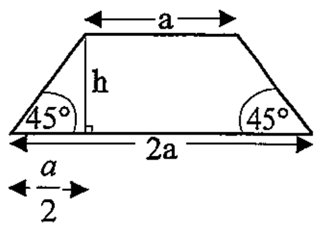
$= 2\pi\left[(16 - 8 - \tfrac{16}{3} + 4) - (-16 - 8 + \tfrac{16}{3} + 4)\right]$

$= 2\pi\left[32 - \tfrac{16}{3}\times 2\right]$

(1 mark) $= \dfrac{128\pi}{3}$ units^3

(c) (i) $\tan 45^\circ = h \div \frac{a}{2}$

$h = \frac{a}{2}$



Area of trapezium at $x = b$ is given by

$\frac{1}{2}(a+2a)\frac{a}{2}$

$= \frac{3a^2}{4}$

(1 mark)

(ii) Consider a slice of width Δx

$$\Delta V = \frac{3a^2}{4}\Delta x \qquad \text{where } 0 \le x \le 1$$

$$= \frac{3}{4}y^2\Delta x$$

$$= \frac{3}{4}\left(1-x^2\right)\Delta x \qquad \text{since } x^2+y^2=1$$

(1 mark)

So $\text{Volume} = \lim_{\Delta x \to 0} \sum_{x=0}^{1} \frac{3}{4}\left(1-x^2\right)\Delta x$

$$= \frac{3}{4}\int_0^1 \left(1-x^2\right)dx$$

$$= \frac{3}{4}\left[x - \frac{x^3}{3}\right]_0^1$$

$$= \frac{3}{4}\left(\frac{2}{3}\right)$$

$$= \frac{1}{2} \text{ unit}^3$$

(1 mark)

(iii) The solid generated when the semicircular base of $\mathfrak{D}$ is rotated through an angle of 90° is a quarter of a sphere. Hence the volume is $\frac{4}{3}\pi r^3 \div 4$. So required volume is $\frac{\pi}{3}$ since $r = 1$.

(1 mark)

(iv) From part (i), area of a trapezium at $x = b$ is given by $\frac{1}{2}(a+2a)h$ where h is the height of the trapezium.

Now $\tan\theta = h \div \frac{a}{2}$

$$= \frac{2h}{a}$$

So area of trapezium at $x = b$

$$= \frac{3a}{2} \times \frac{a}{2} \tan\theta$$

$$= \frac{3a^2}{4} \tan\theta$$

So $Volume = \frac{3}{4}\tan\theta \int_0^1 \left(1 - x^2\right) dx$ where θ is constant for a particular solid

$$= \frac{1}{2}\tan\theta$$ **(1 mark)**

We want to find θ such that

$$\frac{1}{2}\tan\theta > \frac{\pi}{3}$$

$$\tan\theta > \frac{2\pi}{3}$$

$$\theta > 64^\circ 29'$$ **(1 mark)**

However $\theta < 90^\circ$ since we have a trapezium.

So we require $64^\circ 29' < \theta < 90^\circ$. **(1 mark)**

Question 6 (15 marks)

(a) (i) As the coefficients of $\mathrm{P}(x)$ are real then $1+i$ is a further root of $\mathrm{P}(x)$.

$\therefore \mathrm{P}(x) = (x-1+i)(x-1-i)\mathrm{R}(x)$

$= ([x-1]^2 - i^2)\mathrm{R}(x)$ **(1mark)**

$= (x^2 - 2x + 2)\mathrm{R}(x)$

As $1-i$ is a root then $\mathrm{P}(1-i) = 0$

$\therefore (1-i)^3 + a(1-i)^2 + b(1-i) + 6 = 0$

$\therefore 1 + 3(-i) + 3(-i)^2 + (-i)^3 + a(1 - 2i + i^2) + b - bi + 6 = 0$ **(1mark)**

$\therefore 1 - 3i - 3 + i + a - 2ai - a + b - bi + 6 = 0$

$\therefore 4 + b + i(-2 - 2a - b) = 0$

Equating real and imaginary parts:

$4 + b = 0$.................(1)

$-2 - 2a - b = 0$.......(2) **(1mark)**

From (1) $b = -4$ sub.into (2) $\therefore a = 1$ **(1mark)**

(ii) $\therefore \mathrm{P}(x) = x^3 + x^2 - 4x + 6$

$= (x^2 - 2x + 2)(x + 3)$

$= (x - 1 + i)(x - 1 - i)(x + 3)$ over the complex field.

(1mark)

(b) $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$ (De Moivre)

$$(\cos\theta + i\sin\theta)^5 = \cos^5\theta + 5\cos^4\theta(i\sin\theta) + 10\cos^3\theta(i\sin\theta)^2 + 10\cos^2\theta(i\sin\theta)^3$$
$$+5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5$$
$$= \cos^5\theta + i5\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - i10\cos^2\theta\sin^3\theta$$
$$+5\cos\theta\sin^4\theta + i\sin^5\theta$$ **(1 mark)**

By equating real and imaginary parts,

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$
$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$ **(1 mark)**

$$\tan 5\theta = \frac{\sin 5\theta}{\cos 5\theta}$$
$$= \frac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta}$$
$$= \frac{\dfrac{5\sin\theta}{\cos\theta} - \dfrac{10\sin^3\theta}{\cos^3\theta} + \dfrac{\sin^5\theta}{\cos^5\theta}}{1 - \dfrac{10\sin^2\theta}{\cos^2\theta} + \dfrac{5\sin^4\theta}{\cos^4\theta}}$$

So $\tan 5\theta = \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$ **(1 mark)**

(ii)

Let $x = \tan\theta$ and $\tan 5\theta = 1$

So, $\tan 5\theta = \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$

becomes $\dfrac{5x - 10x^3 + x^5}{1 - 10x^2 + 5x^4} = 1$

So, $x^5 - 5x^4 - 10x^3 + 10x^2 + 5x - 1 = 0$

For $\tan 5\theta = 1$

$$5\theta = n\pi + \frac{\pi}{4},\ n \text{ is an integer}$$

$$\theta = \frac{n\pi}{5} + \frac{\pi}{20}$$

$$n = 0,\ \theta = \frac{\pi}{20}$$

$$n = 1,\ \theta = \frac{\pi}{4}$$

$$n = -1,\ \theta = \frac{-3\pi}{20}$$

$$n = 2,\ \theta = \frac{9\pi}{20}$$

$$n = -2,\ \theta = \frac{-7\pi}{20}$$

So, $x = \tan\frac{\pi}{4} = 1,\ \tan\frac{\pi}{20},\ \tan\frac{9\pi}{20},\ \tan\left(\frac{-3\pi}{20}\right) = -\tan\frac{3\pi}{20}$ and $\tan\left(\frac{-7\pi}{20}\right) = -\tan\frac{7\pi}{20}$

(2 marks)

(iii)

$$\text{let } P(x) = x^5 - 5x^4 - 10x^3 + 10x^2 + 5x - 1 = 0$$

$$(x-1)(x^4 - 4x^3 - 14x^2 - 4x + 1) = 0$$ **(1 mark)**

$$x^4 - 4x^3 - 14x^2 - 4x + 1 = x^2(x^2 - 4x - 14 - \frac{4}{x} + \frac{1}{x^2}) = 0$$

So, $x^2\left(x^2 + \frac{1}{x^2} - 4\left(x + \frac{1}{x}\right) - 14\right) = 0$

but $x^2 \neq 0$ so, $\left(x + \frac{1}{x}\right)^2 - 4\left(x + \frac{1}{x}\right) - 16 = 0$

since $P(0) \neq 0$, $\quad x + \frac{1}{x} = \frac{4 \pm \sqrt{16 - 4(1)(-16)}}{2}$

$$= \frac{4 \pm \sqrt{80}}{2}$$

$$= \frac{4 \pm 4\sqrt{5}}{2}$$

So, $x + \frac{1}{x} = 2 + 2\sqrt{5}$, or $x + \frac{1}{x} = 2 - 2\sqrt{5}$

(2 marks)

Now $\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right) = \dfrac{1}{\tan\left(\frac{\pi}{2} - \theta\right)}$,

so, if $x = \tan\frac{\pi}{20}$ then $\tan\frac{\pi}{20} + \dfrac{1}{\tan\frac{\pi}{20}} = 2 + 2\sqrt{5}$

and hence $\tan\frac{\pi}{20} + \tan\frac{9\pi}{20} = 2 + 2\sqrt{5}$

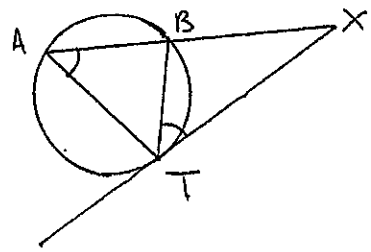
Similarly, if $x = -\tan\frac{3\pi}{20}$ then $-\tan\frac{3\pi}{20} - \dfrac{1}{\tan\frac{3\pi}{20}} = 2 - 2\sqrt{5}$

and hence $\tan\frac{3\pi}{20} + \tan\frac{7\pi}{20} = 2\sqrt{5} - 2$

(2 marks)

Question 7

(a) (i)



In Δs XAT and XTB:

$\angle XAT = \angle XTB$

[Angle between tangent and chord at point of contact equals angle in alternate segment] [1 mark]

$\angle X$ is common

$\angle XTA = \angle XBT$ [Remaining $\angle$s are equal; $\angle$ sum of $\Delta = 180°$]

$\therefore$ As Δs are equiangular

$\therefore \Delta XAT \ ||| \ \Delta XTB$ [1 mark]

(ii) Now $\dfrac{XA}{XT} = \dfrac{XT}{XB}$ [Corresponding sides of similar Δs are in the same ratio.] [1 mark]

$\therefore XT^2 = XA \cdot XB$. [1 mark]

(b)

(i) $x = \sqrt{g}\, t\cos\theta$

So $t = \dfrac{x}{\sqrt{g}\cos\theta}$

In $y = \sqrt{g}t\sin\theta - \dfrac{1}{2}gt^2$

becomes $y = \dfrac{\sqrt{g}x\sin\theta}{\sqrt{g}\cos\theta} - \dfrac{1}{2}g\dfrac{x^2}{g\cos^2\theta}$

So, $y = x\tan\theta - \dfrac{x^2\sec^2\theta}{2}$

(1 mark)

(ii) From (i) we have

$$y = x\tan\theta - \frac{x^2\sec^2\theta}{2}$$

$$= x\tan\theta - \frac{x^2}{2}\left(1+\tan^2\theta\right)$$

At point P, $x = d\cos 30^\circ$ and $y = d\sin 30^\circ$

$$= \frac{\sqrt{3}d}{2} \qquad = \frac{d}{2}$$

(1 mark)

So $\dfrac{d}{2} = \dfrac{\sqrt{3}d}{2}\tan\theta - \dfrac{3d^2}{8}\left(1+\tan^2\theta\right)$

$$4d = 4\sqrt{3}d\tan\theta - 3d^2 - 3d^2\tan^2\theta$$

$$3d^2\tan^2\theta - 4\sqrt{3}d\tan\theta + 3d^2 + 4d = 0$$

(1 mark)

We have a quadratic in $\tan\theta$.

So $\Delta = 48d^2 - 4\times 3d^2\left(3d^2+4d\right)$

$$= 48d^2 - 12d^2\left(3d^2+4d\right)$$

$$= 12d^2\left(4-3d^2-4d\right)$$

If there is one path of trajectory for the particle to land at point P then $\Delta = 0$.

So $12d^2\left(4-3d^2-4d\right)=0$

$$4-3d^2-4d=0 \qquad \left(12d^2 \neq 0\right)$$

$$\left(-3d+2\right)\left(d+2\right)=0$$

$$d=\frac{2}{3} \text{ or } d=-2 \qquad \text{reject this since } d>0$$

So $d=\frac{2}{3}$ **(1 mark)**

So we have, $\frac{4}{3}\tan^2\theta-\frac{8\sqrt{3}}{3}\tan\theta+4=0$

So $\tan\theta=\left(\frac{8\sqrt{3}}{3}\pm\sqrt{0}\right)\div\frac{8}{3}$

$$=\sqrt{3}$$

$$\theta=60^\circ$$ **(1 mark)**

(c)

$$\cos 4\theta+\cos 2\theta=\sqrt{2}\cos^2\theta+\frac{1}{\sqrt{2}}\sin 2\theta$$

$$2\cos 3\theta\cos\theta=\sqrt{2}\cos^2\theta+\sqrt{2}\sin\theta\cos\theta$$ **(1 mark)**

$$\sqrt{2}\cos 3\theta\cos\theta=\cos\theta(\cos\theta+\sin\theta)$$

$$=\cos\theta\left(\sqrt{2}\cos\left(\theta-\frac{\pi}{4}\right)\right)$$ **(1 mark)**

So $\cos 3\theta\cos\theta=\cos\theta\cos\left(\theta-\frac{\pi}{4}\right)$

$$\cos\theta=0 \text{ or } \cos 3\theta=\cos\left(\theta-\frac{\pi}{4}\right)$$ **(1 mark)**

$$\theta=2n\pi\pm\frac{\pi}{2} \text{ or } 3\theta=2n\pi\pm\left(\theta-\frac{\pi}{4}\right) \qquad n \text{ is an integer}$$

$$2\theta=2n\pi-\frac{\pi}{4} \quad \text{or} \quad 4\theta=2n\pi+\frac{\pi}{4}$$

So, $\theta=2n\pi\pm\frac{\pi}{2}$ or $\theta=n\pi-\frac{\pi}{8}$ or $\theta=\frac{n\pi}{2}+\frac{\pi}{16}$

(1 mark) **(1 mark)** **(1 mark)**

Question 8 (15 marks)

(a) (i)

$$(a-b)^2 \geq 0$$
$$a^2+b^2-2ab \geq 0$$
$$a^2+b^2 \geq 2ab \quad \text{(equality iff } a=b\text{)}$$

(1 mark)

Similarly, $b^2+c^2 \geq 2bc$

$$a^2+c^2 \geq 2ac$$

By addition, $2\left(a^2+b^2+c^2\right) \geq 2(ab+bc+ca)$

$$a^2+b^2+c^2 \geq ab+bc+ca \quad \text{(equality iff } a=b=c\text{)}$$

(1 mark)

(ii)

$$(a+b+c)^2 = a^2+b^2+c^2+2(ab+ac+bc)$$

Since $a^2+b^2+c^2 \geq ab+bc+ca$

$$(a+b+c)^2 \geq ab+bc+ca+2(ab+ac+bc)$$
$$(a+b+c)^2 \geq 3(ab+bc+ca)$$

Since $a+b+c=9$,

$$81 \geq 3(ab+bc+ca)$$
$$ab+bc+ca \leq 27 \quad \text{(equality iff } a=b=c\text{)}$$

(2 marks)

$$\frac{1}{abc}(ab+bc+ca) \leq \frac{27}{abc}$$
$$\frac{1}{c}+\frac{1}{a}+\frac{1}{b} \leq \frac{27}{abc}$$

ie $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \leq \frac{27}{abc}$ as required **(1 mark)**

8(b)

Step 1: When $n=1$ $U_1 = 3^1 - 2^1 = 1$ which is true

$n=2$ $U_2 = 3^2 - 2^2 = 5$ which is true

$n=3$ $U_3 = 5U_2 - 6U_1$
$= 25 - 6$
$= 19$
$= 3^3 - 2^3$ which is true

[2 marks]

$\therefore$ it is true for $n = 1, 2$ and 3.

Step 2: Assume it is true for $n=k$ ($k \le n$, $k \in J^+$) and prove it is true for $n = k+1$.

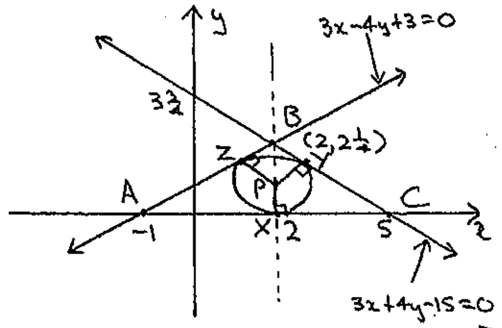
Now if $n = k+1$ $U_n = U_{k+1}$
$= 5U_k - 6U_{k-1}$
$= 5[3^k - 2^k] - 6[3^{k-1} - 2^{k-1}]$
$= 5.3^k - 5.2^k - 6.3^{k-1} + 6.2^{k-1}$
$= 5.3^k - 2.3^k - 5.2^k + 3.2^k$
$= 3.3^k - 2.2^k$
$= 3^{k+1} - 2^{k+1}$

[2 marks]

$\therefore$ if it is true for $n=k$ so it is true for $n=k+1$.

Step 3: It is true for $n = 1, 2$ and 3 so it is true for $n = 3+1 = 4$. It is true for $n=4$ and so it is true for $n = 4+1 = 5$ and so on for all positive integral values of n. [1 mark]

8(c)



[1 mark]

Let $A = (-1, 0)$, $C = (5, 0)$

$3x - 4y + 3 = 0$ — ①
$3x + 4y - 15 = 0$ — ②

①+②: $6x - 12 = 0$ $\therefore x = 2$ sub into ②
$\therefore y = 2\frac{1}{4}$ $\therefore B \doteq (2, 2\frac{1}{4})$

Now $d_{AB} = \sqrt{(2--1)^2 + (2\frac{1}{4})^2} = 3\frac{3}{4}$

$d_{BC} = \sqrt{(5-2)^2 + (-2\frac{1}{4})^2} = 3\frac{3}{4}$

$\therefore \triangle ABC$ is isosceles

$\therefore x = 2$ is the right bisector of side AC. [1 mark]

Let $P(2, y_1)$ be the centre of the inscribed circle.

Let X, Y and Z be the feet of the perpendiculars drawn from P to each line.

Now perp. distance $PX = PY = PZ$

$\therefore |y_1| = \frac{|6 + 4y_1 - 15|}{\sqrt{3^2+4^2}} = \frac{|6 - 4y_1 + 3|}{\sqrt{3^2+4^2}}$ [1 mark]

$\therefore |y_1| = \frac{|4y_1 - 9|}{5} = \frac{|9 - 4y_1|}{5}$

$\therefore 25y_1^2 = 16y_1^2 - 72y_1 + 81$
$\therefore 9y_1^2 + 72y_1 - 81 = 0$ $\therefore 9(y_1 + 9)(y_1 - 1) = 0$
$\therefore y_1 = 1$ (from diagram) [1 mark]

$\Rightarrow$ centre of inscribed circle is $(2, 1)$ and equation is: $(x-2)^2 + (y-1)^2 = 1$. [1 mark]