Term 3, 2009

Year 12 Mathematics Extension 2

Trial HSC Examination

Friday July 31, 2009

Time Allowed: 3 hours, plus 5 minutes reading time

Total Marks: 120

There are 8 questions, all of equal value.

Submit your work in eight 4 Page booklets.

All necessary working should be shown in every question. Full marks may not be awarded if work is careless or badly arranged. Board of Studies approved calculators may be used. A list of standard integrals is attached to the back of this paper.

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Question 1 (15 marks) Use a separate page/booklet **Marks**

(a) Find the indefinite integral:

(i) $\int \frac{dx}{4-9x^2}$ **3**

(ii) $\int \frac{dx}{\sqrt{4-9x^2}}$ **2**

(iii) $\int \frac{dx}{\left(4-9x^2\right)^{3/2}}$ **3**

(b) Evaluate:

(i) $\int_0^{1/2} \sin^{-1}x \, dx$ **3**

(ii) $\int_0^2 \frac{x^2 \, dx}{x^6+64}$ **2**

(iii) $\int_0^a x\sin(a-x) \, dx$ **2**

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Question 2 (15 marks) Use a separate page/booklet **Marks**

(a) If z and w are complex numbers, $\bar{z}$ and $\bar{w}$ are the respective complex conjugates of z and w.

(i) Prove: $\overline{z+w} = \bar{z} + \bar{w}$ **1**

(ii) Show that $|z|^2 = z\bar{z}$ **1**

(iii) Prove: $3|z-1|^2 = |z+1|^2$ if and only if $|z-2|^2 = 3$. **2**

(iv) Draw a rough sketch of the subset of the complex plane, where

$$\sqrt{3}\,|z-1| = |z+1|$$ **2**

(v) (α) Prove that for all complex numbers z and w

$$|z+w|^2 + |z-w|^2 = 2\left[\,|z|^2 + |w|^2\,\right]$$ **2**

(β) Give a geometrical interpretation of this equation in the complex plane. **1**

(b) (i) If $z = \cos\theta + i\sin\theta$, show that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$ **1**

(ii) For $z = r(\cos\theta + i\sin\theta)$, find r and the smallest value of θ which may satisfy the equation $2z^3 = 9 + 3\sqrt{3}\,i$. **2**

(c) Shade the region in the complex plane, for which

$$\frac{\pi}{6} \le \arg z \le \frac{\pi}{3}$$ **1**

(d) Two fixed points z_1 and z_2 and a variable point z represent the complex numbers z_1, z_2 and z respectively. Find the locus if

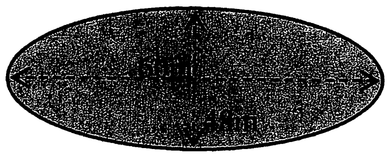
$$\arg\left[\frac{z-z_1}{z-z_2}\right] = \beta$$ **2**

Question 3 (15 marks) Use a separate page/booklet **Marks**

(a) Show that the equation of the circle on the diameter joining the points (x_1, y_1) and (x_2, y_2) is given by **2**

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

(b) The village green of Mathematicus is in the shape of an ellipse with external dimensions 60m by 48m.



NOT TO SCALE

(i) Write an equation to model the shape of the green. Assume that the centre of the green is the origin. **1**

(ii) Write the coordinates of the focii and the equations of the directrices. **2**

(c) (i) In the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $(a > b)$, A and A' are two points where the ellipse cuts the y- *axis*. The tangents at A and A' to the ellipse intersect the tangent at P in Q and Q' respectively. Show that $AQ \times A'Q' = a^2$. **2**

(ii) If the circle on QQ' as diameter meets the x *–axis* at the points R and R', show that $OR \times OR' = a^2 - b^2$, where O is the origin of the ellipse. **3**

(d) If $(2 - \sqrt{3})^n = a_n - b_n\sqrt{3}$ for all positive integers *n*, where a_n and b_n are integers, show that

(i) $a_{n+1} = 2a_n + 3b_n$ and $b_{n+1} = a_n + 2b_n$ **1**

(ii) Calculate: $a_n^{\ 2} - 3b_n^{\ 2}$, for $n = 1, 2$ and 3 **2**

(iii) Guess a formula for $a_n^{\ 2} - 3b_n^{\ 2}$ and prove your guess is true for all positive integers *n*. **2**

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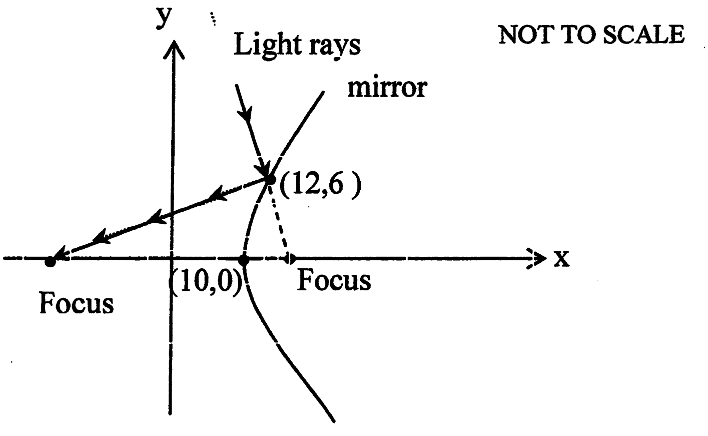
Question 4 (15 marks) Use a separate page/booklet **Marks**

(a) A hyperbolic mirror is used in some telescopes. It has the property that a light ray directed at one focus will be reflected to the other focus. Using the figure given below write an equation to model the hyperbolic mirror's surface. 2



(b) P $\left(2p, \dfrac{2}{p}\right)$ is a point on the hyperbola $xy = 4$

(i) Show that the equation of the normal at P is given by

$$y = p^2x - 2p^3 + \frac{2}{p}$$ 3

(ii) If this normal meets the $x - axis$ at Q, find the coordinates of Q 1

(iii) Find the coordinates the midpoint M of PQ . 2

(iv) Hence find the locus of M. 1

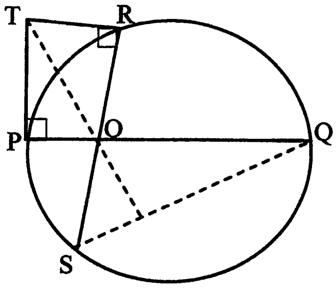
(c) (i) Show that $(\cos\theta + i\sin\theta)^5 = \sum_{r=0}^{5} {}^5c_r \, i^r \cos^{5-r}\theta \sin^r\theta$ 2

(ii) Hence prove that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$ 2

(iii) Deduce that $\cos\dfrac{\pi}{10}\cos\dfrac{3\pi}{10}\cos\dfrac{7\pi}{10}\cos\dfrac{9\pi}{10} = \dfrac{5}{16}$ 2

Question 5 (15 marks) Use a separate page/booklet **Marks**

(a) NOT TO SCALE



In the diagram, PQ and RS are two chords of the circle intersecting at O. TR and TP are perpendicular to RS and PQ respectively. Prove that the line through T and O is perpendicular to SQ. **3**

(b) If $p > 0$ and $q > 0$, and $p + q = 1$, show that $\frac{1}{p} + \frac{1}{q} \geq 4$ **2**

(c) Find the general solution of the equation $\cos 5\theta - \sin 4\theta = 0$
Hence write down the solutions in $0 \leq x \leq 4\pi$. **3**

(d) If ω is a complex root of the equation $x^3 = 1$, show that the other complex root is ω^2 and $1 + \omega + \omega^2 = 0$ **2**

(e) If $y = \dfrac{x^2}{x^3 + 1}$.

(i) Find the coordinates of the stationary points of the curve and determine their nature. **3**

(ii) Sketch the curve. **1**

(iii) Use the sketch to find the number of real roots of the equation $x^3 - 4x^2 + 1 = 0$ **1**

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Question 6 (15 marks) Use a separate page/booklet **Marks**

(a) The angles of elevation of the top of a tower P measured from three points A, B, C are α, β, γ respectively.
A,B,C are in a straight line such that AB = BC = a, but the line AC does not pass through S, the base of the tower.

(i) If $\angle ABS = \theta$, show that

$$CS^2 = a^2 + h^2 \cot^2 \beta + 2ah \cot \beta \cos \theta$$ **2**

(ii) Prove that the height of the tower is

$$\frac{a\sqrt{2}}{\left\{\cot^2 \alpha + \cot^2 \gamma - 2\cot^2 \beta\right\}^{\frac{1}{2}}}$$ **3**

(b) If α, β, γ are the roots of the cubic equation $x^3 + qx + r = 0$, find the value of (in terms of q, r)

(i) $\alpha + \beta + \gamma$ **1**

(ii) $\alpha\beta + \alpha\gamma + \beta\gamma$ **2**

(iii) $\alpha\beta\gamma$ **1**

(iv) Hence find the value of $(\beta - \gamma)^2 + (\gamma - \alpha)^2 + (\alpha - \beta)^2$ **3**

(c) A particle moving with simple harmonic motion has a speed of 32m/s and 24 m/s when its distances from the centre of motion are respectively 3 m and 4 m. Find the periodic time of the motion. **3**

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Question 7 (15 marks) Use a separate page/booklet **Marks**

(a) Use the shell method to find the volume generated by revolving about the x – axis the region in the coordinate plane bounded by

$y = \frac{1}{4}x^3$ *and* $y = \sqrt{2x}$. **4**

(b) If a >0 and b>0, c >0, show that

(i) $\frac{a+b+c}{3} \geq \sqrt[3]{abc}$ **3**

(ii) $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3$ **1**

(c) A circular hole is drilled through the centre of a sphere of radius 1 cm so that the volume of the sphere cut away is half of the volume of the sphere. Find the radius of the drill bit used. **4**

(d) Find the roots of the equation $x^4 + 2x^3 + 6x^2 + 8x + 8 = 0$ given that one of the roots is purely imaginary. **3**

Question 8 (15 marks) Use a separate page/booklet **Marks**

(a) Consider two polynomials $P(x)$ and $F(x)$.
When $P(x)$ is divided by x^2+6x+8 the remainder is $2x-11$.
When $F(x)$ is divided by x^2+6x+8 the remainder is $x+4$.
With each division the quotient is the same.

(i) Show that $P(x)$ and $F(x)$ must have the same degree. **1**

(ii) Write an expression for $P(x)\cdot F(x)$ **1**

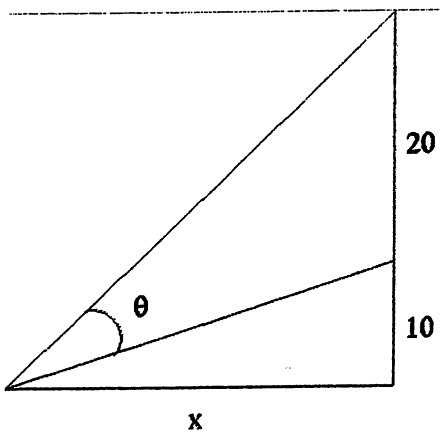
(iii) Find the remainder when $P(x)$ is divided by $x+4$. **1**

(b) Define $I_n = \int_0^{\pi/4} \tan^n x \, dx$.

(i) Show that $I_n + I_{n-2} = \dfrac{1}{n-1}$, where n is an integer and $n \geq 3$ **4**

(ii) Hence evaluate I_7 **2**

(c) A movie screen on a wall at the front of a room is 10 m high and 5 m above the floor. Ralf is to position himself a distance x in from the front of the room where his viewing angle of the movie screen is θ. (See diagram)



(i) Show that

$$\theta = \tan^{-1}\left(\frac{30}{x}\right) - \tan^{-1}\left(\frac{10}{x}\right)$$

2

(ii) Show that

$$\frac{d\theta}{dx} = \frac{10}{x^2+100} - \frac{30}{x^2+900}$$

1

(iii) Find the value of the maximum possible viewing angle, θ, and what the distance x needs to be in order to achieve it. 3

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, n \neq -1;\ x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ \ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ \ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\,dx, = \frac{1}{a}\tan ax,\ \ a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln(x + \sqrt{x^2 - a^2}),\ \ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln(x + \sqrt{x^2 + a^2})$$

NOTE : $\ln x = \log_e x, \qquad x > 0$