Centre Number | 1 | 2 | 5 |

CRANBROOK SCHOOL

Student Number | | | | | | | | |

2017

HSC Trial Examination

Assessment Task 3

Mathematics Extension 2

Reading time	5 minutes
Writing time	3 hours
Total Marks	100
Task weighting	40%

General Instructions

- Write using blue or black pen
- Diagrams drawn using pencil
- A Board-approved calculator may be used
- The Reference sheet is on the last page of this booklet
- Use the Multiple-Choice Answer Sheet provided
- All relevant working should be shown for each question

Additional Materials Needed

- Multiple Choice Answer Sheet
- 6 writing booklets

Structure & Suggested Time Spent

Section I

Multiple Choice Questions

- Answer Q1 – 10 on the multiple choice answer sheet
- Allow about 15 minutes for this section

Section II

Extended response Questions

- Attempt all questions in this section in a separate writing booklet
- Allow about 2 hours and 45 minutes for this section

This paper must not be removed from the examination room

Disclaimer

The content and format of this paper does not necessarily reflect the content and format of the HSC examination paper.

Section I

10 Marks

Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1 - 10.

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1 The polynomial $P(z)$ has real coefficients, and $P(3 - i) = 0$.

Which of the following must be a quadratic factor of $P(z)$?

(A) $z^2 - 6z + 10$

(B) $z^2 + 6z + 10$

(C) $z^2 - 6z + 8$

(D) $z^2 + 6z + 8$

2 Which of the following best describes the locus of $\frac{1}{z} + \frac{1}{\overline{z}} = 1$?

(A) A straight line

(B) A circle

(C) A parabola

(D) An ellipse

3 An ellipse is centred about the origin, and its foci lie on the x axis. The distance between the foci is 10 units, and the distance between the directrices is 50 units. What is the equation of the ellipse?

(A) $\dfrac{x^2}{250}+\dfrac{y^2}{100}=1$

(B) $\dfrac{x^2}{100}+\dfrac{y^2}{250}=1$

(C) $\dfrac{x^2}{125}+\dfrac{y^2}{100}=1$

(D) $\dfrac{x^2}{100}+\dfrac{y^2}{125}=1$

4 What is the equation of the normal to the hyperbola $\dfrac{x^2}{144}-\dfrac{y^2}{36}=1$ at the point $P(12\sec\theta, 6\tan\theta)$, where $\theta=\dfrac{\pi}{3}$?

(A) $x+\sqrt{3}y=6$

(B) $x-\sqrt{3}y=6$

(C) $\sqrt{3}x-y=30\sqrt{3}$

(D) $\sqrt{3}x+y=30\sqrt{3}$

5 What is the gradient of the tangent to curve $x^2 - xy + 2y^2 = 4$ at the point $(2,1)$?

(A) $\frac{3}{2}$

(B) $-\frac{3}{2}$

(C) $\frac{2}{3}$

(D) $-\frac{2}{3}$

6 Which of the following integrals is the greatest in value?

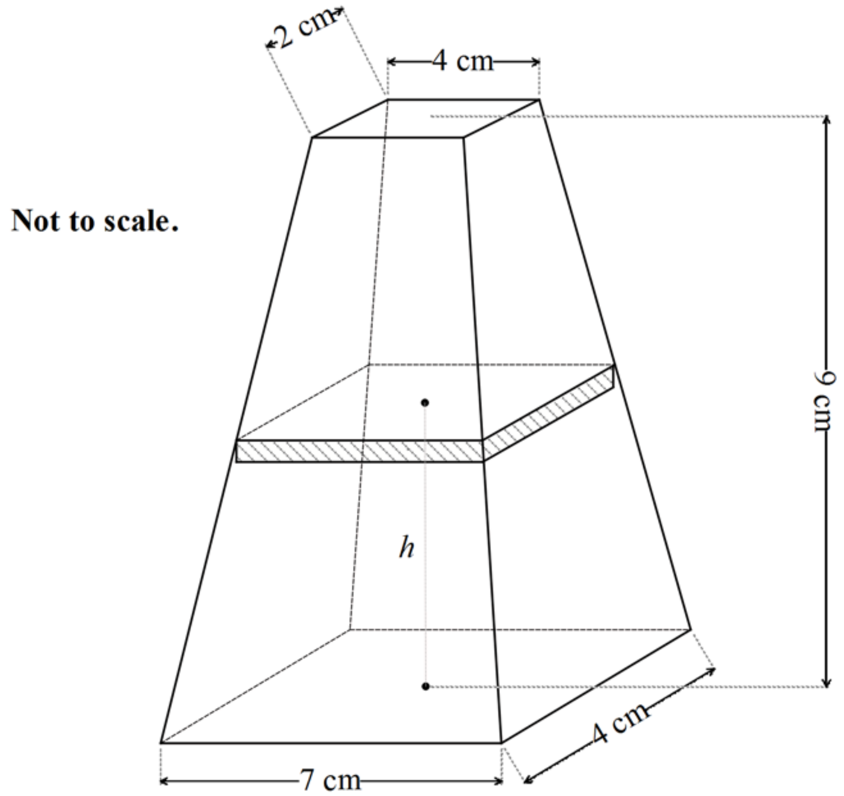
(A) $\int_0^2 \left|2x - x^2\right| dx$

(B) $\int_0^2 2x - x^2 \; dx$

(C) $\int_0^2 \sqrt{2x - x^2} \; dx$

(D) $\int_0^2 \left(2x - x^2\right)^3 dx$

7 Consider the following solid as shown below, which has a rectangular base and a rectangular top. Which expression best represents the volume of the slice taken at a height h cm from the base of the solid?



(A) $\delta V = \left(7 - \frac{h}{2}\right)\left(4 - \frac{h}{4}\right)\delta h$

(B) $\delta V = \left(7 - \frac{h}{3}\right)\left(4 - \frac{2h}{9}\right)\delta h$

(C) $\delta V = \left(4 - \frac{2h}{9}\right)^2 \delta h$

(D) $\delta V = \left(4 - \frac{h}{3}\right)\left(7 - \frac{2h}{9}\right)\delta h$

8 Which of the following is always true for the continuous function $f(x)$?

(A) $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

(B) $\int_{-a}^{0} f(x)\,dx = -\int_{0}^{a} f(-x)\,dx$

(C) $\int_{0}^{a} f(x)\,dx = \int_{a}^{2a} f(2a-x)\,dx$

(D) $2\int_{0}^{a} f(x)\,dx = \int_{-a}^{a} f(a-x)\,dx$

9 What are the roots of the equation $z^2 + z(2-10i) + 2i - 19 = 0$?

(A) $1 + 2i$, $1 - 2i$

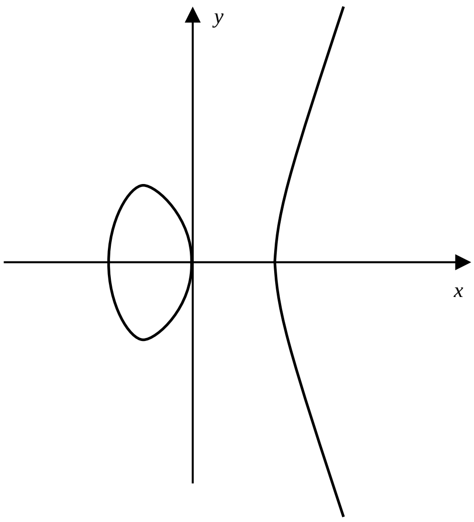
(B) $1 + 2i$, $-3 + 8i$

(C) $1 - 2i$, $-3 + 8i$

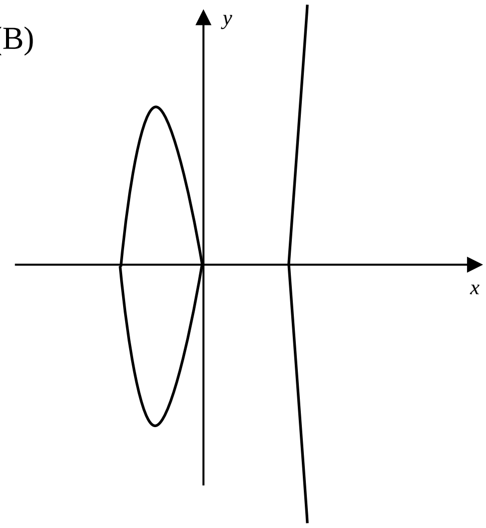
(D) $1 + 2i$, $3 - 8i$

10 Which diagram best represents the curve $y^2 = x^3 - 5x$?

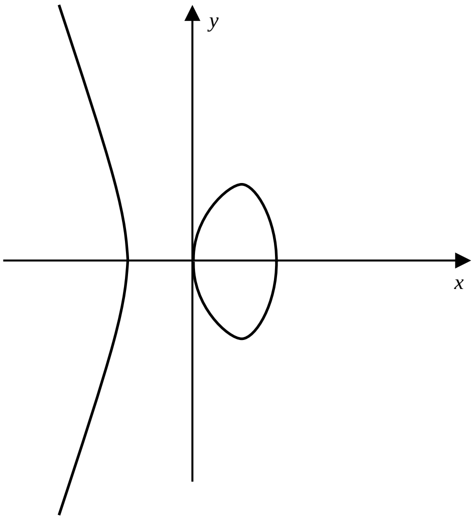
(A)



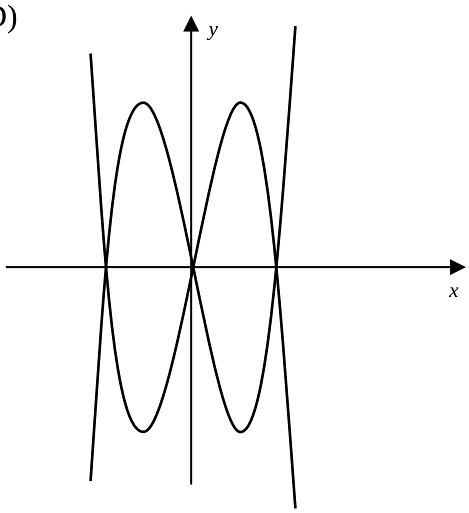
(B)



(C)



(D)



END OF SECTION I

Section II

90 Marks

Allow about 2 hours and 45 minutes for this section

Answer Questions 11 – 16 in separate writing booklets.

Question 11 **Start a new booklet** **15 Marks**

(a) Consider the complex numbers $z = -\sqrt{3} + i$ and $w = 8 + 8i$.

(i) Write z and w in modulus argument form. **2**

(ii) Express $\dfrac{z^3}{(\overline{w})^8}$ in Cartesian form. **2**

(b) Find the solutions to $z^2 + 2(1+i)z + (3+6i) = 0$ in simplest Cartesian Form **3**

(c) Sketch and describe the locus of all points z which satisfy the equation **2**

$|z - 4 - i| = 4$

(d) Find $\displaystyle\int \frac{1}{x\left(x^{2017}+1\right)}\,dx$ **2**

Question 11 continues on page 9.

Question 11 (continued)

(e) Consider the geometric series:

$$1+\frac{1}{3}\text{cis}\,\theta+\frac{1}{9}(\text{cis}\,\theta)^2+\frac{1}{27}(\text{cis}\,\theta)^3+\dots$$

(i) Explain why this series has a limiting sum. **1**

(ii) Hence, show that: **3**

$$\sin\theta+\frac{1}{3}\sin 2\theta+\frac{1}{9}\sin 3\theta+\dots=\frac{9\sin\theta}{10-6\cos\theta}$$

END OF QUESTION 11

Question 12 **Start a new booklet** **15 Marks**

(a) Let there be two arbitrary complex numbers z and w.

(i) Sketch the complex numbers z, w and $z + w$ on the Argand diagram, representing each as a vector. **1**

(ii) Justify why $|z+w| \le |z| + |w|$ **1**

(iii) If $|z| = 10$, prove that $|13z^2 - 2z + 5| \le 1325$ **2**

(b) Let a polynomial $p(x)$ have a root α of multiplicity m. That is, we can represent the polynomial as $p(x) = (x-\alpha)^m \times Q(x)$, where $Q(x)$ is another polynomial.

(i) Explain why $Q(\alpha) \ne 0$ **1**

(ii) Prove that $p'(x)$ has a root α of multiplicity exactly $m - 1$. **2**

(iii) Given that $f(x) = 2x^4 - 15x^3 + 42x^2 - 52x + 24$ has a triple root, find all the solutions to $f(x) = 0$. **2**

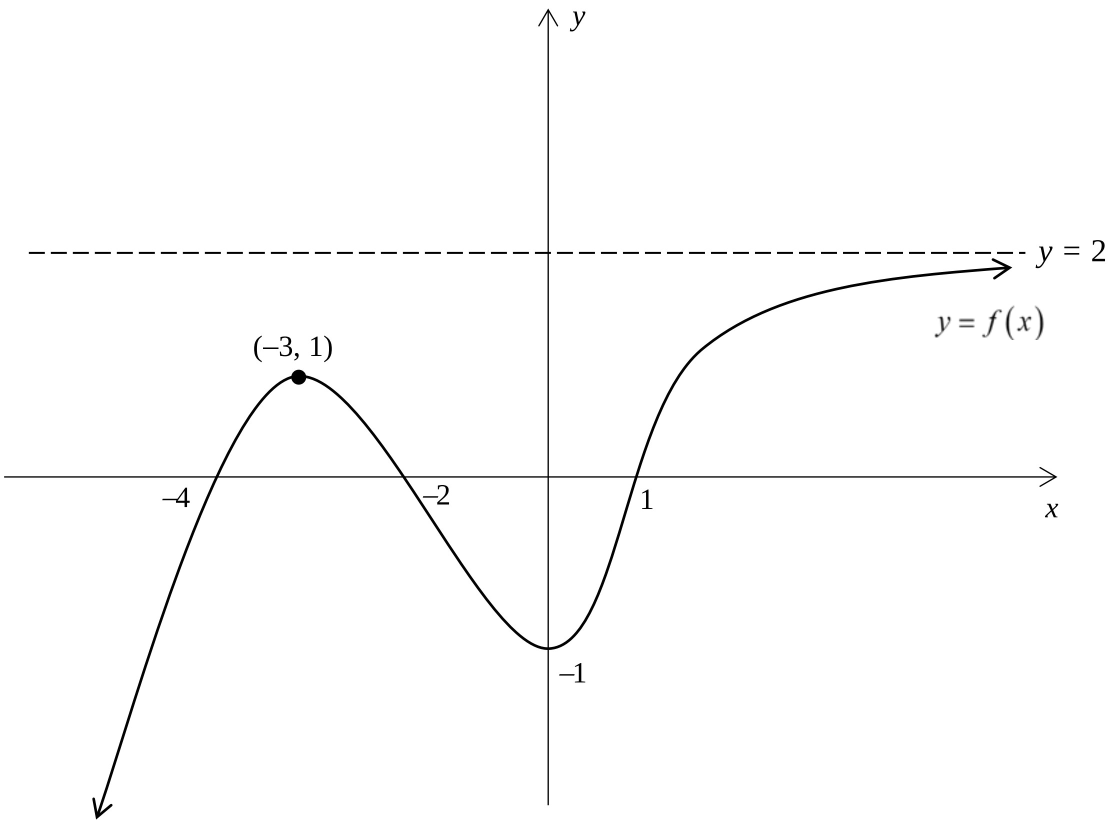
(c) The roots of the polynomial $3x^3 - 9x^2 + 7x - 1$ are α, β and γ. Find the cubic polynomial with roots $\alpha + \beta, \alpha + \gamma$ and $\beta + \gamma$. **3**

(d) It is given that $f(x) = x^3 - 6ax + 3b$ has exactly three distinct real roots. Show that $9b^2 < 32a^3$. **3**

END OF QUESTION 12

Question 13 **Start a new booklet** **15 Marks**

(a) Consider the graph of $y = f(x)$ shown below.



Sketch the following graphs on separate one third of a page diagrams.

(i) $y = \left[f(x)\right]^2$ **2**

(ii) $y = \sqrt{f(x)}$ **1**

(iii) $y = \dfrac{1}{f(x)}$ **2**

(iv) $y = f(|x|)$ **1**

(v) $|y| = |f(x)|$ **2**

(vi) $y = f\left(\dfrac{1}{x}\right)$ **2**

Question 13 continues on page 12

Question 13 (continued)

(b) Consider the area bounded by the curves $y = 6x - x^2$ and $y = x + 4$.

(i) Sketch the two graphs on the Cartesian plane, clearly labelling any points of intersection. **1**

(ii) The area described above is now rotated about the line $x = 1$. **4**

Find the volume of the solid formed.

END OF QUESTION 13

Question 14 **Start a new booklet** **15 Marks**

(a) Consider the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(i) Consider a point P which lies on this ellipse. Find the equation of the normal at the point P. **2**

(ii) The normal at P now intersects the x-axis at the point N. **3**
Prove that $NS = ePS$, where S is the focus of the ellipse.

(b) Let $z = \text{cis}\,\theta$.

(i) Prove that $z^n + z^{-n} = 2\cos n\theta$. **1**

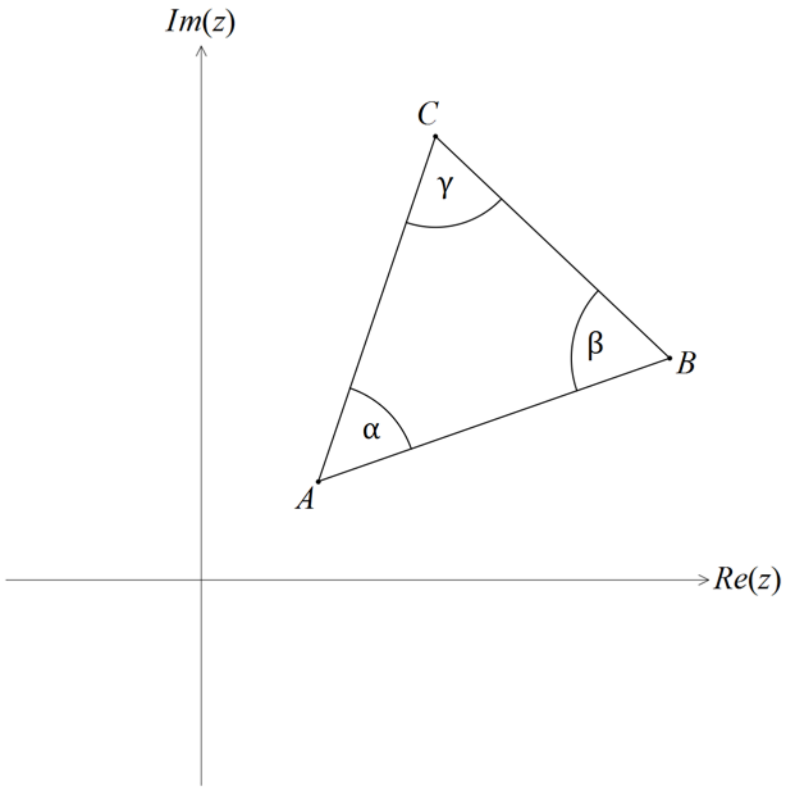
(ii) Given that all the roots of the polynomial $12z^4 - 23z^3 + 34z^2 - 23z + 12 = 0$ have a modulus of 1, find all solutions to the polynomial equation. **3**

(c) Find: $\int \cos\left(\sqrt{x}\right) dx$ **3**

Question 14 continues on page 14

Question 14 (continued)

(d) Let z_1, z_2 and z_3 be three arbitrary complex numbers represented by A, B and C respectively on the Argand diagram shown below:



(i) By considering the vector $\overrightarrow{AC}$ and $\overrightarrow{AB}$, show that: **1**

$$\alpha = \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$$

(ii) Hence, by finding equivalent expressions for β and γ, deduce that the angle sum of a triangle is π radians. **2**

END OF QUESTION 14

Question 15 **Start a new booklet** **15 Marks**

(a) Consider the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$ with the point $P(a\sec\theta, a\tan\theta)$.

(i) Provide a neat sketch of this hyperbola, showing all key features such as the foci, intercepts with the co-ordinate axes and the directrices. **2**

(ii) We can represent the above hyperbola on the Argand diagram with the complex number $z = a\sec\theta + ia\tan\theta$. Explain the geometrical relationship between this complex number z, and $w = z \times \text{cis}\frac{\pi}{4}$. **1**

(iii) Find in Cartesian form the complex number $w = z \times \text{cis}\frac{\pi}{4}$. **1**

(iv) Hence prove that the locus of the complex number $w = x + iy$ is given by: **2**

$$xy = \frac{a^2}{2}$$

(b) Consider the graph of $xy = 8$. The line L is defined to be the line which is perpendicular to the major axis of the hyperbola and passes through the focus in the first quadrant. The area bounded by the curve and L is rotated around the line $y = x$ in order to form a solid. Using part (a) or otherwise, find the volume of the solid. **4**

Question 15 continues on page 16

Question 15 (continued)

(c) Given $I_n = \int e^{ax} \cos^n x \, dx$,

(i) Show that: **3**

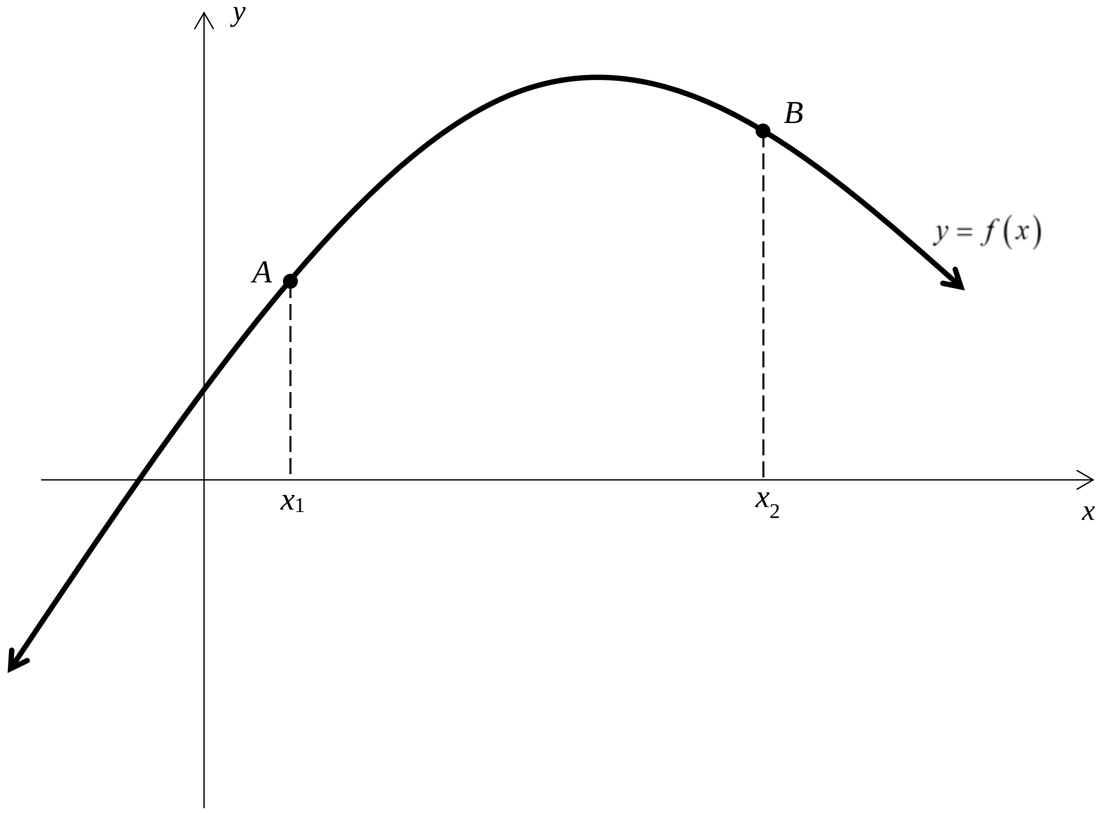
$$\left(a^2 + n^2\right) I_n = e^{ax} \cos^{n-1} x \left(a \cos x + n \sin x\right) + n\left(n-1\right) I_{n-2}$$

(ii) Hence find: $\int e^{3x} \cos^4 x \, dx$ **2**

END OF QUESTION 15

Question 16 **Start a new booklet** **15 Marks**

(a) Consider any concave down function such as $y = f\left(x\right)$ shown below.



A and B are points the curve $y = f\left(x\right)$ with the x values x_1 and x_2 respectively.

Furthermore, it is known that $x_1 < x_2$.

(i) Show that the point P which divides the interval AB in the ratio $\lambda_2 : \lambda_1$ **1**

where $\lambda_1 + \lambda_2 = 1$ is given by:

$$P\left(\lambda_1 x_1 + \lambda_2 x_2, \lambda_1 f\left(x_1\right) + \lambda_2 f\left(x_2\right)\right)$$

Question 16 continues on page 18

Question 16 (continued)

(ii) Hence, or otherwise, explain why **1**

$$\lambda_1 f(x_1)+\lambda_2 f(x_2)\le f(\lambda_1 x_1+\lambda_2 x_2)$$

(iii) Using (ii) or otherwise, prove by mathematical induction that for any **4**

concave down function $g(x)$:

$$\frac{g(x_1)+g(x_2)+\ldots+g(x_n)}{n}\le g\left(\frac{x_1+x_2+\ldots+x_n}{n}\right), \text{ for } n\ge 2$$

(b) Suppose that *A*, *B* and *C* are the angles of a triangle.

Prove that:

(i) $\tan A+\tan B+\tan C=\tan A\tan B\tan C$ **2**

(ii) $\cos A+\cos B+\cos C=1+4\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$ **3**

(c) By considering parts (a) and (b), in conjunction with the function $h(x)=\ln[\sin x]$ **4**

Prove that if $a+b+c=abc$:

$$\frac{1}{\sqrt{1+a^2}}+\frac{1}{\sqrt{1+b^2}}+\frac{1}{\sqrt{1+c^2}}\le\frac{3}{2}$$

END OF EXAM

12 MAY 2017 Trial Solutions

1. As the polynomial has real coefficients,

If $3-i$ is a root, $3+i$ is also a root.

$\therefore\ P(z) = (z-(3-i))(z-(3+i))$

$= z^2 - 6z + 10$ So the answer is (A)

2. $\dfrac{1}{z} + \dfrac{1}{\bar{z}} = 1$

$$\frac{\bar{z}+z}{z\cdot\bar{z}} = 1$$

$\therefore\ 2x = x^2+y^2$, which is a circle. So the answer is (B)

3. As foci are $S(\pm ae, 0)$, distance between foci is $2ae$

$\therefore\ 2ae = 10$

$\therefore\ ae = 5$ (1)

As directrices are $x = \pm\dfrac{a}{e}$, distance between directrices is $\dfrac{2a}{e}$

$\therefore\ \dfrac{2a}{e} = 50$

$\therefore\ \dfrac{a}{e} = 25$ (2)

$\dfrac{(1)}{(2)}:\ e^2 = \dfrac{1}{5}$ $\qquad$ $(1)\times(2):\ a^2 = 125$

But $b^2 = a^2(1-e^2)$

$b^2 = 125\left(1-\frac{1}{5}\right)$

$b^2 = 100$

$\therefore$ Ellipse has equation

$$\frac{x^2}{125} + \frac{y^2}{100} = 1$$

So the answer is (C)

4. Equation of normal : $\dfrac{ax}{\sec\theta} + \dfrac{by}{\tan\theta} = a^2+b^2$

$a=12, \quad b=6, \quad \theta = \dfrac{\pi}{3}.$

$$\therefore \quad \frac{12x}{\sec(\frac{\pi}{3})} + \frac{6y}{\tan(\frac{\pi}{3})} = 12^2+6^2$$

$$\frac{12x}{2} + \frac{6y}{\sqrt{3}} = 144+36$$

$$6x + 2\sqrt{3}y = 180$$

$$6\sqrt{3}x + 6y = 180\sqrt{3}$$

$$\sqrt{3}x + y = 30\sqrt{3}$$

So the answer is D

5. $x^2 - xy + 2y^2 = 4$

$$\frac{d}{dx}(x^2) - \frac{d}{dx}(xy) + \frac{d}{dx}(2y^2) = \frac{d}{dx}(4)$$

$$2x - \left(y + x\frac{dy}{dx}\right) + 4y \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(4y-x) = y-2x$$

$$\frac{dy}{dx} = \frac{y-2x}{4y-x}$$

At $(2,1)$, $\dfrac{dy}{dx} = \dfrac{1-2(2)}{4(1)-2} = \dfrac{-3}{2}$

So the answer is B

6. As $0 \leq 2x - x^2 \leq 1$ for $0 \leq x \leq 2$

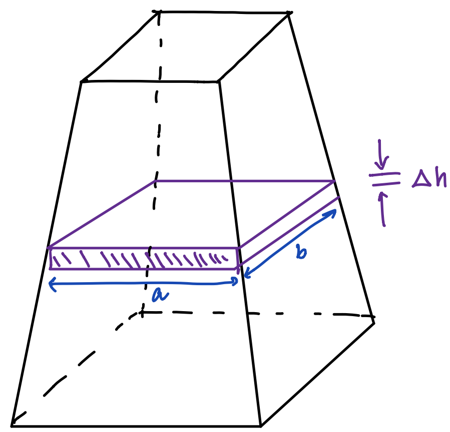
(A) & (B) do not change magnitude

(D) decreases magnitude

(C) increases magnitude

So the answer is C

7. Potentially needs to change as appeared in 2016 HSC



Clearly, a & b follow a linear relation with h.

$\therefore a = m_1 h + C_1$

$b = m_2 h + C_2$

When $h = 0$, $a = 7 \Rightarrow C_1 = 7$

$h = 9$, $a = 4$

$4 = m_1 \times 9 + 7$

$9m_1 = -3$

$\therefore m_1 = -\frac{1}{3}$

$\therefore a = 7 - \frac{h}{3}$

When $h = 0$, $b = 4 \Rightarrow C_2 = 4$

$h = 9$, $b = 2$

$2 = 9m_2 + 4$

$9m_2 = -2$; $m_2 = -\frac{2}{9}$

$\therefore b = 4 - \frac{2h}{9}$

$$\Delta V = a \times b \times \Delta h$$
$$= \left(7 - \frac{h}{3}\right)\left(4 - \frac{2h}{9}\right) \cdot \Delta h$$

So the answer is B

8. As $f(2a-x)$ is simply a reflection about the line $x=a$,
It is clear that ⓒ is correct by inspection

So the answer is ⓒ

9. The sum of roots of the given equation has to be

$-(2-10i) = 10i-2$

The only option that satisfies this condition is Ⓑ

So the answer is Ⓑ

10. Consider $y^2 = x^3 - 5x$ as $y = \pm\sqrt{x^3-5x}$

↳ So vertical tangents at roots.

Considering domain, large values of x must be included

So the answer is Ⓐ

Multiple Choice Solutions.

Q.	1	2	3	4	5	6	7	8	9	10
A.	A	B	C	D	B	C	B	C	B	A

Question 11.

a) i) $z = -\sqrt{3} + i$ $\qquad$ $w = 8 + 8i$

$\therefore z = 2\operatorname{cis}\left(\frac{5\pi}{6}\right)$ $\qquad$ $\therefore w = 8\sqrt{2}\operatorname{cis}\left(\frac{\pi}{4}\right)$

ii) $$\frac{z^3}{(\bar{w})^8} = \frac{\left[2\operatorname{cis}\left(\frac{5\pi}{6}\right)\right]^3}{\left[8\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)\right]^8} = \frac{8\operatorname{cis}\left(\frac{5\pi}{2}\right)}{(8\sqrt{2})^8\operatorname{cis}(-2\pi)}$$, By De Moivre's Theorem

$$= \frac{8i}{8^8 \times 2^4}$$

$$= 0 + \frac{1}{33554432}i$$

b) $z^2 + 2(1+i)z + (3+6i) = 0$

$\Delta = b^2 - 4ac = 4(1+i)^2 - 4(1)(3+6i)$

$= 4(2i) - 12 - 24i$

$= -12 - 16i$

let $a + ib = \sqrt{\Delta} = \sqrt{-12-16i}$; $a, b \in \mathbb{R}$.

$a^2 - b^2 + i(2ab) = -12 - 16i$

Equating Real & imaginary parts.

$a^2 - b^2 = -12$ $\qquad$ $2ab = -16$

$ab = -8$

By inspection, $a = 2,\ b = -4$

$a = -2,\ b = 4$

$\therefore \sqrt{\Delta} = \pm(2 - 4i)$

$$\therefore\ z = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-2(1+i) \pm (2-4i)}{2}$$

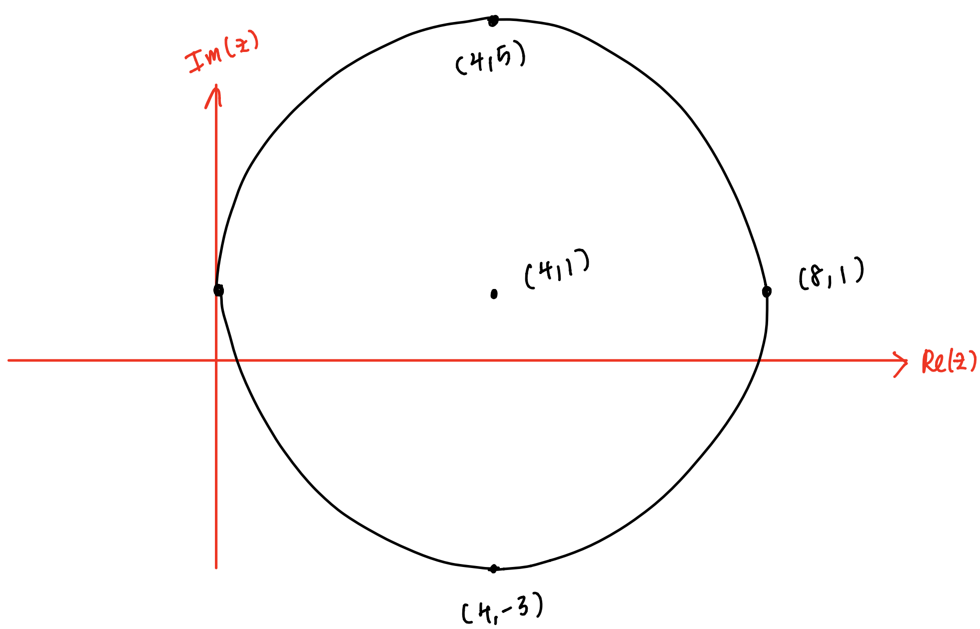
$$= -(1+i) \pm (1-2i)$$

$$= -3i \quad \text{or} \quad -2+i$$

c.) $|z-4-i| = 4$

$|z-(4+i)| = 4$

↳ so the points z lie on a circle with centre $(4,1)$ & radius 4 units



d) $$\int \frac{1}{x(x^{2017}+1)}\,dx = \int \frac{1+x^{2017}-x^{2017}}{x(x^{2017}+1)}\,dx$$

$$= \int \frac{1}{x} - \frac{x^{2016}}{x^{2017}+1}\,dx$$

$$= \ln|x| - \frac{1}{2017}\ln|x^{2017}+1| + C$$

e) i) $1+\frac{1}{3}\text{cis}\theta+\frac{1}{9}(\text{cis}\theta)^2+\frac{1}{27}(\text{cis}\theta)^3+\dots$

Common ratio: $\frac{1}{3}\text{cis}\theta$.

$|r| = |\frac{1}{3}\text{cis}\theta| = \frac{1}{3} < 1$

As $|r|<1$, A limiting sum exists.

ii) $1+\frac{1}{3}\text{cis}\theta+\frac{1}{9}(\text{cis}\theta)^2+\frac{1}{27}(\text{cis}\theta)^3+\dots$

$$= \frac{a}{1-r} = \frac{1}{1-\frac{1}{3}\text{cis}\theta}$$

$$= \frac{3}{3-\text{cis}\theta}$$

$$= \frac{3}{3-\cos\theta-i\sin\theta}$$

$$= \frac{3[3-\cos\theta+i\sin\theta]}{(3-\cos\theta)^2+\sin^2\theta}$$

$$= \frac{3(3-\cos\theta)+i\,3\sin\theta}{9-6\cos\theta+\cos^2\theta+\sin^2\theta}$$

$$= \frac{3(3-\cos\theta) + i(3\sin\theta)}{10 - 6\cos\theta}$$

$$\text{LHS} = 1 + \frac{1}{3}\cos\theta + \frac{1}{9}\cos 2\theta + \frac{1}{27}\cos 3\theta + \cdots$$ (By De Moivre's Theorem)

Equating imaginary parts of both sides.

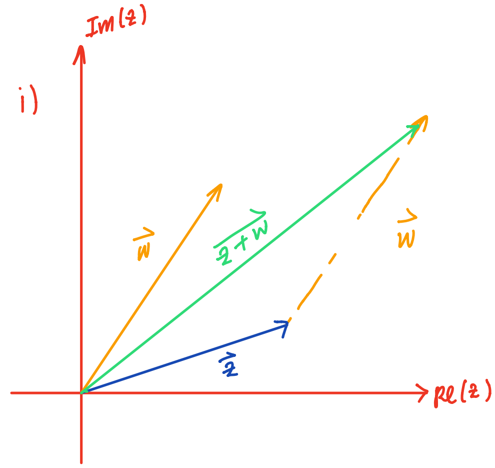
$$\frac{1}{3}\sin\theta + \frac{1}{9}\sin 2\theta + \frac{1}{27}\sin 3\theta + \cdots = \frac{3\sin\theta}{10-6\cos\theta}$$

$\times 3$

$$\therefore \sin\theta + \frac{1}{3}\sin 2\theta + \frac{1}{9}\sin 3\theta + \cdots = \frac{9\sin\theta}{10-6\cos\theta}$$

Question 12

a) i)



ii) As the sum of two sides of a triangle is always longer than the remaining side,

$$|z|+|w| \geq |z+w|$$

$\therefore |z+w| \leq |z|+|w|$

as required.

iii) $|13z^2-2z+5| \leq |13z^2|+|-2z|+|5|$, by Triangle inequality.

$= 13\times|z|^2+2|z|+5$

$= 13\times10^2+2\times10+5$

$= 1325$

$\therefore |13z^2-2z+5| \leq 1325$

b) i) $p(x) = (x-\alpha)^m \cdot Q(x)$

$Q(\alpha) \neq 0$, as $p(x)$ has a root α of multiplicity m.

If $Q(\alpha) = 0$, the root would be of multiplicity $> m$.

ii) $p'(x) = m(x-\alpha)^{m-1} \cdot Q(x) + (x-\alpha)^m \cdot Q'(x)$

$= (x-\alpha)^{m-1}[\underbrace{mQ(x) + (x-\alpha) \cdot Q'(x)}]$

↳ let $S(x) = mQ(x) + (x-\alpha)Q'(x)$

$S(\alpha) = mQ(\alpha) \neq 0$ as $Q(\alpha) \neq 0$

$\therefore p'(x) = (x-\alpha)^{m-1} \cdot S(x)$; $S(\alpha) \neq 0$

↳ $p'(x)$ has a root α of multiplicity $m-1$.

iii) $f(x) = 2x^4 - 15x^3 + 42x^2 - 52x + 24$

Let the triple root of $f(x)$ be α.

$\therefore f(\alpha) = f'(\alpha) = f''(\alpha) = 0$

$f'(x) = 8x^3 - 45x^2 + 84x - 52$

$f''(x) = 24x^2 - 90x + 84$

$= 6(4x^2 - 15x + 14)$

$= 6(4x-7)(x-2)$

$\therefore$ Possible values of α: $\alpha = \frac{7}{4}$; $\alpha - 2$

$f'(\frac{7}{4}) \neq 0$; $f'(2) = 0$

$\therefore \alpha = 2$ is the triple root.

Let the remaining root be β

$\sum\alpha$: $3\alpha + \beta = \frac{15}{2}$

$\beta = \frac{15}{2} - 3\alpha = \frac{3}{2}$

$\therefore$ Roots are $2, 2, 2, 3/2$

c) Old polynomial: $f(x) = 3x^3 - 9x^2 + 7x - 1$

New roots: y : $\sum\alpha - x$

$y = 3 - x$

$x = 3 - y$.

$f(3-y) = 0$

$\therefore 3(3-y)^3 - 9(3-y)^2 + 7(3-y) - 1 = 0$

$3(27 - 27y + 9y^2 - y^3) - 9(9 - 6y + y^2) + 21 - 7y - 1 = 0$

$-3y^3 + 18y^2 - 34y + 20 = 0$

$3x^3 - 18x^2 + 34y - 20 = 0$

d) $f(x) = x^3 - 6ax + 3b$

Let stationary points occur at $x = \alpha$ & $x = \beta$

$f(\alpha) \cdot f(\beta) < 0$ for 3 distinct real roots.

$$f'(x) = 3x^2 - 6a$$

$$f'(x) = 0 \Rightarrow 3x^2 - 6a = 0$$

$$x^2 = 2a$$

$$x = \pm\sqrt{2a}$$

$$f(\sqrt{2a}) = 2a\sqrt{2a} - 6a\sqrt{2a} + 3b$$

$$= 3b - 4a\sqrt{2a}$$

$$f(-\sqrt{2a}) = -2a\sqrt{2a} + 6a\sqrt{2a} + 3b$$

$$= 3b + 4a\sqrt{2a}$$

$$f(\sqrt{2a}) \times f(-\sqrt{2a}) < 0$$

$$\therefore (3b - 4a\sqrt{2a})(3b + 4a\sqrt{2a}) < 0$$

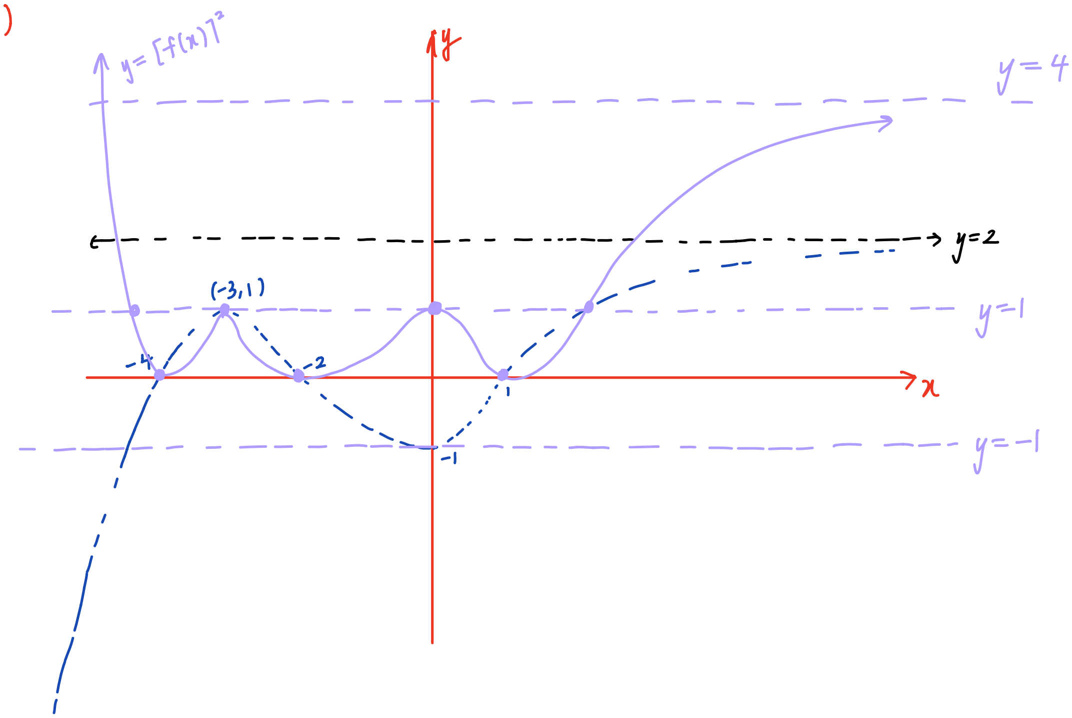
$$9b^2 - 16a^2 \times 2a < 0$$

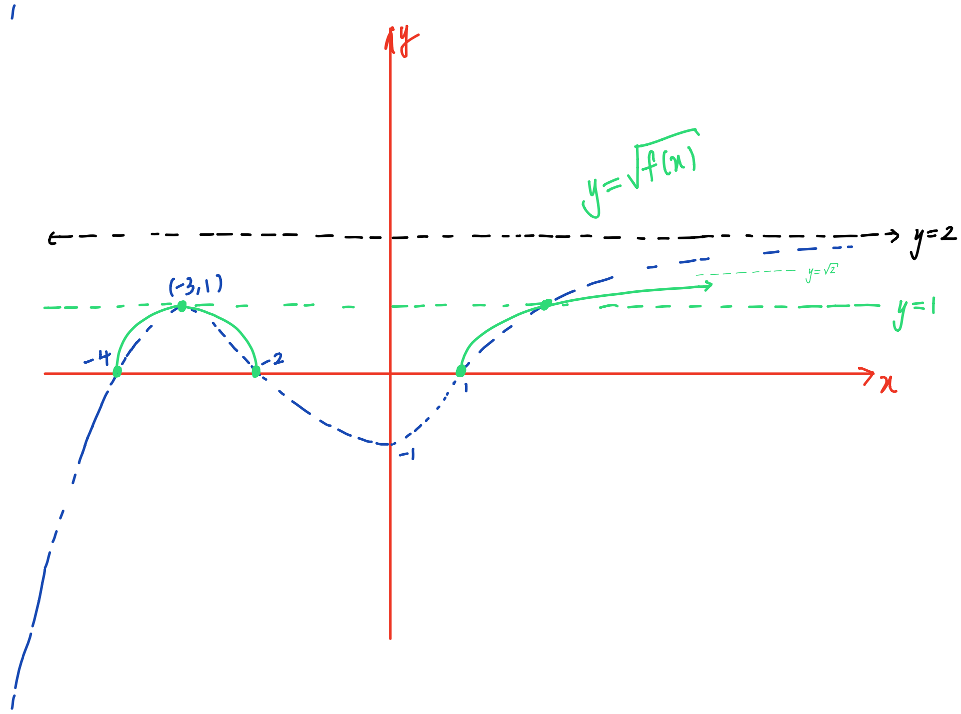
$$9b^2 - 32a^3 < 0$$

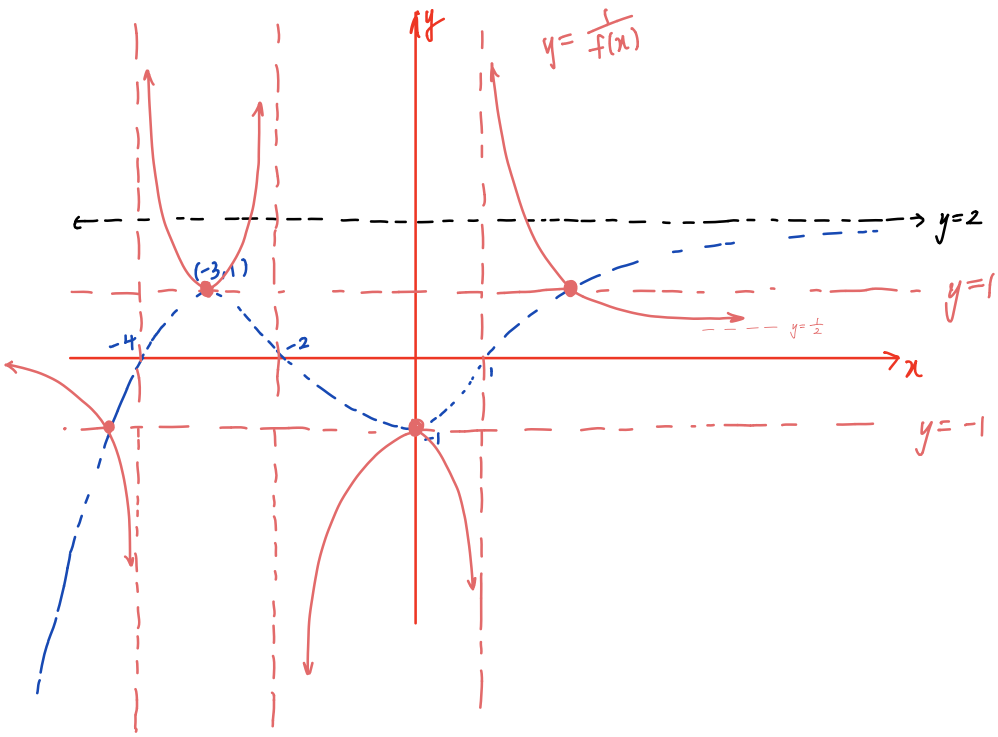
$$9b^2 < 32a^3$$

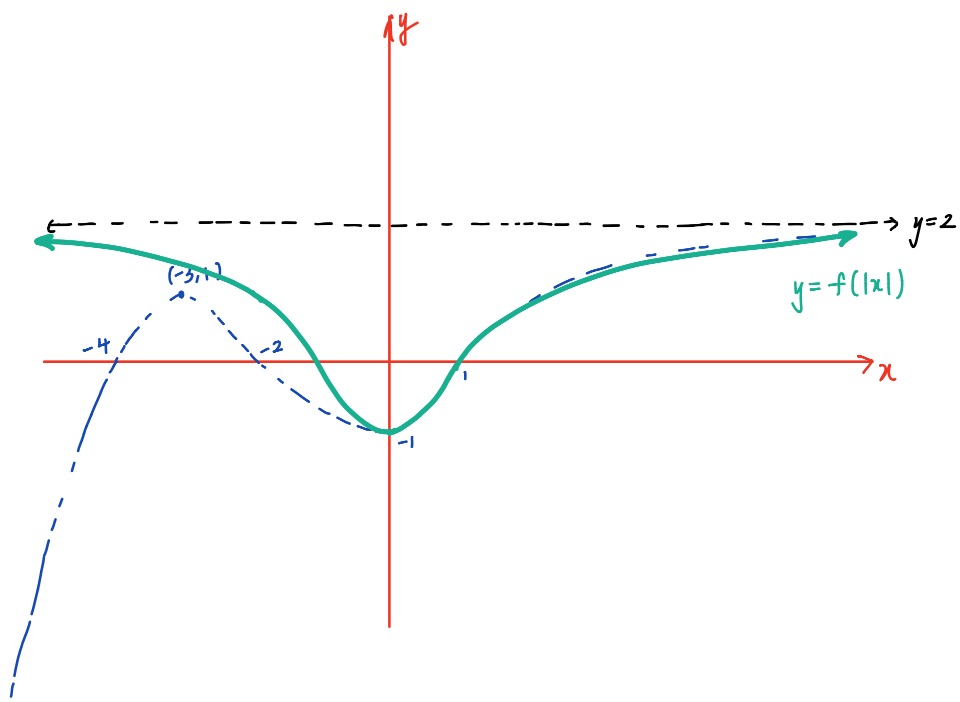
Question 13

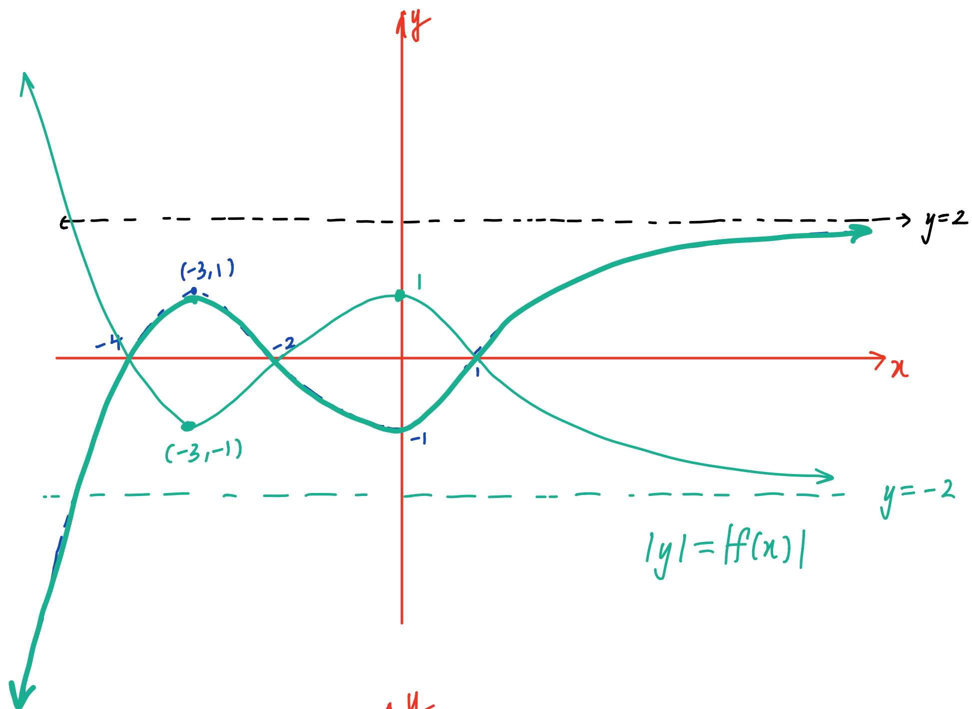
a)



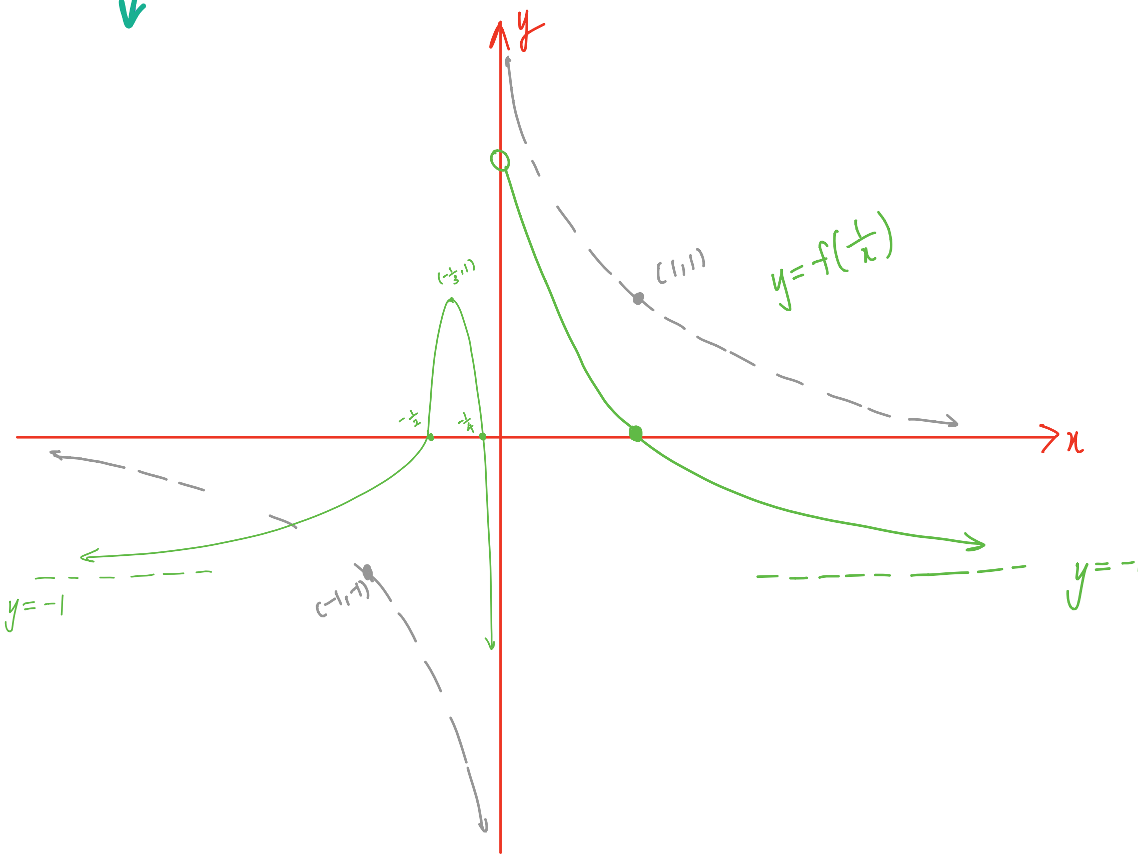



y
y = 1/f(x)
y=2
(-3,1)
y=1
y=1/2
-4
-2
x
1
-1
y=-1


y
y=2
(-3,1)
y=f(|x|)
-4
-2
x
1
-1

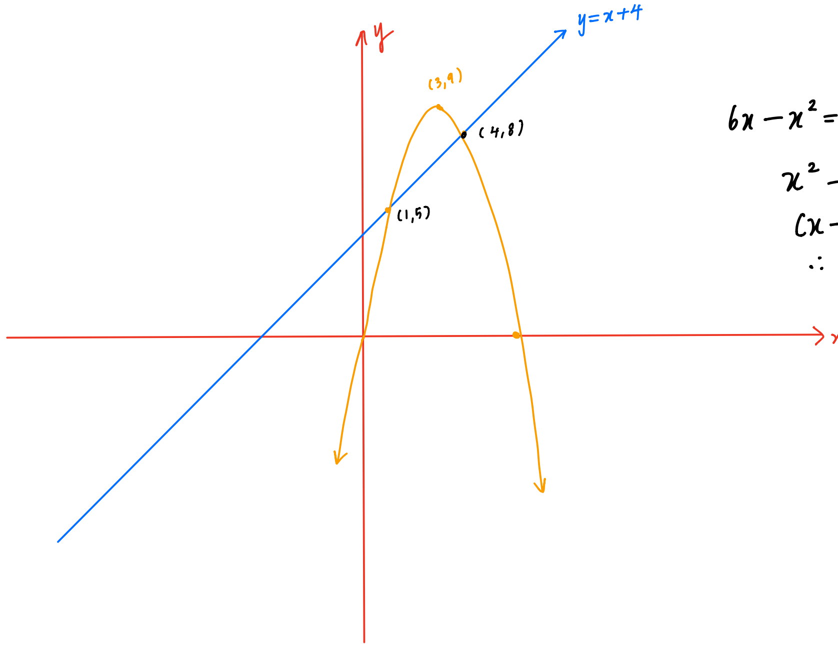


y
x
y=2
(-3,1)
1
-4
-2
1
-1
(-3,-1)
y=-2
|y|=|f(x)|



y
x
(1,1)
y=f(1/x)
(-1/3,1)
-1/2
(-1,-1)
y=-1
y=-1

b) i)

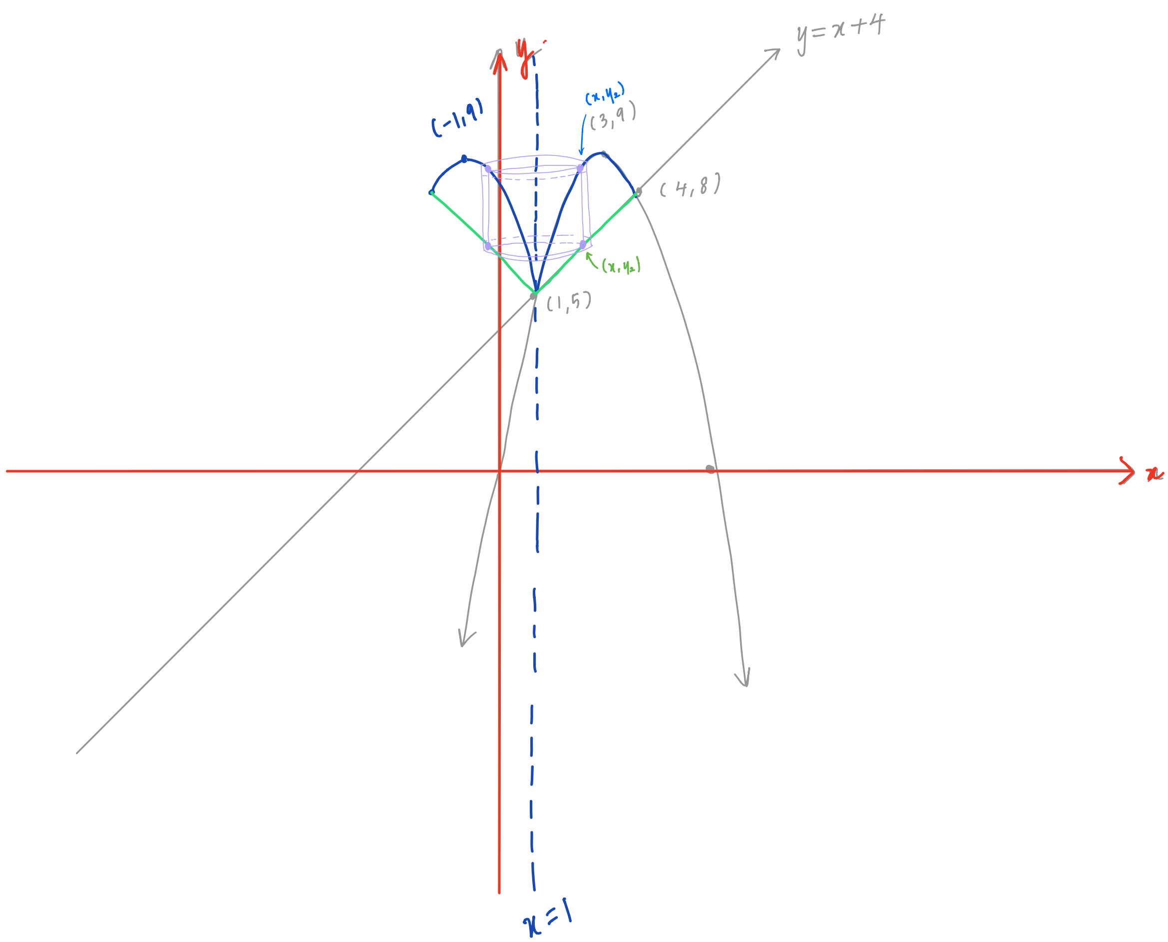


$$6x - x^2 = x + 4$$

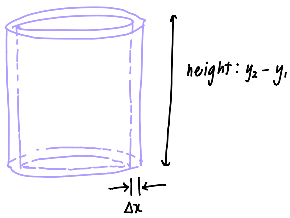
$$x^2 - 5x + 4 = 0$$

$$(x-4)(x-1) = 0$$

$$\therefore x = 1, \; x = 4$$



Arbitrary Shell



Inner radius: $x-1$

$$\Delta V = \pi [OR^2 - IR^2] \times height$$

$$= \pi \left[[(x-1)+\Delta x]^2 - (x-1)^2\right] \times (y_2 - y_1)$$

$$= \pi \left[2(x-1) \cdot \Delta x + (\Delta x)^2\right] \cdot (y_2 - y_1)$$

$$= 2\pi (x-1) \cdot (y_2 - y_1) \cdot \Delta x$$

$$= 2\pi (x-1)\left(6x - x^2 - (x+4)\right) \cdot \Delta x$$

$$= 2\pi (x-1)(-x^2 + 5x - 4) \cdot \Delta x$$

$$V \doteqdot \sum_{x=1}^{4} 2\pi (x-1)(-x^2+5x-4) \cdot \Delta x$$

$$V = \lim_{\Delta x \to 0} \sum_{x=1}^{4} 2\pi (x-1)(-x^2+5x-4)$$

$$= 2\pi \int_1^4 -x^3 + 5x^2 - 4x + x^2 - 5x + 4 \, dx$$

$$= 2\pi \int_1^4 -x^3 + 6x^2 - 9x + 4 \, dx$$

$$= 2\pi \left[-\frac{x^4}{4} + 2x^3 - \frac{9x^2}{2} + 4x\right]_1^4$$

$$= 2\pi \left[[-64 + 128 - 72 + 16] - \left[-\frac{1}{4} + 2 - \frac{9}{2} + 4\right]\right]$$

$$= \frac{27\pi}{2} \text{ units}^3$$

Question 14

a) i) Let the point be $P(a\cos\theta, b\sin\theta)$

$x = a\cos\theta \qquad y = b\sin\theta$

$\frac{dx}{d\theta} = -a\sin\theta \qquad \frac{dy}{d\theta} = b\cos\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{-b\cos\theta}{a\sin\theta}$

$\therefore m_T = -\frac{b\cos\theta}{a\sin\theta} \Rightarrow m_N = \frac{a\sin\theta}{b\cos\theta}$

$y - y_1 = m(x - x_1)$

$y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$

$by\cos\theta - b^2\sin\theta\cos\theta = ax\sin\theta - a^2\sin\theta\cos\theta$

$ax\sin\theta - by\cos\theta = a^2\sin\theta\cos\theta - b^2\sin\theta\cos\theta$

$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$

ii) RTP: $NS = ePS$

$RHS = ePS = e^2 PD$

$= e^2 \left| \frac{a}{e} - a\cos\theta \right|$

$= ae\,|1 - e\cos\theta|$

For N, $y = 0$

$\therefore x = \left(\frac{a^2 - b^2}{a}\right)\cos\theta$

but $b^2 = a^2 - a^2e^2$

$\therefore a^2 - b^2 = a^2e^2$

$\therefore x_N = ae^2\cos\theta$

$NS = |ae^2\cos\theta - ae|$

$= ae|e\cos\theta - 1|$

$= ae|1 - e\cos\theta|$

$\therefore NS = ePS.$

b)i) $z = \text{cis}\,\theta.$

RTP: $z^n + z^{-n} = 2\cos n\theta.$

$LHS = z^n + z^{-n}$

$= (\text{cis}\,\theta)^n + (\text{cis}\,\theta)^{-n}$

$= \text{cis}\,n\theta + \text{cis}(-n\theta)$

$= \text{cis}(n\theta) + \overline{\text{cis}(n\theta)}$

$= 2 \times \text{Re}[\text{cis}(n\theta)]$

$= 2\cos n\theta.$

$= RHS.$

ii) $12z^4 - 23z^3 + 34z^2 - 23z + 12 = 0$

(Dividing both sides by z^2; $z \neq 0$)

$12z^2 - 23z + 34 - 23z^{-1} + 12z^{-2} = 0$

$12(z^2 + z^{-2}) - 23(z + z^{-1}) + 34 = 0$

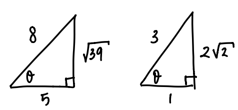
$12(2\cos 2\theta) - 23(2\cos\theta) + 34 = 0$

$12(2\cos^2\theta - 1) - 23\cos\theta + 17 = 0$

$24\cos^2\theta - 23\cos\theta + 5 = 0$

$(8\cos\theta - 5)(3\cos\theta - 1) = 0$

$\therefore \cos\theta = \frac{5}{8}$, $\cos\theta = \frac{1}{3}$



$\therefore \sin\theta = \frac{\sqrt{39}}{8}$ $\qquad \sin\theta = \frac{2\sqrt{2}}{3}$

As $z = \text{cis}\,\theta$; roots are

$z = \frac{5}{8} \pm i \cdot \frac{\sqrt{39}}{8}$ OR $\frac{1}{3} \pm i\frac{2\sqrt{2}}{3}$; As the polynomial has real coefficients, so the conjugates are also roots.

c) $I = \int \cos(\sqrt{x})\, dx$

let $u = \sqrt{x}$

$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$

$dx = 2\sqrt{x}\, du = 2u\, du$

$\therefore I = 2\int u \cos u\, du$

Integration by parts

$u = u \qquad v = \sin u$

$u' = 1 \qquad v' = \cos u$

$I = 2u\sin u - 2\int \sin u\, du$

$= 2u\sin u + 2\cos u + C$

$= 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C.$

d) i) Without loss of generality, we can assume α, β, γ are labelled as shown.

$$\alpha = \angle CAB = \arg(\vec{AC}) - \arg(\vec{AB})$$
$$= \arg(z_3 - z_1) - \arg(z_2 - z_1)$$
$$= \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$$

ii) Similarly;

$$\beta = \arg(\vec{BA}) - \arg(\vec{BC})$$
$$= \arg(z_1 - z_2) - \arg(z_3 - z_2)$$
$$= \arg\left(\frac{z_1 - z_2}{z_3 - z_2}\right)$$

$$\gamma = \arg(\vec{CB}) - \arg(\vec{CA})$$
$$= \arg(z_2 - z_3) - \arg(z_1 - z_3)$$
$$= \arg\left(\frac{z_2 - z_3}{z_1 - z_3}\right)$$

Now, $\alpha + \beta + \gamma = \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) + \arg\left(\frac{z_1 - z_2}{z_3 - z_2}\right) + \arg\left(\frac{z_2 - z_3}{z_1 - z_3}\right)$

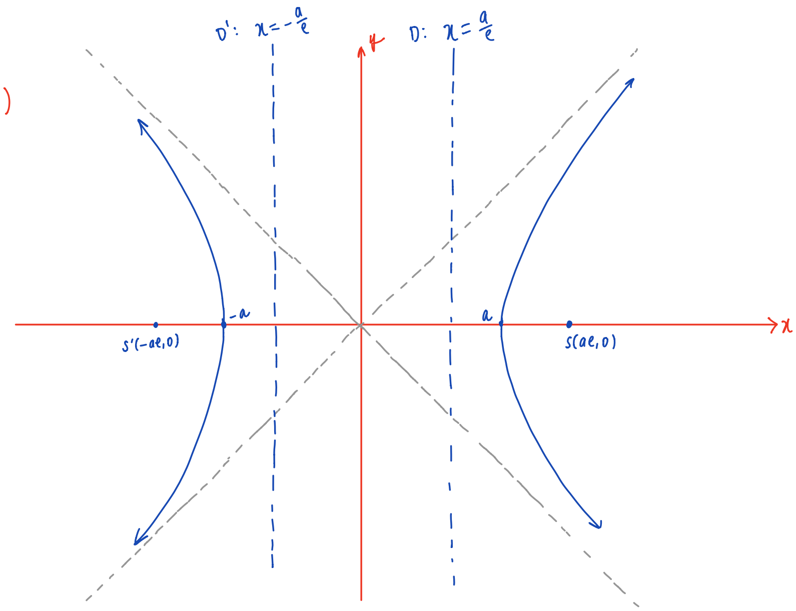
$$= \arg\left(\frac{z_3 - z_1}{z_2 - z_1} \times \frac{z_1 - z_2}{z_3 - z_2} \times \frac{z_2 - z_3}{z_1 - z_3}\right)$$

$$= \arg(-1 \times -1 \times -1) = \arg(-1) = \pi$$

$\therefore$ Angle sum of a triangle is π radians.

Question 15.

a) i)



ii) Multiplying a complex number by $\operatorname{cis}\frac{\pi}{4}$ rotates the complex number anticlockwise by an angle of $\frac{\pi}{4}$ whilst preserving its modulus. So, $w = z \times \operatorname{cis}\frac{\pi}{4}$ is simply the hyperbola $x^2-y^2=a^2$ rotated anticlockwise about the origin by an angle of $\frac{\pi}{4}$.

iii) $w = z \times \operatorname{cis}\frac{\pi}{4}$. $\qquad z = a\sec\theta + ia\tan\theta$

$$w = (a\sec\theta + ia\tan\theta)\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

$$\therefore\ x+iy = \frac{a\sec\theta}{\sqrt{2}} - \frac{a\tan\theta}{\sqrt{2}} + i\left(\frac{a\tan\theta}{\sqrt{2}} + \frac{a\sec\theta}{\sqrt{2}}\right)$$

Equating real & imaginary parts

$$x = \frac{a}{\sqrt{2}}(\sec\theta - \tan\theta) \quad ① \qquad y = \frac{a}{\sqrt{2}}(\sec\theta + \tan\theta) \quad ②$$

① × ② :

$$xy = \frac{a^2}{2}(\sec^2\theta - \tan^2\theta)$$

$$= \frac{a^2}{2}(1)$$

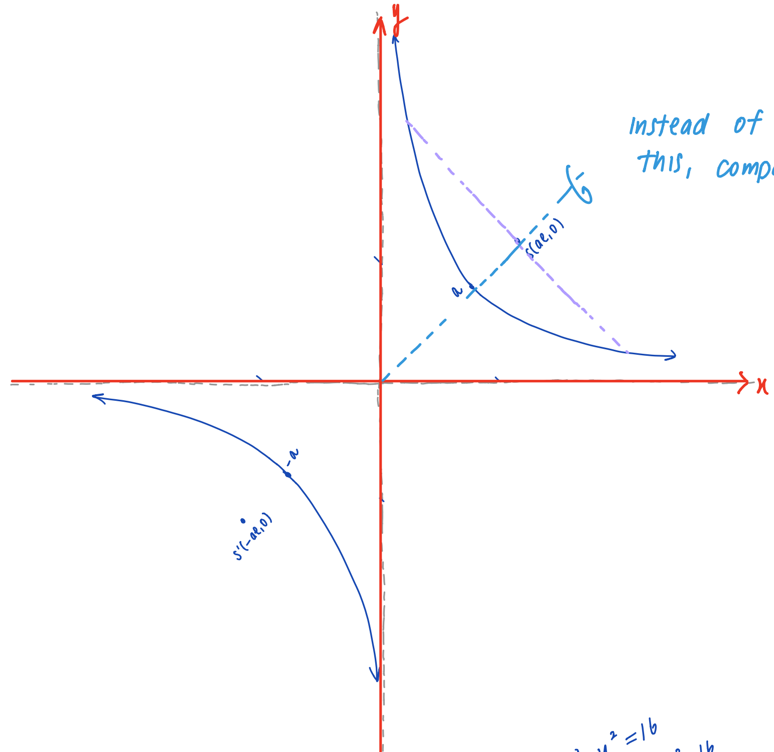
$$\therefore\ xy = \frac{a^2}{2}$$

b) $xy = 8 = \frac{a^2}{2}$

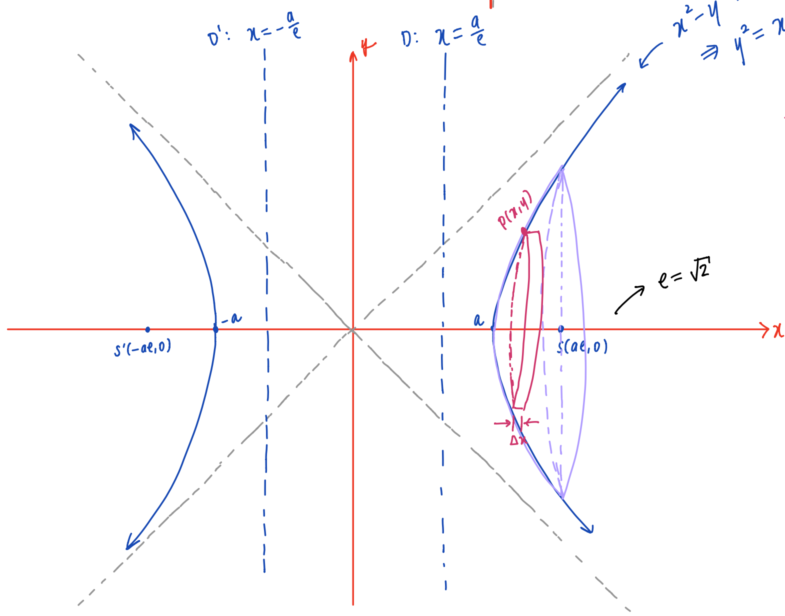
$\therefore a^2 = 16$

$\therefore \frac{x^2}{16} - \frac{y^2}{16} = 1$

is the equivalent hyperbola



Instead of rotating this, compare to $x^2 - y^2 = a^2$!



$x^2 - y^2 = 16$

$\Rightarrow y^2 = x^2 - 16$

$\Delta V = \pi y^2 \cdot \Delta x$

$= \pi (x^2 - 16) \Delta x$

$$V \doteqdot \sum_{x=4}^{4\sqrt{2}} \pi (x^2 - 16) \cdot \Delta x$$

$$V = \lim_{\Delta x \to 0} \sum_{x=4}^{4\sqrt{2}} \pi (x^2 - 16) \cdot \Delta x$$

$$= \pi \int_4^{4\sqrt{2}} x^2 - 16 \, dx$$

$$= \pi \left[\frac{x^3}{3} - 16x \right]_4^{4\sqrt{2}}$$

$$= \pi \left\{ \left[\frac{128\sqrt{2}}{3} - 64\sqrt{2} \right] - \left[\frac{64}{3} - 64 \right] \right\}$$

$$= \pi \left\{ -\frac{64\sqrt{2}}{3} + \frac{128}{3} \right\}$$

$$= \pi \times \frac{128 - 64\sqrt{2}}{3} \quad \text{units}^3$$

c) i) $I_n = \int e^{ax} \cos^n x \, dx$

$u = \cos^n x$ $\quad$ $v = \frac{1}{a} e^{ax}$

$u' = -n \sin x \cdot \cos^{n-1} x$ $\quad$ $v' = e^{ax}$

$$\therefore I_n = \frac{1}{a} e^{ax} \cos^n x + \frac{n}{a} \int e^{ax} \cdot (\sin x \cos^{n-1} x) \, dx$$

$$a I_n = e^{ax} \cos^n x + n \int e^{ax} (\sin x \cos^{n-1} x) \, dx$$

$u = \sin x \cos^{n-1} x$ $\quad$ $V = \frac{1}{a} e^{ax}$

$u' = \cos^n x - (n-1) \sin^2 x \cos^{n-2} x$ $\quad$ $v' = e^{ax}$

$$= \cos^n x - (n-1) \times (1 - \cos^2 x) \cdot \cos^{n-2} x$$

$$= \cos^n x - (n-1) \cos^{n-2} x + (n-1) \cos^n x$$

$$= n \cos^n x - (n-1) \cos^{n-2} x$$

$$\therefore a I_n = e^{ax} \cos^n x + n \left[\frac{1}{a} e^{ax} \sin x \cos^{n-1} x - \frac{1}{a} \int e^{ax} (n \cos^n x - (n-1) \cos^{n-2} x \, dx \right]$$

$$a^2 I_n = a e^{ax} \cos^n x + n e^{ax} \sin x \cos^{n-1} x - n^2 \int e^{ax} \cos^n x + n(n-1) \int e^{ax} \cos^{n-2} x \, dx$$

$$a^2 I_n = e^{ax} \cos^{n-1} x \, (a \cos x + n \sin x) - n^2 I_n + n(n-1) I_{n-2}$$

$$\therefore (a^2 + n^2) I_n = e^{ax} \cos^{n-1} x \, (a \cos x + n \sin x) + n(n-1) I_{n-2}$$

ii) Rearranging result in (i)

$$I_n = \frac{1}{a^2+n^2}\left[e^{ax}\cos^{n-1}x\,(a\cos x + n\sin x) + n(n-1)I_{n-2}\right]$$

Noting that $\int e^{3x}\cos^4 x\,dx$ is I_4 with $a=3$.

$$I_4 = \frac{1}{3^2+4^2}\left[e^{3x}\cos^3 x\,(3\cos x + 4\sin x) + 4(3)I_2\right]$$

$$= \frac{1}{25}\left[e^{3x}\cos^3 x\,(3\cos x + 4\sin x) + 12\times\left[\frac{1}{3^2+2^2}\times\left(e^{3x}\cos x\,(3\cos x + 2\sin x) + 2(1)\cdot I_0\right)\right]\right]$$

$$= \frac{1}{25}e^{3x}\cos^3 x\,(3\cos x + 4\sin x) + \frac{12}{25}\left[\frac{1}{13}\left(e^{3x}\cos x\,(3\cos x + 2\sin x) + 2\int e^{3x}\,dx\right)\right]$$

$$= \frac{1}{25}e^{3x}\cos^3 x\,(3\cos x + 4\sin x) + \frac{12}{325}e^{3x}\cos x\,(3\cos x + 2\sin x) + \frac{24}{325}\times\frac{e^{3x}}{3} + C$$

$$= \frac{1}{25}e^{3x}\cos^3 x\,(3\cos x + 4\sin x) + \frac{12}{325}e^{3x}\cos x\,(3\cos x + 2\sin x) + \frac{8e^{3x}}{325} + C$$

Question 16.

a) i) $A(x_1, f(x_1))$ $\quad B(x_2, f(x_2))$

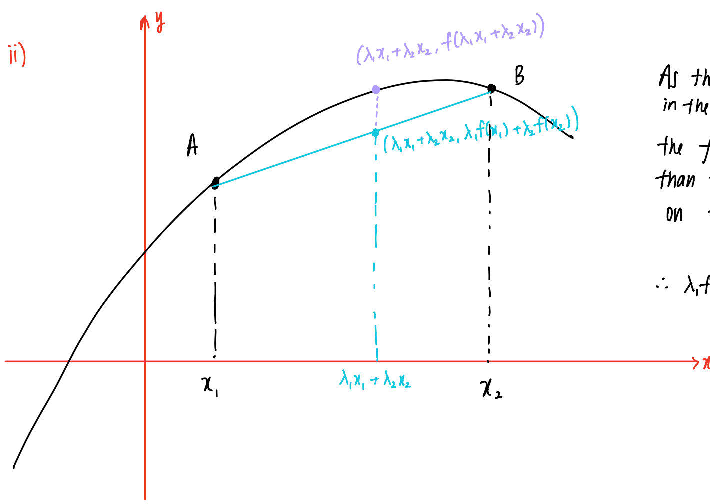
Ratio: $\lambda_2 : \lambda_1$

$$P: \left(\frac{\lambda_1 x_1 + \lambda_2 x_2}{\lambda_1 + \lambda_2} , \frac{\lambda_1 f(x_1) + \lambda_2 f(x_2)}{\lambda_1 + \lambda_2} \right)$$

But $\lambda_1 + \lambda_2 = 1$

$\therefore P(\lambda_1 x_1 + \lambda_2 x_2, \lambda_1 f(x_1) + \lambda_2 f(x_2))$

ii)



As the function is concave down in the interval $x_1 \le x \le x_2$, the function value is larger than the y-value of the point on the secant at $x = \lambda_1 x_1 + \lambda_2 x_2$

$\therefore \lambda_1 f(x_1) + \lambda_2 f(x_2) \le f(\lambda_1 x_1 + \lambda_2 x_2)$ for a concave down function $f(x)$

iii) RTP: $\dfrac{g(x_1) + g(x_2) + \dots + g(x_n)}{n} \le g\left(\dfrac{x_1 + x_2 + \dots + x_n}{n}\right)$ for $n \geqslant 2$

Step 1: Prove true for $n=2$

$$\text{LHS} = \frac{g(x_1) + g(x_2)}{2}$$

$$= \tfrac{1}{2} g(x_1) + \tfrac{1}{2} g(x_2) \le g\left(\tfrac{1}{2} x_1 + \tfrac{1}{2} x_2\right) \qquad \text{(By (ii))}$$

$$= g\left(\frac{x_1 + x_2}{2}\right) = \text{RHS}$$

$\therefore$ Statement is true for $n=2$

Step 2: Assume true for $n=k$; $k \in \mathbb{Z}^+$; $k \geq 2$.

$$\frac{g(x_1)+g(x_2)+\dots+g(x_k)}{k} \leq g\left(\frac{x_1+x_2+\dots+x_k}{k}\right)$$

Step 3: Prove true for $n=k+1$

RTP: $$\frac{g(x_1)+g(x_2)+\dots+g(x_k)+g(x_{k+1})}{k+1} \leq g\left(\frac{x_1+x_2+\dots+x_k+x_{k+1}}{k+1}\right)$$

$$\text{LHS} = \frac{g(x_1)+g(x_2)+\dots+g(x_k)}{k+1} + \frac{g(x_{k+1})}{k+1}$$

Note: $\lambda_1+\lambda_2 = \frac{k}{k+1}+\frac{1}{k+1} = 1$

$$\leq \underbrace{\frac{k}{(k+1)}}_{\lambda_1} \cdot g\Big(\underbrace{\frac{x_1+x_2+\dots+x_k}{k}}_{x_1}\Big) + \underbrace{\frac{1}{k+1}}_{\lambda_2} \cdot g\big(\underbrace{x_{k+1}}_{x_2}\big)$$

(By Assumption)

$$\leq g\left[\frac{k}{(k+1)} \cdot \left(\frac{x_1+x_2+\dots+x_k}{k}\right) + \frac{1}{k+1}(x_{k+1})\right]$$

using (ii)

$$= g\left(\frac{x_1+x_2+\dots+x_{k+1}}{k+1}\right) = \text{RHS}$$

$\therefore$ true for $n=k+1$

Step 4: Conclusion.

Hence, the statement is true for $n \in \mathbb{Z}^+$; $n \geq 2$, by induction

b) i) If A, B & C are angles of a triangle, $C = \pi - (A+B)$ {Angle sum of a triangle is π}.

$$\begin{aligned}
LHS &= \tan A + \tan B + \tan C \\
&= \tan A + \tan B + \tan[\pi - (A+B)] \\
&= \tan A + \tan B - \tan(A+B) \\
&= \tan A + \tan B - \left[\frac{\tan A + \tan B}{1 - \tan A \tan B}\right] \\
&= \frac{(\tan A + \tan B)(1 - \tan A \tan B) - \tan A - \tan B}{(1 - \tan A \tan B)} \\
&= \frac{-\tan A \tan B\,(\tan A + \tan B)}{1 - \tan A \tan B} \\
&= -\tan A \tan B \tan(A+B) \\
&= \tan A \tan B \tan[\pi - (A+B)] \\
&= \tan A \tan B \tan C = RHS.
\end{aligned}$$

ii) RTP: $\cos A + \cos B + \cos C = 1 + 4\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$

Note: $C = \pi - (A+B)$

$\frac{C}{2} = \frac{\pi}{2} - \frac{(A+B)}{2}$

$\therefore \frac{A+B}{2} = \frac{\pi}{2} - \frac{C}{2}$

Note: $\cos(X+Y) + \cos(X-Y) = 2\cos X \cos Y$

let $X = \frac{A+B}{2}$, $Y = \frac{A-B}{2}$

$\therefore \cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$$\begin{aligned}
\text{So } LHS &= \cos A + \cos B + \cos C \\
&= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + \cos(\pi - (A+B)) \\
&= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) - \cos(A+B) \\
&= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) - \left(2\cos^2\left(\frac{A+B}{2}\right) - 1\right) \\
&= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) - 2\cos^2\left(\frac{A+B}{2}\right) + 1 \\
&= 1 + 2\cos\left(\frac{A+B}{2}\right)\left[\cos\left(\frac{A}{2} - \frac{B}{2}\right) - \cos\left(\frac{A}{2} + \frac{B}{2}\right)\right] \\
&= 1 + 2\cos\left(\frac{\pi}{2} - \frac{C}{2}\right) \times \left[2\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\right] \\
&= 1 + 4\sin\left(\frac{C}{2}\right)\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right) = RHS.
\end{aligned}$$

c) If $a+b+c = abc$,

then we make the substitution

$a = \tan A$, $b = \tan B$, $c = \tan C$; where A, B & C are the angles of a triangle.

[the constraint is satisfied by (i)]

So; $LHS = \dfrac{1}{\sqrt{1+\tan^2 A}} + \dfrac{1}{\sqrt{1+\tan^2 B}} + \dfrac{1}{\sqrt{1+\tan^2 C}}$

$= \cos A + \cos B + \cos C$

$= 1 + 4\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$ (using b) ii))

Consider $h(x) = \ln[\sin x]$

$h'(x) = \dfrac{\cos x}{\sin x} = \cot x$

$h''(x) = -\operatorname{cosec}^2 x \leq 0$ for all x

$\therefore h(x) = \ln[\sin x]$ is concave down.

$$\frac{\ln\left[\sin\left(\frac{A}{2}\right)\right] + \ln\left[\sin\left(\frac{B}{2}\right)\right] + \ln\left[\sin\left(\frac{C}{2}\right)\right]}{3} \leq \ln\left[\sin\left(\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3}\right)\right]$$

$\therefore \ln\left[\sin\left(\frac{A}{2}\right) \times \sin\left(\frac{B}{2}\right) \times \sin\left(\frac{C}{2}\right)\right] \leq 3\ln\left[\sin\left(\frac{\pi}{6}\right)\right]$ as $A+B+C = \pi$

$\ln\left[\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\right] \leq \ln\left[\left(\frac{1}{2}\right)^3\right]$

$\therefore \sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right) \leq \frac{1}{8}$.

$\therefore$ As $LHS = 1 + 4\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$

$LHS \leq 1 + 4\left(\frac{1}{8}\right) = 1 + \frac{1}{2} = 3/2$

$$\therefore \frac{1}{\sqrt{1+a^2}} + \frac{1}{\sqrt{1+b^2}} + \frac{1}{\sqrt{1+c^2}} \leq \frac{3}{2}.$$