FORT STREET HIGH SCHOOL
4 Unit Mathematics
1999 Trial HSC Examination

Question 1

(a) Find the exact value of: **(i)** $\int_0^1 \frac{e^x}{e^{2x}+1}\, dx$ **(ii)** $\int_e^{e^2} x^2 \log x\, dx$ **(iii)** $\int_4^5 \frac{x+5}{x^2-2x-3}\, dx$

(b) If $I = \int_0^\pi \frac{x \sin x}{1+\cos^2 x}\, dx$ use the substitution $x = \pi - y$ to:

(i) show that $I = \frac{\pi}{2} \int_0^\pi \frac{\sin y}{1+\cos^2 y}\, dy$;

(ii) hence or otherwise show that $I = \frac{\pi^2}{4}$.

Question 2

(a) If $z_1 = 1 + i\sqrt{3}$ and $z_2 = \sqrt{3} + i$

(i) Express z_1 and z_2 in mod-Arg form

(ii) Hence, or otherwise, write $\frac{z_1}{z_2}$ and $\left(\frac{z_1}{z_2}\right)^5$ in the form $a + ib$, where a, b are real.

(b) If $w = 2 + 3i$, illustrate on an Argand diagram the points w and iw clearly, labelling the size of the angle $\arg iw - \arg w$

(c) Describe and sketch the locus defined by

(i) $2 \le |z + 2 - i| \le 4$ **(ii)** $-\frac{\pi}{2} < \arg z < \frac{\pi}{6}$

(d) Show the locus of z defined by $w = \frac{z-i}{z-2}$, where w is purely imaginary, is a circle. Give the centre and radius of this circle.

Question 3

(a) If $P(x) = x^2(x-2)(x+2)$ then sketch the following on separate graphs (indicate clearly the coordinates of turning points and asymptotes).

(i) $y = P(x)$ **(ii)** $y = \frac{1}{P(x)}$

(b) **(i)** Evaluate $\lim\limits_{x \to 0} \frac{\sin x}{x}$.

(ii) Consider $f(x) = \frac{\sin x}{x}$ for $x \geq 0$. Sketch this curve showing intercepts (but do not calculate the coordinates of turning points).

(c) Find the equation of the tangent to the curve $3x^2y^3 + 4xy^2 = 6 + y$ at the point $(1, 1)$.

Question 4

(a) If z is a complex number such that $|z-2| + |z+2| = 6$ explain why the locus of z is an ellipse. For this ellipse find the:

(i) co-ordinates of the foci;

(ii) equations of the directrices;

(iii) eccentricity.

(b) A conic is a rectangular hyperbola with eccentricity $\sqrt{2}$, focus $(2, 0)$ and directrix $x = 1$.

(i) Find the equation of this hyperbola.

(ii) Sketch this hyperbola indicating the asymptotes and vertices.

(iii) Prove the equation of the <u>normal</u> at a point $P(a\sec\theta, a\tan\theta)$ is
$x\tan\theta + y\sec\theta = 2\sqrt{2}\sec\theta\tan\theta$.

(iv) This normal meets the x-axis at $Q(x, 0)$ and the y-axis at $R(0, y)$. Find the locus of the point $T(x, y)$ and describe this locus geometrically.

Question 5

(a) **(i)** Show that the area cut off by the *latus rectum* of the parabola $x^2 = 4Ay$ is $\frac{8A^2}{3}$ square units.

(ii) A solid is now formed such that its base is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the cross-section taken perpendicular to the major axis of the ellipse is a parabola with its *latus rectum* in this base (i.e., the base of the cross section is the *latus rectum*). Find this volume in terms of a and b.

(iii) A cylindrical hole is bored through the centre of a sphere of unknown radius. However, the length of the hole is known to be $2L$. Using cylindrical shells show that the volume of the portion of the sphere that remains is equal to the volume of a sphere of diameter $2L$.

Question 6

(a) Given that $x^4 - 3x^3 - 6x^2 + 28x - 24 = 0$, has a triple root (i.e., a root of multiplicity 3) solve the equation completely.

(b) The polynomial $P(x)$ is given by $P(x) = x^5 - 5cx + 1$ where c is a real number

(i) By considering the turning points, prove that if $c < 0, P(x)$ has just one real root which is negative.

(ii) Prove that $P(x)$ has three distinct real roots if and only if $c > \left(\frac{1}{4}\right)^{4/5}$.

Question 7

(a) Simplify the square of $\frac{1}{4}(\sqrt{6} - \sqrt{2})$.

(i) Hence state the positive square root of $\frac{1}{4}(2 - \sqrt{3})$ and

(ii) Given that θ is acute and that $\cos\theta = \frac{1}{4}(\sqrt{6} + \sqrt{2})$, find $\sin\theta$.

(iii) Hence, or otherwise, evaluate $\sin 2\theta$ and deduce the exact value(s) of θ expressing your answer in radians.

(b) A particle of mass m kg is projected vertically upwards from the ground with a velocity u m.s^{-1} in a medium whose resistance is given by mkv^2 Newtons, where v is the speed at that instant (in m.s^{-1}) and k is a positive constant.

(i) Prove that the time taken to reach the highest point is $\frac{1}{\sqrt{kg}} \tan^{-1}\left(u\sqrt{\frac{k}{g}}\right)$ seconds, where g m.s^{-1} is the acceleration due to gravity.

(ii) Prove that the greatest height reached is $\frac{1}{2k} \ln\left(1 + \frac{ku^2}{g}\right)$ metres.

(iii) How fast is the particle going when it reaches the ground again?

Question 8

(a) Draw a neat sketch of the curve $3y^2 = x(x-1)^2$ and show that the area enclosed by the loop of the curve is $\frac{8\sqrt{3}}{45}$ unit^2.

(b) Show that to hit a target h metres above what was its maximum range position on a horizontal plane, the initial speed of a projectile projected at the same angle as before, must be increased from V to $\frac{V^2}{\sqrt{V^2 - gh}}$ m.s^{-1} (air resistance is neglected.)