2000 - Ext. II Maths Trial

QUESTION 1

Marks

(a) If $z = 4 + 3i$ and $w = 2 - i$. Find in simplest exact form: 3

(i) $\frac{z}{w}$.

(ii) $\text{Im}(wz)$

(iii) $|3z - 3iw|$

(b) Indicate on an Argand diagram the region which contains the point P representing z when: 3

(i) $|z| > 2$ *and* $\arg z \le \frac{\pi}{2}$.

(ii) $|z - 3 - i| > 2$ *and* $\text{Re}(z) \ge 3\,\text{Im}(z)$.

(c) w is a complex cube root of unity. 9

(i) Show that w^2 is also a root and that $1 + w + w^2 = 0$.

(ii) Prove that $(1 + w)(1 + 2w)(1 + 3w)(1 + 5w) = 21$.

(iii) If w and w^2 are roots of $P(x) = x^3 + px^2 + qx + m = 0$, deduce that $p = q = m + 1$.

(iv) If r is a cube root of a, where $a \in C$, show that rw and rw^2 are the two cube roots of a.

(v) Hence, determine the complex numbers z such that

$$\left(\frac{z+1}{z}\right)^3 + 8 = 0.$$

QUESTION 2

Begin a new page.

Marks

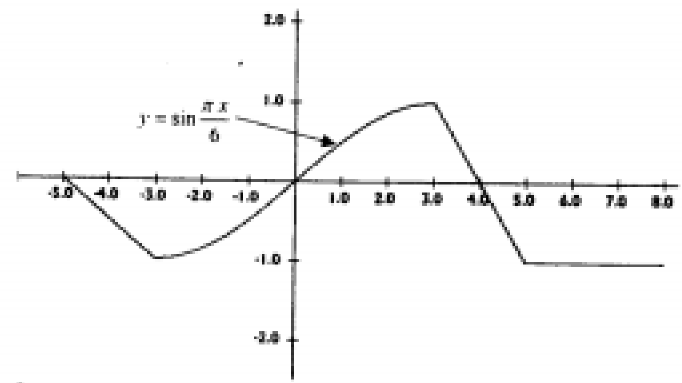
(a) α, β *and* δ are the roots of $x^3 + px + q = 0$. 3

Evaluate $(\alpha - \beta)^2 + (\beta - \delta)^2 + (\delta - \alpha)^2$.

(b) (i) Prove that if a polynomial P(x) has a root of multiplicity m, then P'(x) has a root of multiplicity (m – 1). 4

(ii) Find the values of k so that the equation $5x^3 - 3x^5 + k = 0$ has two equal roots, both positive.

(c) The diagram is a sketch of $y = f(x)$ which includes part of the curve $y = \sin\frac{\pi x}{6}$. 8



On separate diagrams, sketch each of the following:

(i) $y = -f(x)$

(ii) $y = |f(x)|$

(iii) $y = [f(x)]^2$

(iv) $|y| = f(x)$.

QUESTION 7 Begin a new page.

Marks

(a) (i) Sketch $y = e^x - 1 - x$ showing all stationary point(s) and asymptote(s). **4**

(ii) Hence, solve $1 + x < e^x$.

(b) P and Q are on the same branch of the rectangular hyperbola $xy = c^2$. **11**

(i) Show that the equation of the chord joining the points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ is $x + pqy = c(p+q)$.

(ii) Deduce the equation of the tangent at P.

(iii) The tangent at P meets the x and y axis at L and M respectively. O is the origin and POD is a diameter. The line MD meets the x axis at T.

Prove that the area of triangle DOT is equal to $\frac{c^2}{3}$ square units.

(iv) The normal at Q meets the x axis at A and the tangent at Q meets the y axis at B. Find the equation of the locus of the mid-point of AB.

QUESTION 8 Begin a new page.

Marks

(a) (i) Find the derivative of $y = \ln\sqrt{\frac{1 - \sin x}{1 + \sin x}}$. **3**

(ii) Hence, find $\int \sec x \, dx$.

(b) (i) Find $\frac{d}{dx}(\cot^{-1} x)$. **3**

(ii) Prove that the function $f(x) = \cot^{-1} x + \tan^{-1} x$ is constant and find the value of this constant.

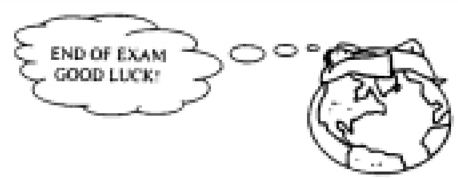
(c) Find the coordinates of the point on the graph $x^2y + xy^2 = 16$ at which the tangent is parallel to the x axis. **4**

(d) Consider the complex number $z = x + iy$ represented by point A on the Argand diagram. **5**

Let $Z = \frac{z - \bar{z} + 2}{z + 1 - i}$ be represented by point M.

(i) Find the locus of A for which Z is a real number.

(ii) Find the locus of M as A moves on the x axis.



2000 4 UNIT TRIAL -FSHS- Solution

Question 1.

(a) $z = 4 + 3i$ $\quad w = 2 - i,\quad \bar{w} = 2 + i.$

(i) $\frac{z}{w} = \frac{4+3i}{2+i} \cdot \frac{(2-i)}{(2-i)} = \frac{8-4i+6i+3}{4+1} = \frac{11}{5} + \frac{2i}{5}.$ 1

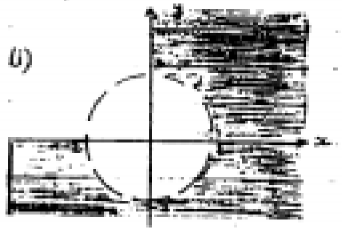
/3

(ii) $wz = (3+4i)(2-i) = 11 + 2i.$ $\quad \text{Im}(wz) = 2.$ 1

(iii) $|3z - 3iw| = |3(4+3i) - 3i(2-i)| = |12 + 9i - 6i - 3i^2|$
$= |9 + 3i| = \sqrt{9^2 + 3^2} = \sqrt{90} = 3\sqrt{10}.$ 1

(b) (i)

/3



$|z| > 2$ and $\arg z \le \pi/2.$
1

(ii) 1 Locus
1 region.

$|z-3-i| > 2$ is region outside circle of centre $(3,1)$, and $r = 2$.

$\text{Re}(z) \ge 3\,\text{Im}(z) \Rightarrow x \ge 3y \Rightarrow y \le \frac{x}{3}$

(c) $w^3 = 1$, w is complex $(w \ne 1)$

(i) $(w^2)^3 = (w^3)^2 = (1)^2 = 1$

$\therefore w^2$ is also a root of $w^3 = 1.$ ½

/1 $\quad (w^3 - 1) = 0 \Rightarrow (w-1)(w^2 + w + 1) = 0$ ½

Since $w \ne 1 \Rightarrow w^2 + w + 1 = 0$ $\quad$ * (See $\underline{\ }$)

(ii) $(1+w)(1+2w)(1+3w)(1+5w)$

$= (1 + 3w + 2w^2)(1 + 8w + 15w^2)$

/2 $\quad = (1 + \underline{2w + 2w^2} + w)(1 + \underline{8w + 8w^2} + 7w^2)$ $\quad$ But $w^2 + w = -1.$

$= (1 + 2(-1) + w)(1 + 8(-1) + 7w^2)$ 1

$= (w-1)(7w^2 - 7) = 7(w-1)(w^2-1)$ $\quad$ Since $w^3 = 1.$

$= 7(w^3 - w^2 - w + 1) = 7(1 - (w^2 + w) + 1)$ $\quad$ [OR: $(w-1)(-7w-14)$ expansion $= 21$].

$= 7(1+1+1) = 7 \times 3 = 21.$ 1

Full expansion: $1 + 41w^2 + 11w + (15+16)w^2 + 30w = 1 + 41w + 41w^2 + 61.$
$= 62 + 41(w + w^2) = 62 - 41 = 21$

2000 FSHS 4 Unit Trial Solutions.

* Alternative s/n for (i): $z^3 = 1$ Cube Roots of unity: $\text{cis}\,3\theta = \text{cis}\,0.$

$3\theta = 2k\pi \Rightarrow \theta = 2k\pi/3.$ $\quad k = 0, \pm 1.$

$\therefore$ cube roots of unity are: $1,\ \text{cis}\frac{2\pi}{3},\ \text{cis}\frac{4\pi}{3} = \text{cis}(-2\pi/3).$

$z^3 - 1 = 0 \Rightarrow (z-1)(z^2 + z + 1) = 0$

$\therefore$ complex roots must satisfy $z^2 + z + 1 = 0.$

$w = \text{cis}\,2\pi/3 \Rightarrow w^2 = \text{cis}\,2 \times \frac{2\pi}{3} = \text{cis}\frac{4\pi}{3}.$ (By De Moivre's).

But $\text{cis}\frac{4\pi}{3} = \text{cis}(-2\pi/3).$

OR: $\left.\begin{array}{l} w = \text{cis}\,2\pi/3 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}. \\ w^2 = \text{cis}(-2\pi/3) = -1/2 - \frac{i\sqrt{3}}{2}. \end{array}\right\} \Rightarrow 1 + w + w^2 = 0.$

(iii) $P(w) = w^3 + pw^2 + qw + m = 0. \Rightarrow 1 + pw^2 + qw + m = 0.$

$P(w^2) = (w^2)^3 + pw^4 + qw^2 + m = 0$ $\quad$ Since $(w^2)^3 = (w^3)^2 = 1$;

/2 $\quad = 1 + pw + qw^2 + m = 0.$

$P(w) - P(w^2) = 0 \Rightarrow p(w^2 - w) + q(w - w^2) = 0.$

$p(w^2 - w) - q(w^2 - w) = 0.$

$(p - q)(w^2 - w) = 0. \Rightarrow p = q$ since $w^2 \ne$

But $1 + pw^2 + qw + m = 0 \Rightarrow 1 + pw^2 + pw + m = 0.$

$1 + p(w^2 + w) + m = 0.$

$1 + p(-1) + m = 0. \Rightarrow p = m + 1.$

$\therefore p = q = m + 1.$

OR: Sums $-p = w + w^2 + \alpha = -1 + \alpha.$ $\therefore p = -\alpha + 1.$ $\quad \Sigma\alpha\beta = q = w^3 + \alpha w + \alpha w^2 = 1 + \alpha(w + w^2) = 1 - \alpha.$
Products $w w^2 \alpha = w^3\alpha = -m \Rightarrow \alpha = -m \Rightarrow p = q = 1 - \alpha = 1 + m.$

(iv) $r^3 = a$.

/1 $\quad (rw)^3 = r^3 w^3 = r^3(1) = a$ since $w^3 = 1.$ $\Rightarrow (rw)$ is a cube root

$(rw^2)^3 = r^3 w^6 = r^3 (w^3)^2 = r^3(1)^2 = r^3 = a \Rightarrow (rw^2)$ is also a cube roo

OR: If r is a cube root, next root is $\text{cis}\frac{2\pi}{3} r = wr$; other root is $\text{cis}\frac{4\pi}{3} r = w^2$

½

(v) $r^3 = -8$ $\therefore$ Roots are -2; $-2w = -2\,\text{cis}\frac{2\pi}{3} = -2\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$; $-2w^2 = -2\left(-\frac{1}{2}\right.$

$\frac{z+1}{z} = -2$ $\quad \frac{z+1}{z} = -2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$ $\quad \frac{z+1}{z} = -2\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$

/3 $\quad z + 1 = -2z$ $\quad \frac{z+1}{z} = 1 - \sqrt{3}i$ $\quad \frac{z+1}{z} = 1 + i\sqrt{3}.$

$3z = -1$ $\quad z + 1 = z - \sqrt{3}zi.$ $\quad z + 1 = z + i\sqrt{3}z.$

$z = -\frac{1}{3}.$ $\quad 1 = -\sqrt{3}iz.$ $\quad z = \frac{1}{i\sqrt{3}} = \frac{i}{i^2\sqrt{3}}$

½ $\quad z = -\frac{1}{\sqrt{3}i} = \frac{i}{\sqrt{3}}.$ $\quad z = -\frac{i}{\sqrt{3}}.$

1 $\quad$ 1

Question 2.

(a). $x^3+px+q=0$. $\rightarrow \alpha+\beta+\delta=0$.
$\alpha\beta+\alpha\delta+\beta\delta=+p$.

/3 $I=(\alpha-\beta)^2+(\beta-\delta)^2+(\delta-\alpha)^2$

$=(\alpha^2+\beta^2-2\alpha\beta)+(\beta^2+\delta^2-2\beta\delta)+(\delta^2+\alpha^2-2\alpha\delta)$.

$=2(\alpha^2+\beta^2+\delta^2)-2(\alpha\beta+\beta\delta+\alpha\delta)$ 1

But $(\alpha+\beta+\delta)^2=\alpha^2+\beta^2+\delta^2+2(\alpha\beta+\beta\delta+\alpha\delta)$

$\Rightarrow \alpha^2+\beta^2+\delta^2=(\alpha+\beta+\delta)^2-2(\alpha\beta+\beta\delta+\alpha\delta)$

$\therefore I=2[(\alpha+\beta+\delta)^2-2(\alpha\beta+\beta\delta+\alpha\delta)]-2(\alpha\beta+\beta\delta+\alpha\delta)$ 1

$=2[(\underbrace{\alpha+\beta+\delta}_{0})^2-2(+p)]-2(+p)$

$=-4p-2p$.

$=-6p$. 1

(b) (i) Let a be the m-fold root of P(x).

$P(x)=(x-a)^m\,Q(x)$.

$P'(x)=m(x-a)^{m-1}.\,Q(x)+(x-a)^m.\,Q'(x)$. 1

$=(x-a)^{m-1}[\underbrace{mQ(x)+Q'(x)(x-a)}]$.

/4 $=(x-a)^{m-1}[S(x)]$.

Hence "a" is a root of multiplicity (m-1) of P'(x). 1

(ii) $P(x)=5x^5-3x^3+k=0$.

For 2 = roots $\Rightarrow P'(x)=0$.

$P'(x)=25x^4-9x^2=x^2(25x^2-9)=0$. ½

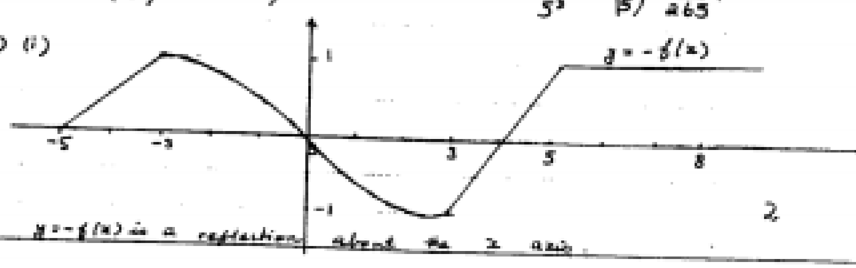
$x=0$ or $x=\pm 3/5$.

But root must be $>0 \Rightarrow x=3/5$. ½

$\therefore P(\frac{3}{5})=0$

$5(\frac{3}{5})^5-3(\frac{3}{5})^3+k=0$. $\Rightarrow k=\frac{3\times3^3}{5^3}-5(\frac{3}{5})^5=\frac{162}{625}$. 1

(c) (i)



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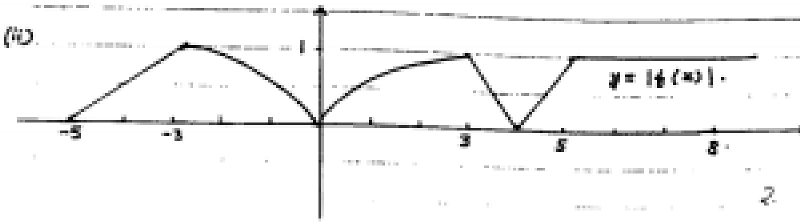
y = -f(x) is a reflection about the x axis.

(ii)



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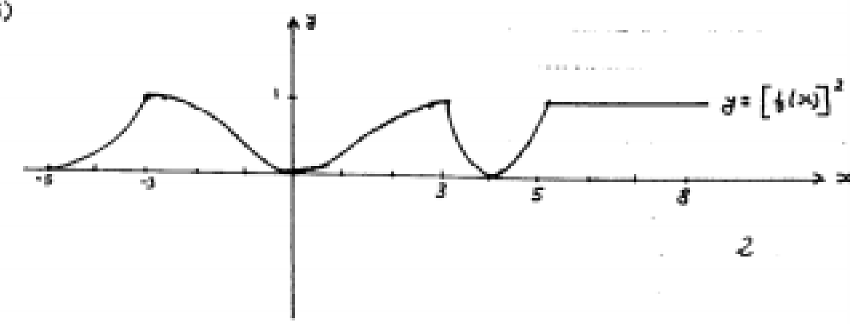
(iii)



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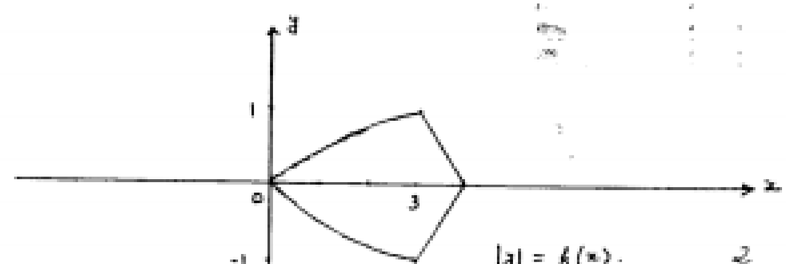
(iv)



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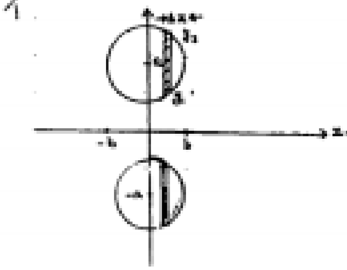
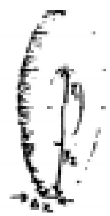
consists of the part of f(x) above the x-axis and its reflection in the x axis.

Question 3.

(a)(i) $x^2 + (y-a)^2 = b^2$ where $a > b$.

1





(ii) $A = \pi(y_2^2 - y_1^2)$. 1

½ To find y_2^2, y_1^2 ????

$(y-a)^2 = b^2 - x^2$.

$y - a = \pm\sqrt{b^2 - x^2}$.

$y = a \pm \sqrt{b^2-x^2}$. $\therefore y_2 = a + \sqrt{b^2-x^2}$, $y_1 = a - \sqrt{b^2-x^2}$. 1

$y_2^2 - y_1^2 = (y_2 + y_1)(y_2 - y_1)$.

$= (a + \sqrt{b^2-x^2} + a - \sqrt{b^2-x^2})(a + \sqrt{b^2-x^2} - a + \sqrt{b^2-x^2})$

$= 2a\,(2\sqrt{b^2-x^2})$.

$= 4a\sqrt{b^2-x^2}$. 1

$\therefore A = \pi\left[4a\sqrt{b^2-x^2}\right] = 4\pi a\sqrt{b^2-x^2}$. 1½

(iii) $\Delta V = 4\pi a\sqrt{b^2-x^2}\,.\,\Delta x$. is volume of a slice.

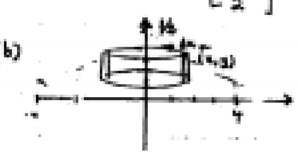
½ $V = \sum_{x=-b}^{b} 4\pi a\sqrt{b^2-x^2}\,.\,\Delta x$.

$= \int_{-b}^{b} 4\pi a\sqrt{b^2-x^2}\,dx = 4\pi a\int_{-b}^{b}\sqrt{b^2-x^2}\,.\,dx$. 1

But $\int_{-b}^{b}\sqrt{b^2-x^2}\,dx$ is area of semi-circle of radius $b = \frac{\pi b^2}{2}$.

$\therefore V = 4\pi a.\left[\frac{\pi b^2}{2}\right] = 2\pi^2 a b^2$. * (*see page 6) 1

(b)



$A_x = 2\pi x y = 2\pi x \times \frac{1}{x+2} = \frac{2\pi x}{x+2}$. 1½

$\Delta V = A_x . \Delta x$.

$= \frac{2\pi x}{x+2}\,\Delta x$ is volume of slice.

$V = \sum_{x=0}^{4}\frac{2\pi x}{x+2}\,\Delta x = \int_0^4 \frac{2\pi x}{x+2}\,dx = 2\pi\int_0^4\frac{x\,dx}{x+2}$.

$= 2\pi\int_0^4\left(\frac{x+2}{x+2} - \frac{2}{x+2}\right)dx = 2\pi\left[x - 2\ln|x+2|\right]_0^4$

$= 2\pi\left[4 - 2\ln 6 - 0 + 2\ln 2\right] = 2\pi\left[4 + 2\ln 2 - 2\ln 6\right]$.

$= 2\pi\left[4 + 2(\ln 2 - \ln 6)\right] = 2\pi\left[4 + 2\ln\frac{2}{6}\right]$

$= 8\pi + 4\pi\ln\frac{1}{3}$. $= 4\pi\,(2 + \ln\frac{1}{3}) = 4\pi\,(2 + \cancel{\ln 1} - \ln 3)$

$= 4\pi(2 - \ln 3)$. or $2\pi\left[4 - \ln 9\right]$. 1

(c)(i) $I_n = \int_0^1 x^n e^x\,dx$. $u = x^n$ $du = nx^{n-1}$. $dv = e^x dx$ $v = e^x$.

$I_n = \left[x^n e^x\right]_0^1 - n\int x^{n-1}e^x\,dx$. 1

$I_n = e - 0 - n\,I_{n-1}$.

$\therefore e = I_n + n\,I_{n-1}$ for $n \geq 1$. 1

(ii) $I_n = e - n\,I_{n-1}$.

$I_4 = e - 4\,I_3$

$= e - 4(e - 3I_2) = -3e + 12\,I_2$.

$= -3e + 12(e - 2I_1)$

$= -3e + 12e - 24\,I_1$ But $I = \int_0^1 e^x dx = \left[e^x\right]_0^1 = e.$

$= 9e - 24(e - I_0)$

$= -13e + 24\,I_0$ $= -13e + 24(e-1) = 9e - 24$. 1

(a)(iii) Long way to find volume:

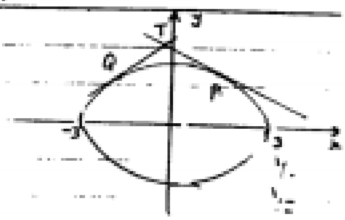
* $\int_{-b}^{b}\sqrt{b^2-x^2}\,dx$ $x = b\sin\theta \Rightarrow dx = b\cos\theta\,d\theta$.

$V = 4\pi a\sqrt{b^2(1-\sin^2\theta)}\,.\,b\cos\theta\,d\theta = 4\pi b^2 a\int\cos^2\theta\,d\theta = 4\pi ab^2\times 2\int_0^{\pi/2}\cos^2$

$= 4\pi ab^2\times 2\int_0^{\pi/2} 1+\cos 2\theta\,d\theta$ ...

Question 4

$a=3 \quad b=2 \quad \therefore \frac{x^2}{9}+\frac{y^2}{4}=1.$



(i) $P(3\cos\theta,\ 2\sin\theta)$.

$x_Q = 3\cos\left(\theta+\frac{\pi}{2}\right) = -3\sin\theta.$ ½

$y_Q = 2\sin\left(\theta+\frac{\pi}{2}\right) = 2\cos\theta.$ ½

$\therefore\ Q(-3\sin\theta,\ 2\cos\theta).$

(ii) $OP^2+OQ^2 = (3\cos\theta)^2+(2\sin\theta)^2+(-3\sin\theta)^2+(2\cos\theta)^2$

$= 9\cos^2\theta+4\sin^2\theta+9\sin^2\theta+4\cos^2\theta$ ½

$= 9(\cos^2\theta+\sin^2\theta)+4(\sin^2\theta+\cos^2\theta)$

$= 9\times1+4\times1 = 13.$ ½

(iii) $\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$ is equation of tangent.

at P: $\frac{3\cos\theta\cdot x}{9}+\frac{2\sin\theta\, y}{4}=1 \Rightarrow \frac{\cos\theta\, x}{3}+\frac{y\sin\theta}{2}=1.$

$\therefore (2\cos\theta)x+(3\sin\theta)y=6 \quad \ldots\ldots (1)$ 1

at Q: $-\frac{3\sin\theta\, x}{9}+\frac{(2\cos\theta)y}{4}=1.$

$-\frac{(\sin\theta)x}{3}+\frac{(\cos\theta)y}{2}=1.$

$(-2\sin\theta)x+(3\cos\theta)y=6 \quad \ldots\ldots 2.$ 1

(iv) Solve (1) and (2) simultaneously:

(1) $\times\sin\theta$. $\quad 2\cos\theta\sin\theta\, x+3\sin^2\theta\, y = 6\sin\theta.$

(2) $\times\cos\theta$. $\quad -2\sin\theta\cos\theta\, x+3\cos^2\theta\, y = 6\cos\theta.$

Add: $\quad 3(\sin^2\theta+\cos^2\theta)y = 6(\sin\theta+\cos\theta).$

$\therefore\ 3y = 6(\sin\theta+\cos\theta).$

$y = 2(\sin\theta+\cos\theta).$ 1½

Sub y into (1) $\quad 2\cos\theta\, x+3\sin\theta\,(2)(\sin\theta+\cos\theta)=6.$

$2\cos\theta\, x+6\sin^2\theta+6\sin\theta\cos\theta=6.$

$2\cos\theta\, x+6(1-\cos^2\theta)+6\sin\theta\cdot\cos\theta=6.$

$2\cos\theta\, x+\cancel{6}-6\cos^2\theta+6\sin\theta\cdot\cos\theta=\cancel{6} \quad \div 2.$

$\cos\theta\, x = 3\cos^2\theta-3\sin\theta\cdot\cos\theta. \quad \div\cos\theta.$

$x = 3\cos\theta-3\sin\theta. \quad \Rightarrow x=3(\cos\theta-\sin\theta).$

$T\left[3(\cos\theta-\sin\theta),\ 2(\sin\theta+\cos\theta)\right].$ 1½

(vi) $\frac{x^2}{9}+\frac{y^2}{4}=1 \Rightarrow \frac{y^2}{4}=1-\frac{x^2}{9} \Rightarrow y^2=4\left(1-\frac{x^2}{9}\right).$

$\therefore\ y=\pm\frac{2}{3}\sqrt{(9-x^2)}$

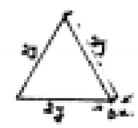
$A=\int_{-3}^{3} y\,dx = 2\int_{-3}^{3}\frac{2}{3}\sqrt{9-x^2}\,dx = \frac{4}{3}\int_{-3}^{3}\sqrt{9-x^2}\,dx$ 1

But $\int_{-3}^{3}\sqrt{9-x^2}\,dx$ is a semi-circle of area $=\frac{\pi 3^2}{2}$

$\therefore\ A=\frac{4}{3}\times\pi\times\frac{9}{2}=\frac{36\pi}{6}=6\pi.$ 1

(vii)

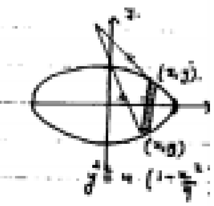
slice:



$A=\frac{1}{2}(2y)(2y)\sin 60^\circ.$

$=\frac{1}{2}(4y^2)\left(\frac{\sqrt{3}}{2}\right).$

$=y^2\sqrt{3}. \quad =4\sqrt{3}\left(1-\frac{x^2}{9}\right).$



$y^2=4\left(1-\frac{x^2}{9}\right)$

$\Delta V = y^2\sqrt{3}\,\Delta x = 4\sqrt{3}\left(1-\frac{x^2}{9}\right)\Delta x.$

$V=\lim_{\Delta x\to0}\sum_{x=-3}^{3}4\sqrt{3}\left(1-\frac{x^2}{9}\right)\Delta x = 4\sqrt{3}\int_{-3}^{3}\left(1-\frac{x^2}{9}\right)dx$

$=4\sqrt{3}\left[x-\frac{x^3}{27}\right]_{-3}^{3} = 4\sqrt{3}\left[3-\frac{27}{27}-\left(-3-\frac{-27}{27}\right)\right].$

$=4\sqrt{3}\left[3-1+3-1\right]=4\sqrt{3}\times4=16\sqrt{3}\ \text{unit}^3.$

or $V=8\sqrt{3}\int_{0}^{3}\left(1-\frac{x^2}{9}\right)dx = 8\sqrt{3}\left[x-\frac{x^3}{27}\right]_0^3 = 8\sqrt{3}\times[3-1]=16\sqrt{3}\ \text{unit}^3.$

Question 5.

(a) $\int (9-x)\sqrt{4-x}\,dx = \int (4-x+5)\sqrt{4-x}\,dx = \int (4-x)\sqrt{4-x}\,dx + 5\int \sqrt{4-x}\,dx$

/2 $= \int (4-x)^{3/2}\,dx + 5\int\sqrt{4-x}\,dx = \dfrac{(4-x)^{5/2}}{-5/2} + 5\dfrac{(4-x)^{3/2}}{-3/2} + C$

$= -\dfrac{2}{5}(4-x)^2\sqrt{4-x} - \dfrac{10}{3}(4-x)\sqrt{4-x} + C.$

(* See page 11)

(b) $I = \int_0^{\pi/2} \dfrac{dx}{2-\cos x+\sin x}$ let $t = \tan\dfrac{x}{2}$.

/3 $dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)dx = \frac{1}{2}(1+t^2)\,dx.$

$\Rightarrow dx = \dfrac{2\,dt}{1+t^2}.$ ½

$I = \int \dfrac{\frac{2\,dt}{1+t^2}}{2-\frac{(1-t^2)}{1+t^2}+\frac{2t}{1+t^2}} = \int \dfrac{2\,dt}{2+2t^2-1+t^2+2t} = \int\dfrac{2\,dt}{3t^2+2t+1} = \dfrac{2}{3}\int\dfrac{dt}{t^2+\frac{2t}{3}+\frac{1}{3}}$

$= \dfrac{2}{3}\int\dfrac{dt}{t^2+\frac{2t}{3}+\left(\frac{1}{3}\right)^2+\frac{1}{3}-\left(\frac{1}{3}\right)^2} = \dfrac{2}{3}\int\dfrac{dt}{\left(t+\frac{1}{3}\right)^2+\frac{1}{3}-\frac{1}{9}} = \dfrac{2}{3}\int\dfrac{dt}{\left(t+\frac{1}{3}\right)^2+\frac{2}{9}}.$ ½

let $u = t+\frac{1}{3}$. $\Rightarrow I = \dfrac{2}{3}\int\dfrac{du}{u^2+\frac{2}{9}} = \dfrac{2}{3}\cdot\dfrac{1}{\left(\frac{\sqrt{2}}{3}\right)}\left[\tan^{-1}\dfrac{u}{\frac{\sqrt{2}}{3}}\right]_{1/3}^{4/3}$ ½

$du = dt$

at $x=0 \Rightarrow u = \tan^{-1}0+\frac{1}{3} = \frac{1}{3}$.

$x = \frac{\pi}{2} \Rightarrow u = \tan\frac{\pi}{4}+\frac{1}{3} = 1+\frac{1}{3} = \frac{4}{3}.$ $= \sqrt{2}\tan^{-1}\left[\dfrac{3u}{\sqrt{2}}\right]_{1/3}^{4/3}.$

$\therefore I = \sqrt{2}\tan^{-1}\left[\dfrac{3\times\frac{4}{3}}{\sqrt{2}}\right] - \sqrt{2}\tan^{-1}\left[\dfrac{3\times\frac{1}{3}}{\sqrt{2}}\right] = \sqrt{2}\tan^{-1}\left(\dfrac{4}{\sqrt{2}}\right) - \sqrt{2}\tan^{-1}\dfrac{1}{\sqrt{2}}.$

$= \sqrt{2}\left[\tan^{-1}2\sqrt{2} - \tan^{-1}\dfrac{1}{\sqrt{2}}\right].$ 1

(c) $A = \int_4^6 \dfrac{16x}{x^4-16}\,dx = \int_4^6\dfrac{16x\,dx}{(x^2-4)(x^2+4)} = \int\dfrac{16x\,dx}{(x-2)(x+2)(x^2+4)}.$

/4 $\dfrac{16x}{(x-2)(x+2)(x^2+4)} = \dfrac{a}{(x-2)} + \dfrac{b}{(x+2)} + \dfrac{cx+d}{x^2+4}.$

$16x = a(x+2)(x^2+4) + b(x-2)(x^2+4) + (cx+d)(x^2-4).$

let $x=+2$: $16(+2) = a(4)(8)+0 \Rightarrow 32 = 32a \Rightarrow \underline{a=1}.$

$x=-2$: $-32 = b(-4)(8)+0 \Rightarrow -32=-32b \Rightarrow \underline{b=1}.$

$\therefore 16x = (x+2)(x^2+4) + (x-2)(x^2+4) + (cx+d)(x^2-4)$

Let $x=0$: $0 = 2\times4 + (-2)(4) + d(-4) \Rightarrow \underline{d=0}.$

$x=1$: $16 = 3\times5 + (-1)(5) + c(-3) \Rightarrow 16-10 = 6 = -3c \Rightarrow \underline{c=-2}.$

$\therefore \dfrac{16x}{(x^2-4)(x^2+4)} = \dfrac{1}{(x-2)} + \dfrac{1}{(x+2)} - \dfrac{2x}{x^2+4}.$ 2 for d = and c

$A = \int_4^6\left[\dfrac{1}{(x-2)}+\dfrac{1}{(x+2)}\right]dx - \int\dfrac{2x}{(x^2+4)}\,dx \to \int\dfrac{du}{u}.$ $u = x^2+4$, $du = 2x$.

$= \Big[\ln|x-2| + \ln|x+2|\Big]_4^6 - \Big[\ln u\Big]_{20}^{40}$ at $x=4 \Rightarrow u=20$, $x=6 \Rightarrow u=40$.

$= \ln 4 + \ln 8 - \ln 2 - \ln 6 - \ln 40 + \ln 20.$ 1

$= \ln 2^2 + \ln 2^3 - \ln 2 - \ln 6 - [\ln 40 - \ln 20].$

$= 2\ln 2 + 3\ln 2 - \ln 2 - \ln 6 - \ln\frac{40}{20}.$

$= 4\ln 2 - \ln 6 - \ln 2.$

$= 3\ln 2 - \ln 6 = \ln 2^3 - \ln 6 = \ln 8 - \ln 6.$

$= \ln\frac{8}{6} = \ln\frac{4}{3}.$ 1

OR: $\dfrac{16x}{(x^2-4)(x^2+4)} = \dfrac{-2x}{x^2+4} + \dfrac{2x}{x^2-4}$

$\therefore A = \int_4^6\left(-\dfrac{2x}{x^2+4} + \dfrac{2x}{x^2-4}\right)dx = \Big[\ln(x^2-4) - \ln(x^2+4)\Big]_4^6 = \left[\ln\dfrac{x^2-4}{x^2+4}\right]_4^6$

$= \ln\frac{32}{40} - \ln\frac{12}{20} = \ln\frac{32}{40}\times\frac{20}{12} = \ln\frac{4}{3}.$

(d). $I = \int \sin^{-1}\frac{x}{a}\,dx.$ $dv = dx$, $v = x$. $u = \sin^{-1}\frac{x}{a}$.

$du = \dfrac{1}{a}\times\dfrac{1\,dx}{\sqrt{1-\frac{x^2}{a^2}}} = \dfrac{dx}{\sqrt{a^2-x^2}}.$

$I = uv - \int v\,du = x\sin^{-1}\dfrac{x}{a} - \int\dfrac{x\,dx}{\sqrt{a^2-x^2}}$ Let $U = a^2-x^2$. $dU = -2x\,dx$ $\Rightarrow x\,dx = \frac{dU}{-2}$.

$= x\sin^{-1}\dfrac{x}{a} + \int\dfrac{dU}{2\sqrt{U}} = x\sin^{-1}\dfrac{x}{a} + \dfrac{1}{2}\cdot\dfrac{U^{1/2}}{1/2} + C.$

$= x\sin^{-1}\dfrac{x}{a} + \sqrt{a^2-x^2} + C.$

OR: $u^2 = a^2-x^2$

$2u\,du = -2x\,dx.$

$-u\,du = x\,dx.$

$\int\dfrac{x\,dx}{\sqrt{a^2-x^2}} = \int\dfrac{-u\,du}{u} = \int -du = -u.$

$\therefore I = x\sin^{-1}\dfrac{x}{a} + u + C.$

$= x\sin^{-1}\dfrac{x}{a} + \sqrt{a^2-x^2} + C.$

OR: $x = a\sin\theta$.

$dx = a\cos\theta\,d\theta.$

$-\int\dfrac{x}{\sqrt{a^2-x^2}}\,dx = -\int\dfrac{a\sin\theta\cdot a\cos\theta\,d\theta}{a\cos\theta}.$

$= a\cos\theta.$

$= a\left[\dfrac{\sqrt{a^2-x^2}}{a}\right]$

$= \sqrt{a^2-x^2}.$

$\sqrt{a^2-x^2}$.

$\cos\theta = \dfrac{\sqrt{a^2-x^2}}{a}$

2000 FSHS 4 Unit Trial Solutions.

(c)(i) R.T.P: $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

4 Let $u = a - x$. $x = 0 \Rightarrow u = a$. $\therefore x = a - u$.

$du = -dx$. $x = a \Rightarrow u = 0$. $dx = -du$.

$$\int_0^a f(x)\,dx = -\int_a^0 f(a-u)\,du = \int_0^a f(a-u)\,du = \int_0^a f(a-x)\,dx. \quad 2$$

(ii) $$I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx = \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2 - x)}}{\sqrt{\sin(\pi/2 - x)} + \sqrt{\cos(\pi/2 - x)}}\,dx = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\,dx. \quad \tfrac{1}{2}$$

$$\therefore \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx = \int_0^{\pi/2} \frac{\sqrt{\cos x}\,dx}{\sqrt{\cos x} + \sqrt{\sin x}}$$ Add I to both sides.

$$\Rightarrow \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx + \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx = \int_0^{\pi/2} \frac{\sqrt{\cos x}\,dx}{\sqrt{\cos x} + \sqrt{\sin x}} + \int_0^{\pi/2} \frac{\sqrt{\sin x}\,dx}{\sqrt{\cos x} + \sqrt{\sin x}}$$

$$2I = \int_0^{\pi/2} \left[\frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} + \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}}\right] dx \quad 1$$

$$2I = \int_0^{\pi/2} \frac{\sqrt{\cos x} + \sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}}\,dx = \int_0^{\pi/2} 1\,dx.$$

$$\therefore 2I = \left[x\right]_0^{\pi/2} = \pi/2$$

$$\Rightarrow I = \pi/4. \quad \tfrac{1}{2}$$

(4) Alternative Sln's: (Longer ways!)

$u^2 = 4 - x$.

$x = 4 - u^2$

$dx = -2u\,du$.

$$\int 9 - (4 - u^2)\,u\,(-2u\,du)$$

$$= -2\int (9u^2 - 4u^2 + u^4)\,du.$$

$$= -2\int (5u^2 + u^4)\,du.$$

$$= -2\left[\frac{5u^3}{3} + \frac{u^5}{5}\right].$$

$$= -\frac{10}{3}u^3 + \frac{2u^5}{5} + C.$$

$$= -\frac{10}{3}\sqrt{(4-x)^3} - \frac{2}{5}\sqrt{(4-x)^5} + C.$$

$$(9-x)\sqrt{4-x} = 9\sqrt{4-x} - x\sqrt{4-x}.$$

$$\int 9(4-x)^{1/2}\,dx - \int x\sqrt{4-x}\,dx.$$ Let $u = 4 - x$. $x = 4 - u$. $du = -dx$.

$$= \frac{9(4-x)^{3/2}}{-3/2} + \int (4-u)\sqrt{u}.$$

$$= -6(4-x)^{3/2} + \int (4\sqrt{u} - u^{3/2})\,du.$$

$$= -6(4-x)^{3/2} + \frac{4u^{3/2}}{3/2} - \frac{u^{5/2}}{5/2}$$

$$= -6(4-x)^{3/2} + \frac{8}{3}(4-x)^{3/2} - \frac{2}{5}(4-x)^{5/2}.$$

$$= -\frac{10}{3}(4-x)^{3/2} - \frac{2}{5}(4-x)^{5/2} + C.$$

2000 FSHS 4 Unit Trial Solutions.

Question 6.

(a) (i) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

Step 1: Prove true for $n = 1$.

$(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta \Rightarrow$ true for $n = 1$. $\tfrac{1}{2}$

3 Step 2: Assume true for $n = k$: $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$.

Prove true for $n = k+1$ i.e. $(\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta$.

Proof: $(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k \cdot (\cos\theta + i\sin\theta)$

$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$ from assumption.

$= \cos k\theta.\cos\theta - \sin\theta.\sin k\theta + i(\cos k\theta.\sin\theta + \sin k\theta.\cos\theta)$.

$= \cos(k\theta + \theta) + i\sin(\theta + k\theta)$

$= \cos(k+1)\theta + i\sin(k+1)\theta$. 2

Step 3: Since it's true for $n = 1$, it is true for $n = 2$; Since it's true for $n = k$, it's true for $n = k+1$. $\tfrac{1}{2}$

$\therefore$ It is true for all positive integers k.

(ii) Let $c = \cos\theta$ and $s = \sin\theta$.

$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$.

3 $(c + is)^5 = c^5 + 5c^4 is + 10c^3(is)^2 + 10c^2(is)^3 + 5c(is)^4 + (is)^5$

$= c^5 + 5c^4 s\,i - 10c^3s^2 - 10c^2s^3 i + 5cs^4 + is^5$. 1

$= (c^5 - 10c^3s^2 + 5cs^4) + i(5c^4s - 10c^2s^3 + s^5)$. $= \cos 5\theta + i\sin$

$\therefore \cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$.

$= \cos^5\theta - 10\cos^3\theta(1 - \cos^2\theta) + 5\cos\theta(1 - \cos^2\theta)^2$

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta(1 + \cos^4\theta - 2\cos^2\theta)$

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta + 5\cos^5\theta - 10\cos^3\theta$.

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. 2

(iii) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. Let $x = \cos\theta$.

$= 16x^5 - 20x^3 + 5x = \frac{\sqrt{3}}{2}$.

2 as given as $32x^5 - 40x^3 + 10x - \sqrt{3} = 0$.

$\therefore \cos 5\theta = \frac{\sqrt{3}}{2} = \cos\frac{\pi}{6}$. $\tfrac{1}{2}$

$\therefore 5\theta = \frac{\pi}{6} + 2k\pi$ where $k = 0, \pm 1, \pm 2$.

$\theta = \frac{\pi}{30} + \frac{2k\pi}{5}$.

$\theta = \frac{\pi}{30}$; $\frac{\pi}{30} + \frac{2\pi}{5} = \frac{13\pi}{30}$; $\frac{\pi}{30} - \frac{2\pi}{5} = \frac{-11\pi}{30}$ or $\frac{49\pi}{30}$; $\frac{\pi}{30} + \frac{4\pi}{5} = \frac{5\pi}{6}$

$\frac{\pi}{30} - \frac{4\pi}{5} = -\frac{23\pi}{30}$ or $37\pi/30$.

$\therefore$ solutions are: $\cos\frac{\pi}{30}$, $\cos\frac{13\pi}{30}$, $\cos\left(\frac{-11\pi}{30}\right)$, $\cos\frac{5\pi}{6} = \frac{-\sqrt{3}}{2}$, $\cos($

P.S: You can also use: $5\theta = 2k\pi \pm \pi/6$ (general formula of cos. soln). 1

(b)(i) For $x \neq 0$ $[H(x)]^2 = y^2 = (1-x)^2(4-x)$.

/3 $2y \cdot y' = -2(1-x)(4-x) - 1(1-x)^2$

$= -(1-x)[8-2x+1-x] = (x-1)(9-3x) = 3(x-1)(3-x)$.

$\therefore y' = \dfrac{3(x-1)(3-x)}{\pm 2(1-x)\sqrt{4-x}} = \pm \dfrac{3(3-x)}{2\sqrt{4-x}} = 0$.

$\therefore x = 3$ is s/n of $y' = 0$. $\Rightarrow y = \pm\sqrt{4} = \pm 2$. $\Rightarrow (3,2)$ & $(3,-2)$. Since $e^x \neq 0$ 1

For $y = (1-x)\sqrt{4-x}$. $\Rightarrow y' = \dfrac{3(x-1)(3-x)}{2(1-x)\sqrt{4-x}} = -\dfrac{3(3-x)}{2\sqrt{4-x}}$. $\therefore y \neq -1-x$

x	3^-	3	3^+
y'	$-$	0	$+$

Minimum. $(3,2)$ 1

For $y = -(1-x)\sqrt{4-x}$ $\Rightarrow y' = \dfrac{3(3-x)}{2\sqrt{4-x}}$.

x	3^-	3	3^+
y'	$+$	0	$-$

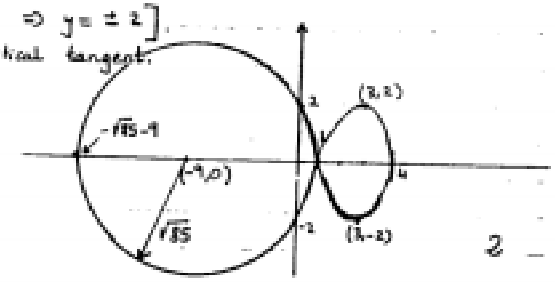
Max $(3,-2)$ 1

(ii) $y^2 = 85 - (9+x)^2$ for $x < 0$ is the equation of a circle

$\therefore$ $(x+9)^2 + y^2 = 85$ of centre $(-9,0)$ and radius $= \sqrt{85}$.

[at $x=0$ $81 + y^2 = 85 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2$].

For $x = 4$ $y' = \infty \Rightarrow$ vertical tangent.



2

(iii) $V = \pi\int_1^4 y^2\,dx = \pi\int_1^4 (1-x)^2(4-x)\,dx$.

/2 As $(1-x)^2(4-x) = (x^2-2x+1)(4-x) = (4x^2-8x+4-x^3+2x^2-x) = -x^3+6x^2-9x+4$.

$V = \pi\int_1^4(-x^3+6x^2-9x+4)\,dx = \pi\left[-\frac{x^4}{4}+\frac{6x^3}{3}-\frac{9x^2}{2}+4x\right]_1^4$ 1

$V = \pi\left[\left(-\frac{4^4}{4}+\frac{6\times4^3}{3}-\frac{9\times4^2}{2}+16\right)-\left(-\frac{1}{4}+2-\frac{9}{2}+4\right)\right]$

$V = \pi\left[-64+128-72+16-1\frac{1}{4}\right] = 6\frac{3}{4}\pi$ or $\frac{27}{4}\pi$ 1

Question 7

(a) (i) $y = e^x - 1 - x = e^x - (1+x)$.

$y' = e^x - 1 = 0 \Rightarrow e^x = 1 \Rightarrow x = 0$ is a stationary pt. $\Rightarrow y = 1-1 = 0$

/3 Nature of $(0,0)$: $y'' = e^x = e^0 = 1 > 0 \Rightarrow$ concave up $\Rightarrow$ minimum $(0,0)$

Limiting values: as $x \to +\infty$, $y = e^x\left[1 - \frac{(1+x)}{e^x}\right] \to +\infty$.

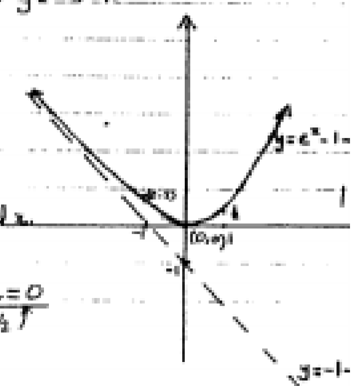
$x \to -\infty$, $y = e^{-\infty} - (-\infty) = 0 + \infty \to +\infty$.

The line $y = -x-1$ is an oblique asymptote as can be shown from addition of the 2 curves: $y = e^x$ and $y = -x-1$.

More points:

$x = 1 \Rightarrow y = e - 2 \doteqdot 0.718$

$x = -1 \Rightarrow y = \frac{1}{e} \doteqdot 0.368$.



(ii) From the graph we can see:

$e^x - 1 - x \geq 0$ for all real x.

$1\frac{1}{2}$ $\Rightarrow e^x \geq 1 + x$ or $1 + x \leq e^x$ for all real x.

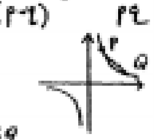
But: $1 + x = e^x$ for $x = 0$.

$\therefore 1 + x < e^x$ for all real x except $x = 0$

1 $\frac{1}{2}$

(b) $xy = c^2 \Rightarrow y = \frac{c^2}{x}$.

(i) $m = \dfrac{\frac{c}{q} - \frac{c}{p}}{cq - cp} = \dfrac{cp - cq}{pq} \times \dfrac{1}{cq - cp} = \dfrac{c(p-q)}{pq\cdot(-c)(p-q)} = -\dfrac{1}{pq}$



/2 $\therefore y - \frac{1}{cp} = -\frac{1}{pq}(x - cp)$ $\times pq$.

$pqy - cq = -x + cp$.

$\Rightarrow x + pqy = cp + cq$

$\therefore x + pqy = c(p+q)$ is the equation of chord. PQ.

(ii) P approaches Q and the chord becomes tangent i.e. $p \to q$

/1 $\therefore$ tangent at P for $p = q$ $\Rightarrow x + p^2y = c(p+p) = 2cp$.

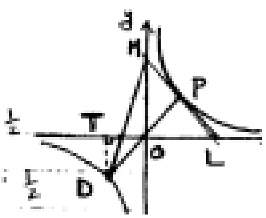
$\therefore$ tangent: $x + p^2y = 2cp$.

(iii) $D\left(-cp, -\frac{c}{p}\right)$ by symmetry of hyperbola.

/4 Tangent at P meets y axis at M $\therefore x = 0$.

$p^2y = 2cp \Rightarrow y = \frac{2cp}{p^2} = \frac{2c}{p}$ $\therefore M\left(0, \frac{2c}{p}\right)$. $\frac{1}{2}$

Gradient of MD: $m = \dfrac{\frac{2c}{p} + \frac{c}{p}}{0 + cp} = \frac{3c}{p} \times \frac{1}{cp} = \frac{3}{p^2}$. $\frac{1}{2}$



2000 FSHS 4 UNIT TRIAL Solutions

Question 8.

(a) (i) $y = \ln\left(\frac{1-\sin x}{1+\sin x}\right)^{\frac14} = \frac14 \ln\left(\frac{1-\sin x}{1+\sin x}\right) = \frac14\left[\ln(1-\sin x) - \ln(1+\sin x)\right]$

$y' = \frac{dy}{dx} = \frac14\left[\frac{-\cos x}{1-\sin x} - \frac{\cos x}{1+\sin x}\right] = \frac14\left[\frac{-\cos x(1+\sin x) - \cos x(1-\sin x)}{1-\sin^2 x}\right]$

½ $= \frac14\left[\frac{-\cos x - \cancel{\cos x \sin x} - \cos x + \cancel{\cos x \sin x}}{\cos^2 x}\right]$ as $1-\sin^2 x = \cos^2 x$.

$= -\frac{2\cos x}{4\cos^2 x} = -\frac{1}{2\cos x} = -\frac12 \sec x$.

(ii) $\frac{d}{dx}\left[\ln\sqrt[4]{\frac{1-\sin x}{1+\sin x}}\right] = -\frac12 \sec x$. (Take primitive of both sides)

1 $\therefore \int -\frac12 \sec x\,dx = \ln\sqrt[4]{\frac{1-\sin x}{1+\sin x}} + c_1$ $\quad \times -2$.

$\int \sec x\,dx = -2\ln\sqrt[4]{\frac{1-\sin x}{1+\sin x}} + C$.

$\therefore \int \sec x\,dx = -\frac24 \ln\left(\frac{1-\sin x}{1+\sin x}\right) + C = \ln\left(\frac{1-\sin x}{1+\sin x}\right)^{-\frac12} + C$.

$\therefore \int \sec x\,dx = \ln\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right) + C$.

(b) (i) Let $y = \cot^{-1}(x) \Rightarrow x = \cot y$.

1 $\therefore 1 = -\operatorname{cosec}^2 y \cdot \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -\frac{1}{\operatorname{cosec}^2 y} = -\frac{1}{1+\cot^2 y} = -\frac{1}{1+x^2}$. 1½

(ii) $f(x) = \cot^{-1}(x) + \tan^{-1}(x)$

$f'(x) = -\frac{1}{1+x^2} + \frac{1}{1+x^2} = 0$.

1: $\Rightarrow f(x)$ is a constant i.e. $f(x) = c$ as its derivative $= 0$. 1/2

To find the constant, we know:

$\tan \pi/4 = 1 \Rightarrow \tan^{-1}(1) = \pi/4$.

$\cot \pi/4 = 1 \Rightarrow \cot^{-1}(1) = \pi/4$.

$f(x) = \cot^{-1}(1) + \tan^{-1}(1) = c$.

$\frac{\pi}{4} + \frac{\pi}{4} = c \Rightarrow c = \pi/2$.

$\therefore f(x) = \tan^{-1}(x) + \cot^{-1}(x) = \pi/2$.

2000 4 UNIT FSHS Trial Solutions.

Equation of MD: $y - \frac{2c}{p} = \frac{3}{p^2}(x-0) \Rightarrow y - \frac{2c}{p} = \frac{3}{p^2}x$. ½

MD cuts the x axis at T $\therefore y=0$ $\therefore -\frac{2c}{p} = \frac{3}{p^2}x$.

2 for T $\therefore x = -\frac{2cp^2}{3p} = -\frac{2cp}{3}$ $\therefore T\left(-\frac{2cp}{3}, 0\right)$. ½

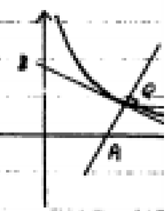
$\therefore |OT| = \frac{2cp}{3}$ units.

height of Δ ODT = y coordinate of point D = $\left|-\frac{c}{p}\right| = \frac{c}{p}$.

$\therefore$ Area of Δ ODT $= \frac12 \times |OT| \times$ height

$= \frac12 \times \frac{\cancel{2}cp}{3} \times \frac{c}{p} = \frac{c^2}{3}$ units. 2

(iv) Tangent at Q: $x + q^2 y = 2cq$.



¼ Meets y axis at B: $x=0$ $\therefore y = \frac{2cq}{q^2} = \frac{2c}{q}$ $\therefore B\left(0, \frac{2c}{q}\right)$.

$m_{\text{tangent}} = -\frac{1}{q^2}$.

$\Rightarrow m_{\text{normal}} = q^2$ at $Q\left(cq, \frac{c}{q}\right)$.

$\therefore$ Equation: $y - \frac{c}{q} = q^2(x - cq)$. $\Rightarrow q^2 x - y = c\left(q^3 - \frac1q\right)$.

cuts x axis at A $\Rightarrow y=0$. $\Rightarrow q^2 x = c\left(q^3 - \frac1q\right)$

$\Rightarrow x = \frac{c}{q^2}\left[q^3 - \frac1q\right] = cq - \frac{c}{q^3}$. $\therefore A\left[cq - \frac{c}{q^3}, 0\right]$

Let N be mid-pt. of AB:

mid-pt 2½ $X_N = \frac{cq - \frac{c}{q^3}}{2} = \frac{cq}{2} - \frac{c}{2q^3}$.

$Y_N = \frac{\frac{2c}{q} + 0}{2} = \frac{c}{q}$ $\Rightarrow q = \frac{c}{Y}$

Sub $q = \frac{c}{Y}$ into $X = \frac{cq}{2} - \frac{c}{2q^3} = \frac{c \times c}{2 \times Y} - \frac{c}{2\left[\frac{c^3}{Y^3}\right]}$.

$\therefore X = \frac{c^2}{2Y} - \frac{Y^3}{2c^2}$. $\quad \times 2Yc^2$.

Locus 1½ $2YXc^2 = c^4 - Y^4$.

$\therefore 2c^2XY + Y^4 = c^4$ is the equation of the locus. 1

2000 FSHS 4 Unit Trial Solution.

(c). $x^2y + xy^2 = 16$. .... (1)

/4 $2xy + x^2\frac{dy}{dx} + y^2 + 2xy\frac{dy}{dx} = 0$.

$(x^2 + 2xy)\frac{dy}{dx} = -(2xy + y^2)$.

$\frac{dy}{dx} = \frac{-(2xy+y^2)}{x^2+2xy} = \frac{-y(2x+y)}{x(x+2y)}$ 1

tangent // to x axis $\Rightarrow m = 0 = \frac{dy}{dx}$.

$\frac{-(2xy+y^2)}{x^2+2xy} = 0. \Rightarrow y(y+2x) = 0.$

$\therefore y = 0$ or $y = -2x$. ..... (2)

But $y = 0$ is not on the curve $\therefore y = 0$ is not a tangent.

[Since $y = 0$ is a horizontal asymptote as: $x = \frac{-y^2 \pm \sqrt{y^4+64y}}{2y} \to$ horizontal asymptote].

Sub. (2) into (1).

$-2x^3 + (-2x)^2x = 16 \Rightarrow -2x^3 + 4x^3 = 16 \Rightarrow 2x^3 = 16$.

$\Rightarrow x^3 = 8 \Rightarrow x = \sqrt[3]{8} = 2$.

Sub $x = 2$ into $x^2y + xy^2 = 16$.

$4y + 2y^2 - 16 = 0. \Rightarrow 2[y^2 + 2y - 8] = 0$.

$2[(y-2)(y+4)] = 0$.

$\Rightarrow y - 2 = 0 \Rightarrow y = 2$. or $y = -4$. 2

at $(2,2)$: $\frac{dy}{dx} = \frac{-2(4+2)}{2(2+4)} = -1. \neq 0$.

at $(2,-4)$: $\frac{dy}{dx} = \frac{4(4-4)}{2(2-8)} = 0$. 1

$\therefore (2,-4)$ is the only pt. where tangent is // to x axis.

2nd method: (smarter!)

When tangent // to x axis, its equation is $y = a$.

$\therefore x^2y + xy^2 = 16$ and $y = a. \Rightarrow ax^2 + a^2x - 16 = 0$.

is equation of $\cap$. But for tangency: $\Delta = 0$.

$\Delta = a^4 + 64a = 0. \Rightarrow a(a^3+64) = 0 \Rightarrow a = 0$ or $a^3 = -64$.

But $a = 0$ rejected as it is x axis $\Rightarrow a^3 = -64 \Rightarrow a = -4$

$\therefore y = -4$ is the required tangent.

$-4x^2 + 16x - 16 = 0 \Rightarrow -4(x^2-4x+4) = 0 \Rightarrow -4(x-2)^2 = 0$.

$\Rightarrow x = 2$ ...

2000 4 Unit FSHS Trial Solutions.

(d). $Z = \frac{z - \bar{z} + 2}{z + 1 - i} = \frac{x+iy-x+iy+2}{x+iy+1-i} = \frac{2iy+2}{(x+1)+i(y-1)}$

(i) $Z = \frac{2(iy+1)}{(x+1)+i(y-1)} \times \frac{[(x+1)-i(y-1)]}{[(x+1)-i(y-1)]} = \frac{2y(x+1)i + 2(x+1) + 2y(y-1) - 2i(y-1)}{(x+1)^2+(y-1)^2}$

$= \frac{2x+2+2y^2-2y+i[2yx+2y-2y+2]}{(x+1)^2+(y-1)^2} = \frac{2(x+1+y^2-y)+2i(xy+1)}{(x+1)^2+(y-1)^2}$.

Z is real $\Rightarrow \text{Im}(Z) = 0$. as $Z = \frac{2(x+1+y^2-y)}{(x+1)^2+(y-1)^2} + \frac{2(xy+1)i}{(x+1)^2+(y-1)^2}$

$\therefore \text{Im}(Z) = 2(xy+1) = 0 \Rightarrow xy = -1. \Rightarrow y = -\frac{1}{x}$ ... But $x \neq -1$, $y \neq 1$

$\therefore$ Locus of A is a hyperbola $y = -\frac{1}{x}$ except the point $(-1,1)$.

(ii) A moves on the x axis $\Rightarrow y = 0$ and $z = x = \bar{z}$.

$\therefore Z = \frac{2(x+1)}{(x+1)^2+1} + \frac{2i}{(x+1)^2+1} = \frac{2(x+1)}{x^2+2x+2} + \frac{2i}{(x^2+2x+2)}$.

$\therefore X = \frac{2(x+1)}{(x+1)^2+1}$. $Y = \frac{2}{(x+1)^2+1}$.

$\Rightarrow \frac{X}{Y} = \frac{2(x+1)}{2} = x + 1$. .... (1) 1

But $Y = \frac{2}{(x+1)^2+1} \Rightarrow (x+1)^2 + 1 = \frac{2}{Y} \Rightarrow (x+1)^2 = \frac{2}{Y} - 1$. ..... (2).

Square (1): $\frac{X^2}{Y^2} = (x+1)^2 = \left(\frac{2}{Y} - 1\right)$

$\therefore \frac{X^2}{Y^2} = \frac{2}{Y} - 1 \quad \times Y^2$. 1/2

$X^2 = 2Y - Y^2 \Rightarrow X^2 + Y^2 - 2Y + 1 = 1$.

$X^2 + (Y-1)^2 = 1$.

$\therefore$ Locus of M is a circle of radius 1 and centre $(0,1)$. 1