Name: ______________________________

Teacher: ____________________________

Class: ______________________________

FORT STREET HIGH SCHOOL

2006

HIGHER SCHOOL CERTIFICATE COURSE

## ASSESSMENT TASK 4: TRIAL HSC

# Mathematics Extension 2

**TIME ALLOWED: 3 HOURS**
**(PLUS 5 MINUTES READING TIME)**

| Outcomes Assessed | Questions | Marks |
|---|---|---|
| Determines the important features of graphs of a wide variety of functions, including conic sections | 4, 6 | |
| Applies appropriate algebraic techniques to complex numbers and polynomials | 2, 3 | |
| Applies further techniques of integration, such as slicing and cylindrical shells, integration by parts and recurrence formulae, to problems | 1, 5 | |
| Synthesises mathematical solutions to harder problems and communicates them in appropriate form | 7, 8 | |

| Question | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | Total | % |
|---|---|---|---|---|---|---|---|---|---|---|
| Marks | /15 | /15 | /15 | /15 | /15 | /15 | /15 | /15 | /120 | |

## Directions to candidates:

- Attempt all questions
- The marks allocated for each question are indicated
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board – approved calculators may be used
- Each new question is to be started on a new page

# STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \, \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

## Question 1: (15 marks)

a) Find $\int \frac{dx}{x^2 - 4x + 9}$ **2**

b)

i. Express $\frac{4x-2}{(x^2-1)(x-2)}$ in the form $\frac{Ax+B}{x^2-1} + \frac{C}{x-2}$,

where $A$, $B$ and $C$ are constants. **3**

ii. Hence evaluate

$$\int_3^6 \frac{4x-2}{(x^2-1)(x-2)}dx$$ **2**

c) Find $\int \frac{e^{2x}}{e^x - 1}\, dx$ by using the substitution $u = e^x$. **3**

d) Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx$, where $n$ is a non-negative integer.

i. Show that $I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x dx$, where $n \geq 2$. **2**

ii. Deduce that $I_n = \left(\frac{n-1}{n}\right) I_{n-2}$ when $n \geq 2$. **2**

iii. Evaluate $I_4$ **1**

## Question 2: (15 marks)

a) Let $z=\sqrt{3}+i$

i. Express $z$ in modulus/argument form. **1**

ii. Show that $z^7+64z=0$ **2**

b) Find the complex number $z=a+ib$, where $a$ and $b$ are real, such that

$$\operatorname{Im}(z)+\bar{z}=\frac{1}{1+i}$$ **3**

c) The complex number $z$ satisfies the condition $|z-8|=2\operatorname{Re}(z-2)$

i. Sketch the locus defined by this equation on an Argand diagram, showing any important features of the curve. State the type of curve and write down its equation. **3**

ii. Write down the value of $|z+8|-|z-8|$ **1**

iii. Find the possible values of $\arg z$ **2**

d) $P,Q$ represent complex numbers $\alpha,\beta$ respectively in an Argand diagram, where $O$ is the origin and $O,P$ and $Q$ are not collinear. In $\Delta OPQ$, the median from $O$ to the midpoint $M$ of $PQ$ meets the median from $Q$ to the midpoint $N$ of $OP$ in the point $R$, where $R$ represents the complex number $z$.

i. Show this information on a sketch **1**

ii. Explain why there are positive real numbers $k,L$ so that

$$kz=\frac{1}{2}(\alpha+\beta) \text{ and } L(z-\beta)=\frac{1}{2}\alpha-\beta$$ **2**

## Question 3: (15 marks)

a) The equation $x^3 + bx^2 + x + 2 = 0$ where $b$ is a real number has roots $\alpha, \beta, \gamma$

i. Obtain an expression in terms of $b$ for $\alpha^2 + \beta^2 + \gamma^2$ **2**

ii. Hence determine the set of possible values of $b$ if the roots of the above equation are real. **1**

iii. Write down the equation whose roots are $2\alpha,\ 2\beta,\ 2\gamma$. **2**

b)

i. Show that if $a$ is a multiple root of the polynomial equation $f(x) = 0$ then $f(a) = f'(a) = 0$. **2**

ii. The polynomial $\alpha x^{n+1} + \beta x^n + 1$ is divisible by $(x-1)^2$.

Show that $\alpha = n$ and $\beta = -(1+n)$. **4**

iii. Prove that $1 + x + \frac{x^2}{2!} + \cdots\cdots + \frac{x^n}{n!}$ has no multiple roots for $n \geq 1$. **4**

## Question 4: (15 marks)

The functions $S(x)$ and $C(x)$ are defined by the formulae

$$S(x) = \frac{1}{2}\left(e^{x} - e^{-x}\right) \text{ and } C(x) = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

a)

i. Verify that $S'(x) = C(x)$ **1**

ii. Show that $S(x)$ is an increasing function for all real $x$. **2**

iii. Prove that $\{C(x)\}^2 = 1 + \{S(x)\}^2$. **2**

b)

i. $S(x)$ has an inverse function, $S^{-1}(x)$ for all values of $x$. Briefly justify this statement **2**

ii. Let $y = S^{-1}(x)$. Prove that $\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$ **4**

iii. Hence, or otherwise, show that

$$S^{-1}(x) = \ln\left\{x + \sqrt{1+x^2}\right\}$$ **1**

iv. Show that $\int_0^1 \frac{dx}{\sqrt{x^2+2x+2}} = \ln\left\{\frac{2+\sqrt{5}}{1+\sqrt{2}}\right\}$ **3**

## Question 5: (15 marks)

a) Using the method of cylindrical shells find the volume of the solid formed when the region bounded by the curve $y = x^2 + 1$ and the $x$-axis between $x = 0$ and $x = 2$ is rotated about the $y$ axis. **4**

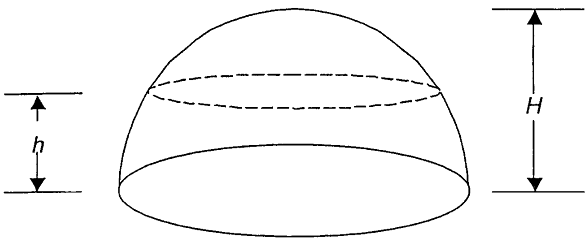
b)

i. Using the substitution $x = a\sin\theta$, or otherwise, verify that

$$\int_0^a \left(a^2 - x^2\right)^{\frac{1}{2}} dx = \frac{1}{4}\pi a^2$$ **4**

ii. Deduce that the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\pi ab$. **3**

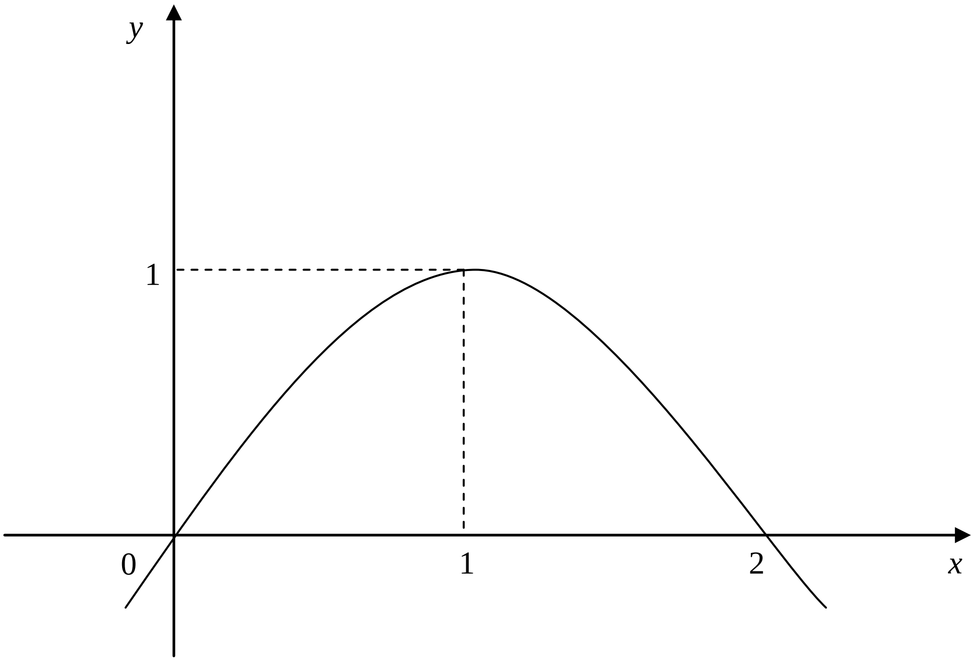
iii. The diagram below shows a mound of height $H$. At height $h$ above the horizontal base, the horizontal cross-section of the mound is elliptical in shape with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \lambda^2$ where $\lambda = 1 - \frac{h^2}{H^2}$ and $x$ and $y$ are appropriate coordinates in the plane of the cross-section.




Show that the volume of the mound is $\frac{8\pi abH}{15}$. **4**

## Question 6: (15 marks)

a) The graph below shows the curve $y = f(x)$ where $f(x) = x(2-x)$.




Without the use of calculus, sketch the following curves. Show any intercepts, asymptotes, end points and turning points.

i. $y = f(2x)$ **1**

ii. $y = \dfrac{1}{f(x)}$ **2**

iii. $|y| = f(x)$ **2**

iv. $y = \ln f(x)$ **2**

v. $y = f(e^x)$ **2**

(continued over)

b) Consider the function $f(x) = |1+x| + |1-x|$.

i. Show that $f(x)$ is an even function. **1**

ii. Sketch the graph of $y = f(x)$ clearly showing essential features. **2**

iii. Use the graph to find the set of values of the real number $k$ for which $f(x) = k$ has exactly 2 real solutions. **1**

c) On separate diagrams sketch the graphs of the following curves, showing the equations of any asymptotes

i. $y = \left(\tan^{-1} x\right)^2$ **1**

ii. $y^2 = \tan^{-1} x$ **1**

## Question 7: (15 marks)

a) The ellipse $E$ has equation $\frac{x^2}{100}+\frac{y^2}{75}=1$

i. Sketch the curve $E$, showing on your diagram the coordinatesof the foci and the equation of each directrix. **2**

ii. Find the equation of the normal to the ellipse at the point $P(5,7.5)$. **2**

iii. Find the equation of the circle that is tangential to the ellipse at $P$ and $Q\ (5,-7.5)$ **3**

b) The hyperbola $H$ has equation $xy=4$

i. Sketch $H$ and indicate on your diagram the positions and coordinates of all points at which $H$ intersects the axes of symmetry. **1**

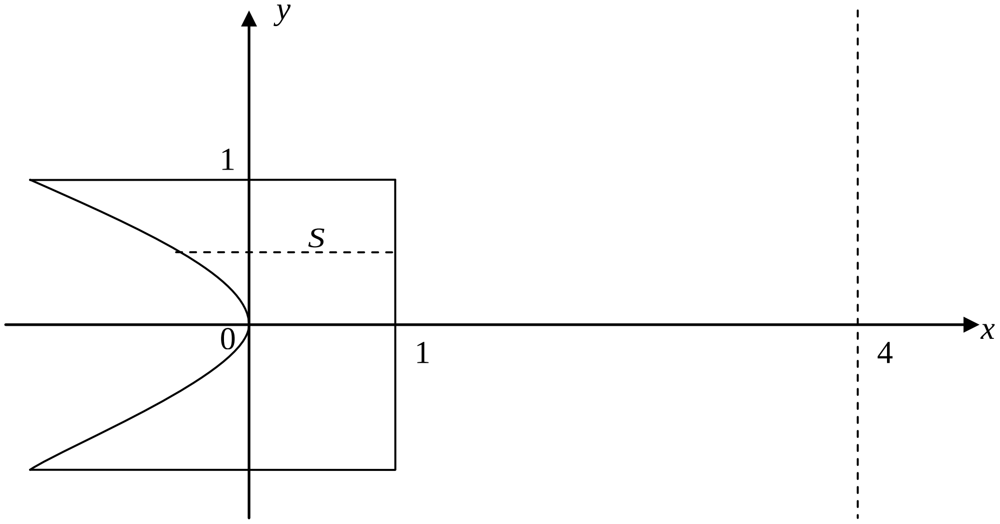
ii. Show that the equation of the tangent to $H$ at $P\left(2t,\frac{2}{t}\right)$ where $t\neq 0$, is $x+t^2y=4t$. **2**

iii. If $s\neq 0$ and $s^2\neq t^2$, show that the tangents to $H$ at $P$ and $Q\left(2s,\frac{2}{s}\right)$ intersect at $M\left(\frac{4st}{s+t},\frac{4}{s+t}\right)$ **2**

iv. Suppose that in (iii) the parameter $s=-\frac{1}{t}$. Show that the locus of M is a straight line through, but excluding the origin. **3**

## Question 8: (15 marks)

a)




The region bounded by the lines $x=1$, $y=1$, and $y=-1$ and by the curve $x+y^2=0$ is rotated through $360^\circ$ about the line $x=4$ to form a solid. As the region is rotated, the line segment $S$ sweeps out an annulus.

i. Show that the area of the annulus swept by $S$ at height $y$ is equal to $\pi\left(y^4+8y^2+7\right)$ **3**

ii. Hence find the volume of the solid. **2**

b) The point $P(x_1, y_1)$ lies on the hyperbola of equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

i. Find the equation of the normal at $P$. **3**

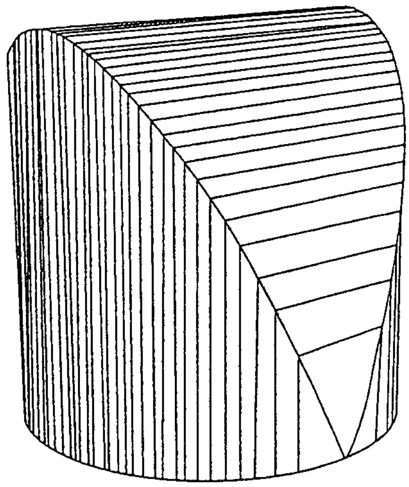
ii. The normal at $P$ meets the $x$ axis at $G$, and $N$ is the foot of the perpendicular from $P$ to the $x$ axis. Show that

$$NG:ON=b^2:a^2$$

where $O$ is the origin. **2**

c) Trieu took a wooden cylinder and carved it into the shape shown in the diagram below. The base of her shape is a circle with radius 8 cm.

Each vertical cross-section shown in the diagram is a square.



i. Show that the area $A$ of the cross-section distance $x$ cm from the centre of the base is $A = 4\left(64 - x^2\right)$. **3**

ii. Hence show that the volume $V$ of the art project is given by

$$V = 8\int_0^8 \left(64 - x^2\right)dx$$

and evaluate the integral. **2**

END OF EXAMINATION

# 2006 Ext 2 Trial Solutions

## Question 1

(a) $\int \frac{dx}{x^2-4x+9} = \int \frac{dx}{x^2-4x+4+5}$

$= \int \frac{dx}{(x-2)^2+5}$ ①

$= \frac{1}{\sqrt{5}}\tan^{-1}\frac{x-2}{\sqrt{5}} + c$ ①

b)(i) $\frac{4x-2}{(x^2-1)(x-2)} \equiv \frac{Ax+B}{x^2-1} + \frac{c}{x-2}$

$4x-2 = (x-2)(Ax+B) + (x^2-1)C$

$= Ax^2+Bx-2Ax-2B+Cx^2-C$ ①

$= (A+C)x^2+(B-2A)x-(2B+C)$

$\therefore A+C=0 \Rightarrow A=-C$

$B-2A=4$ — ①

$2B+C=2$ ①

$\therefore 2B-A=2$ — ②

②×2 $4B-2A=4$ — ③

①−③ $-3B=0 \Rightarrow B=0$

$A=-2, \quad C=2$ ①

Hence $\frac{4x-2}{(x^2-1)(x-2)} = \frac{-2x}{x^2-1} + \frac{2}{x-2}$

(ii) $\int_3^6 \frac{4x-2}{(x^2-1)(x-2)}dx = \int_3^6 \frac{-2x}{x^2-1} + \frac{2}{x-2}\,dx$

$= \left[-\ln(x^2-1) + 2\ln(x-2)\right]_3^6$

$= \left[\ln\left\{\frac{(x-2)^2}{(x^2-1)}\right\}\right]_3^6$ ①

$= \ln\frac{16}{35} - \ln\frac{1}{8}$

$= \ln\frac{128}{35}$ ①

(c) $\int \frac{e^{2x}}{e^x-1}dx \qquad u=e^x$

$du = e^x dx$

$= \int \frac{u}{u-1}du$ ①

$= \int \frac{u-1}{u-1} + \frac{1}{u-1}\,du$

$= \int 1 + \frac{1}{u-1}\,du$ ①

$= u + \ln(u-1) + c$

$= e^x + \ln(e^x-1) + c$ ①

(d)(i) $I_n = \int_0^{\frac{\pi}{2}} \sin^{n-1}x\,\sin x\,dx$

$= \left[-\cos x\sin^{n-1}x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} -\cos x(n-1)\sin^{n-2}x\cos x\,dx$

$= 0 + (n-1)\int_0^{\frac{\pi}{2}}\cos^2 x\sin^{n-2}x\,dx$

$\therefore I_n = (n-1)\int_0^{\frac{\pi}{2}}\cos^2 x\sin^{n-2}x\,dx$ ①

(ii) $I_n = (n-1)\int_0^{\frac{\pi}{2}}(1-\sin^2 x)\sin^{n-2}x\,dx$

$= (n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x - \sin^n x\,dx$

$\therefore I_n = (n-1)(I_{n-2} - I_n)$ ①

$= (n-1)I_{n-2} - (n-1)I_n \quad (n\geq 2)$

$nI_n = (n-1)I_{n-2}$

$I_n = \frac{n-1}{n}I_{n-2}$ ①

(iii) $I_4 = \frac{4-1}{4}I_2$

$= \frac{3}{4}\cdot\frac{1}{2}I_0$

$= \frac{3}{8}\int_0^{\frac{\pi}{2}}dx$

$= \frac{3\pi}{16}$ ①

## Question 2

a) (i) $\sqrt{3}+i = 2\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$

$= 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ ①

(ii) $z^7 + 64z = 2^7\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) + 64.\,2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$ ①

$= 2^7\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right) + 2^7\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$

$= 0$ ①

b) $\text{Im}(z) + \bar{z} = b + a - ib$

$= (a+b) - ib$

$(a+b) - ib = \frac{1}{1+i} \times \frac{1-i}{1-i}$ ①

$= \frac{1-i}{2}$

①

$\therefore\ a+b = \frac{1}{2},\quad b = \frac{1}{2} \Rightarrow a = 0$

Hence $z = \frac{1}{2}i$ ①

c) $|z-8| = 2\,\text{Re}(z-2)$

Distance from $z$ to $S(8,0)$ is twice distance from $z$ to line $L$ with equation $x=2$ and $\text{Re}\,z \geq 2$.

Locus is right hand branch of the hyperbola with focus $S$, directrix $L$ and eccentricity $e=2$.

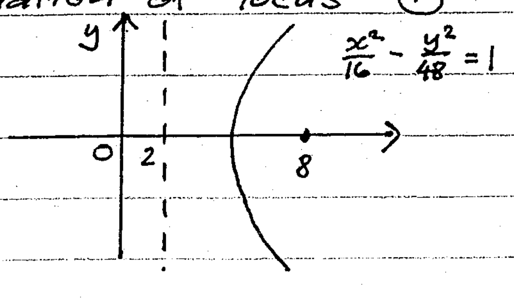
$L: x = 2 = \frac{4}{2}\qquad S: (8,0) = (4e, 0)$

Hence hyperbola is centred on the origin with $a=4$, $b^2 = 4^2(2^2-1) = 48$ ①

and equation $\frac{x^2}{16} - \frac{y^2}{48} = 1$

name & equation of locus ①

sketch ①




(ii) If $P$ represents $z$, then

$|z+8| - |z-8| = PS' - PS$

$= 2\,(PM' - PM)$

$= 8$ ①

where $S'(-8,0)$ is the second focus of the hyperbola

$L': x=-2$ is the second directrix and $M'$, $M$ are the feet of the perpendiculars from $P$ to $L'$, $L$ resp.

(iii) $\frac{b}{a} = \frac{4\sqrt{3}}{4} = \sqrt{3}$

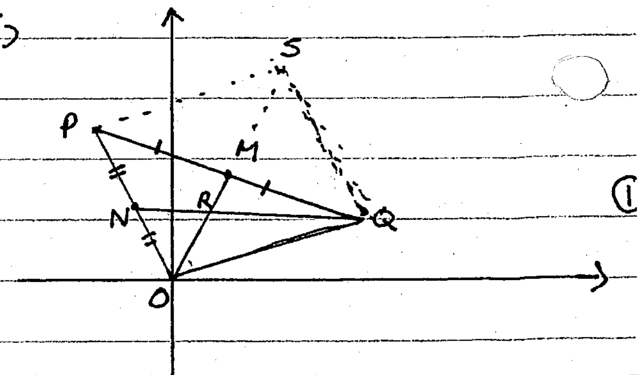
Hence the asymptotes of the hyperbola have equations

$y = \pm\sqrt{3}\,x$ ①

The asymptotes each make angle $\frac{\pi}{3}$ with the $x$ axis

Hence $-\frac{\pi}{3} < \arg z < \frac{\pi}{3}$ ①

(d) (i)




①

(ii) By completing the parallelogram POQS, $\overrightarrow{OM} = \frac{1}{2}(\alpha+\beta)$ (diagonals bisec-

Also $\overrightarrow{OM} = k\,\overrightarrow{OR} = kz$ (k const

$\therefore\ kz = \frac{1}{2}(\alpha+\beta)$ ①

$\overrightarrow{QN} = \frac{1}{2}\alpha - \beta$

$\overrightarrow{QR} = z - \beta$ (L const

$\overrightarrow{QN} = L(z-\beta) = \frac{1}{2}\alpha - \beta$ (

$\therefore\ kz = \frac{1}{2}(\alpha+\beta)$ and $L(z-\beta) = \frac{1}{2}\alpha -$

Question 3

(a) $x^3 + bx^2 + x + 2 = 0$

(i) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$\alpha+\beta+\gamma = -b$

$\alpha\beta + \alpha\gamma + \beta\gamma = 1$ ①

$\therefore \alpha^2+\beta^2+\gamma^2 = (-b)^2 - 2$

$= b^2 - 2$ ①

(ii) For real roots, $\alpha^2+\beta^2+\gamma^2 \geq 0$

$b^2 - 2 \geq 0$

i.e., $b \leq -\sqrt{2}$, $b \geq \sqrt{2}$ ①

(iii) $\left(\frac{x}{2}\right)^3 + b\left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right) + 2 = 0$ ①

$\frac{x^3}{8} + \frac{bx^2}{4} + \frac{x}{2} + 2 = 0$

① $\therefore x^3 + 2bx^2 + 4x + 16 = 0$ has roots $2\alpha, 2\beta, 2\gamma$

①

b) (i) Let $f(x) = (x-a)^r Q(x)$, $Q(a) \neq 0$

$f'(x) = r(x-a)^{r-1}Q(x) + (x-a)^r Q'(x)$

$= (x-a)^{r-1} Q_1(x)$ ①

$\therefore f(a) = f'(a) = 0$

(ii) Let $f(x) = \alpha x^{n+1} + \beta x^n + 1$

then $f'(x) = \alpha(n+1)x^n + \beta n x^{n-1}$ ①

$f(1) = \alpha + \beta + 1 = 0$ — ①

$f'(1) = \alpha(n+1) + \beta n = 0$ — ② ①

$= \alpha n + \alpha + \beta n = 0$

$= (\alpha+\beta)n + \alpha = 0$

From ① $\alpha + \beta = -1$ ①

$\therefore -n + \alpha = 0 \Rightarrow \alpha = +n$

$\beta = -1 - n$

$= -(1+n)$ ①

$\therefore \alpha = n$ and $\beta = -(1+n)$

(iii) Let $P(x) = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$

$P'(x) = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^{n-1}}{(n-1)!}$

$\therefore P(x) - P'(x) = \frac{x^n}{n!}$ ①

Let $\alpha$ be a multiple zero of $P(x)$. Then $P(\alpha) = P'(\alpha) = 0$, ①
and $P(\alpha) - P'(\alpha) = 0 \Rightarrow \alpha = 0$.
But $P(0) \neq 0$. ①
Hence $P(x)$ has no multiple zero.

## Question 4

(a) (i) $S(x) = \frac{1}{2}(e^{x} - e^{-x})$

$S'(x) = \frac{1}{2}(e^{x} - (-e^{-x}))$

$= \frac{1}{2}(e^{x} + e^{-x})$

$= C(x)$ (1)

(ii) $S'(x) = \frac{1}{2}(e^{x} + e^{-x})$

$e^{x} > 0$ for all $x$ (1)

$e^{-x} > 0$ for all $x$

$\therefore S'(x) > 0$ for all $x$

Hence $S(x)$ is an increasing function for all real $x$ (1)

(iii) $\{C(x)\}^2 = \frac{1}{4}(e^{2x} + 2 + e^{-2x})$ (1)

$1 + \{S(x)\}^2 = 1 + \frac{1}{4}(e^{2x} - 2 + e^{-2x})$

$= \frac{1}{4}(e^{2x} + 2 + e^{-2x})$ (1)

$\therefore \{C(x)\}^2 = 1 + \{S(x)\}^2$

(b) (i) $S(x)$ is a monotonic increasing function (1)

i.e., it is a one-one function. (1)

$\therefore S(x)$ has an inverse function, $S^{-1}(x)$ for all $x$

(ii) $S(x) = \frac{1}{2}(e^{x} - e^{-x})$

$S^{-1}(x)$: $x = \frac{1}{2}(e^{y} - e^{-y})$ (1)

$\frac{dx}{dy} = \frac{1}{2}(e^{y} + e^{-y})$ (1)

$= \sqrt{1 + x^2}$ from (a) (iii) (1)

$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$ (1)

(iii) $\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$

$y = \ln\left[x + \sqrt{1+x^2}\right]$ using table of standard integrals

i.e., $S^{-1}(x) = \ln\left[x + \sqrt{1+x^2}\right]$

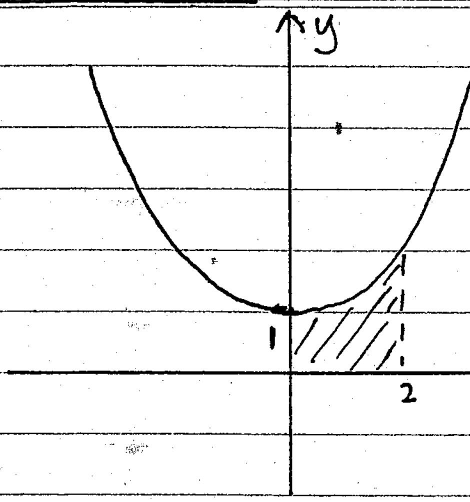
(iv) $\int_0^1 \frac{dx}{\sqrt{x^2+2x+2}} = \left[\ln\{(x+1) + \sqrt{1+(x+1)^2}\}\right.$ (1)

$= \ln\{2+\sqrt{5}\} - \ln\{1+\sqrt{2}\}$ (1)

$= \ln\left\{\frac{2+\sqrt{5}}{1+\sqrt{2}}\right\}$ (1)

## Question 5

a)




$y = x^2+1$

$r_1 = x+\delta x$ ①

$r_2 = x$

$h = x^2+1$

$\delta V = \pi(x^2+1)\,2x\,\delta x$ ①

$V = 2\pi \int_0^2 x^3 + x \quad dx$

$= 2\pi \left[\frac{x^4}{4} + \frac{x^2}{2}\right]_0^2$ ①

$= 2\pi\,[4 + 2 - 0]$

$\therefore$ Volume $= 12\pi$ units$^3$ ①

b) (i) $x = a\sin\theta$

$\frac{dx}{d\theta} = a\cos\theta$

$a = a\sin\theta, \quad \theta = \frac{\pi}{2}$ ①

$0 = a\sin\theta, \quad \theta = 0$

$\therefore \int_0^a (a^2-x^2)^{\frac{1}{2}}\,dx$

$= \int_0^{\frac{\pi}{2}} (a^2 - a^2\sin^2\theta)^{\frac{1}{2}}\, a\cos\theta\, d\theta$ ①

$= a^2 \int_0^{\frac{\pi}{2}} \cos^2\theta \quad d\theta$

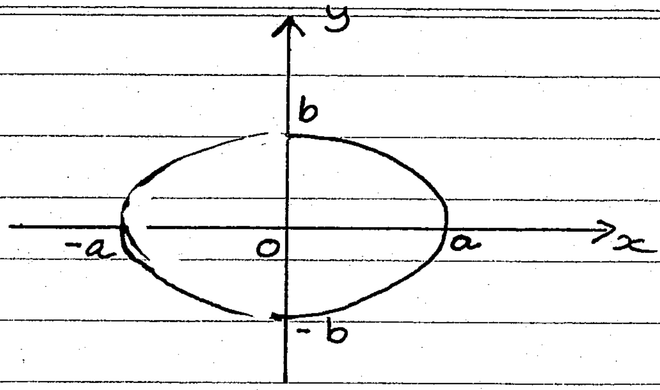
$= \frac{a^2}{2} \int_0^{\frac{\pi}{2}} \cos 2\theta + 1 \quad d\theta$ ①

$= \frac{a^2}{2}\left[\frac{\sin 2\theta}{2} + \theta\right]_0^{\frac{\pi}{2}}$

$= \frac{a^2}{2}\left[\frac{\pi}{2} - 0\right]$

$= \frac{\pi a^2}{4}$ ①

(ii)




$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$

$y = \frac{b}{a}(a^2-x^2)^{\frac{1}{2}}$

$A = 4\int_0^a \frac{b}{a}(a^2-x^2)^{\frac{1}{2}}\,dx$ ①

$= 4 \cdot \frac{b}{a} \times \frac{\pi a^2}{4}$

$= \pi ab$ ①

(iii)

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \lambda^2$

$\frac{x^2}{\lambda^2 a^2} + \frac{y^2}{\lambda^2 b^2} = 1$

A ellipse $= \pi.\lambda a.\lambda b$ from (i

$= \pi\lambda^2 ab$ (

$= \pi ab\left(1 - \frac{h^2}{H^2}\right)^2$

$\therefore \quad V = \pi ab \int_0^H 1 - \frac{2h^2}{H^2} + \frac{h^4}{H^4}$

$= \pi ab\left[h - \frac{2h^3}{3H^2} + \frac{h^5}{5H^4}\right]_0^H$

$= \pi ab \times \frac{8H}{15}$

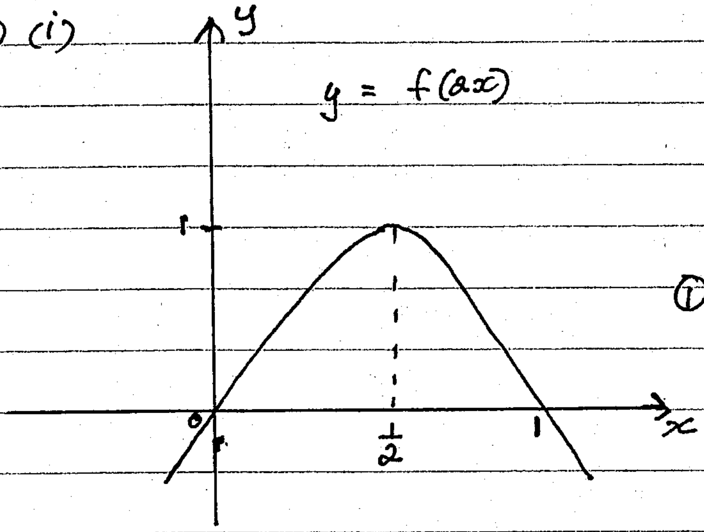
$= \frac{8\pi abH}{15}$ (i

$H - \frac{2H}{3} + \frac{H}{5}$

$\frac{15H - 10H + 3H}{15}$

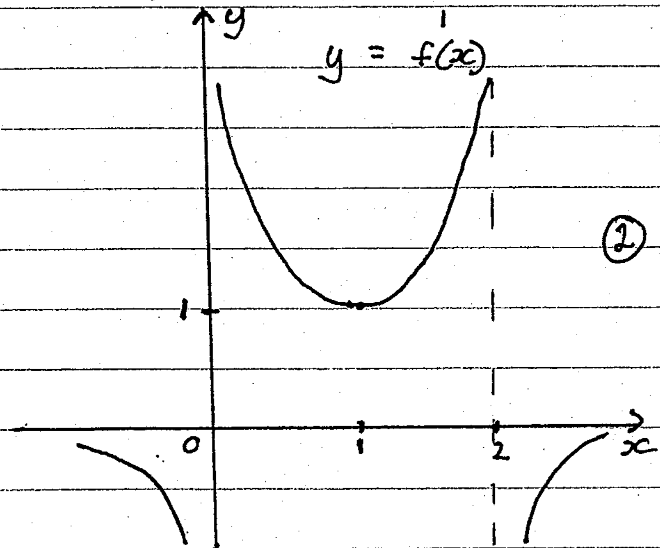
# Question 6

(a) (i)



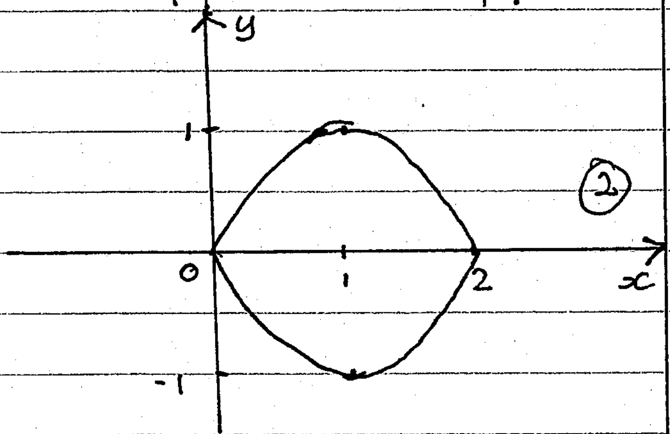

①

(ii)



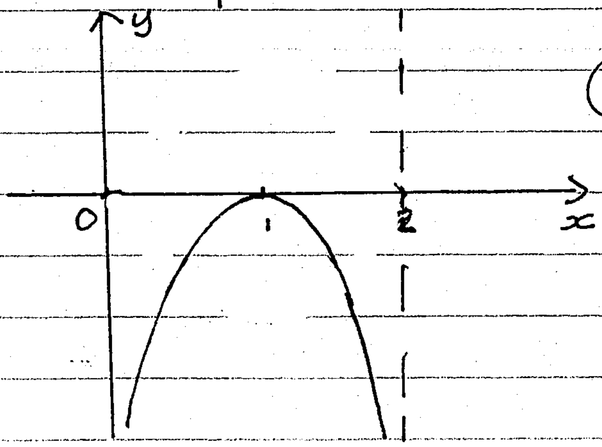

②

iii



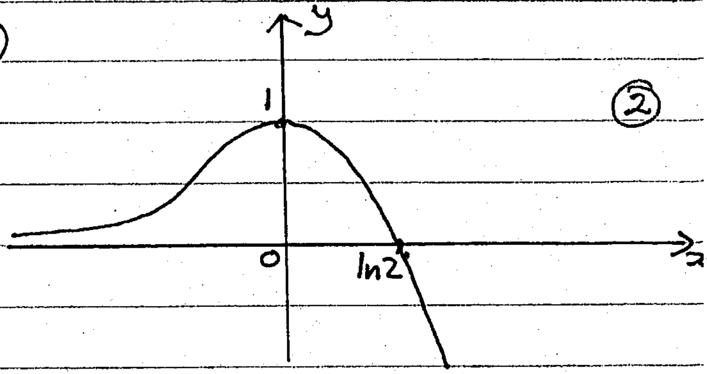

②

(iv)




②

(v)




②

(b) (i) $f(x) = |1+x| + |1-x|$

$f(-x) = |1+(-x)| + |1-(-x)|$

$= |1-x| + |1+x|$

$= f(x)$ ①

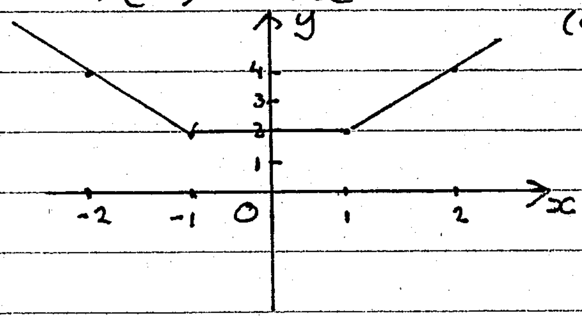
$\therefore f(x)$ is even. ○

(ii) $f(x) = 1+x+1-x = 2 \quad -1 < x < 1$

$f(x) = 2x \quad x > 1$

$f(x) = -2x \quad x < -1$

(even f…

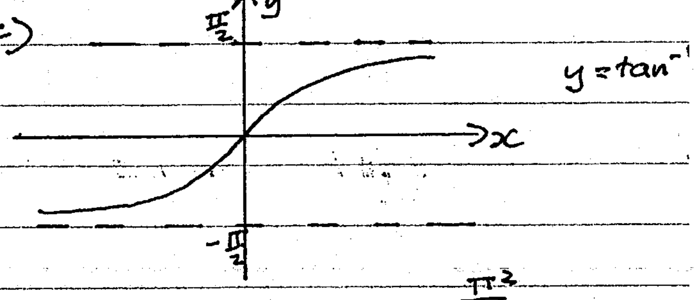


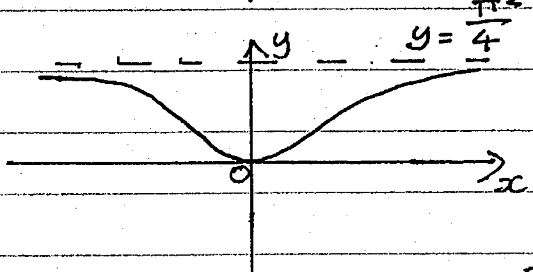

②

(iii) $f(x) > 2$ for 2 solns.

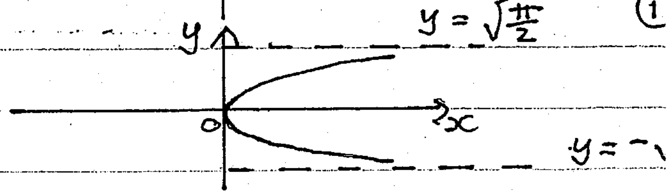
i.e., $k > 2$ ○

(c)




①




①

## Question 7

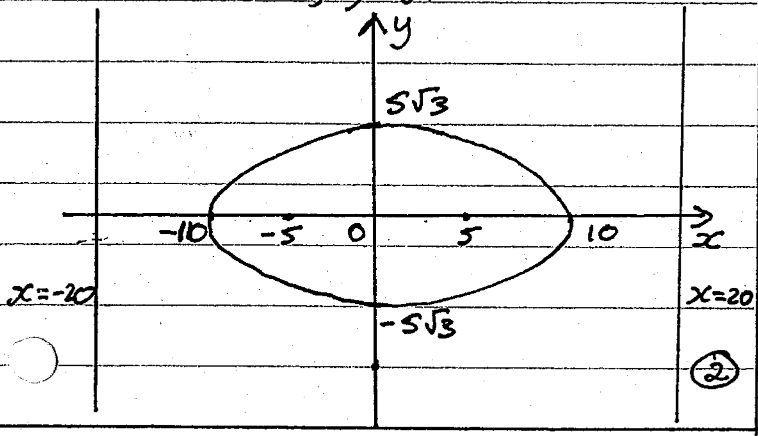
(a) (i) $\frac{x^2}{100} + \frac{y^2}{75} = 1$

$a = 10, \quad b = 5\sqrt{3}, \qquad b^2 = a^2(1-e^2)$

$e^2 = 1 - \frac{75}{100}$

$e = \frac{1}{2}$

foci: $(\pm 5, 0)$ directrices: $x = \pm 20$




(2)

(ii) $\frac{2x}{100} + \frac{2y}{75}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{3x}{4y}$ (1)

$= -\frac{1}{2}$ at $(5, 7.5)$

$\therefore$ gradient normal $= 2$

$y - 7.5 = 2(x-5)$

(any form) $\therefore y = 2x - 2.5$ (1)

is the equation of the normal.

(iii) Centre: pt intersection of normals at P & Q.

P: $y = 2x - 2.5$

Q: $y = -2x + 2.5$

$2y = 0 \Rightarrow y = 0$

$x = 1.25$

Centre $(1.25, 0)$ (1)

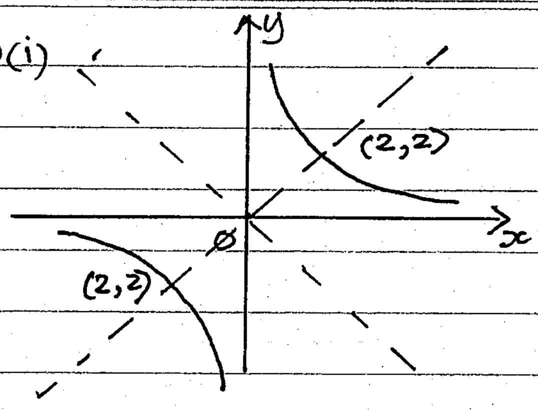
Radius: distance from centre to P

$r^2 = (5-1.25)^2 + (7.5-0)^2$

$= \frac{1125}{16}$ (1)

Circle: $\left(x - \frac{5}{4}\right)^2 + y^2 = \frac{1125}{16}$ (1)

(b) (i)




(1)

(ii) $x = 2t \qquad y = 2t^{-1}$

$\frac{dx}{dt} = 2 \qquad \frac{dy}{dt} = -2t^{-2} = -\frac{2}{t^2}$

$\frac{dy}{dx} = -\frac{2}{t^2} \times \frac{1}{2} = -\frac{1}{t^2}$ (1)

$y - \frac{2}{t} = -\frac{1}{t^2}(x - 2t)$

$t^2 y - 2t = -x + 2t$ (1)

$x + t^2 y = 4t$

(iii) $x + t^2 y = 4t$ — (1)

$x + s^2 y = 4s$ — (2)

(1) − (2) $= (t^2 - s^2)y = 4(t-s)$

$y = \frac{4(t-s)}{(t-s)(t+s)}$

$= \frac{4}{s+t} \quad (t \neq s)$

From (1) $x = 4t - t^2\left(\frac{4}{s+t}\right)$

$= \frac{4t(s+t) - 4t^2}{s+t}$

$= \frac{4st}{s+t}$

$\therefore M = \left(\frac{4st}{s+t}, \frac{4}{s+t}\right)$

(iv) $s = -\frac{1}{t} \Rightarrow x = \frac{-4}{t - \frac{1}{t}} = \frac{-4t}{t^2-1}$ (

$t \neq 0 \qquad y = \frac{4}{-\frac{1}{t}+t} = \frac{4t}{t^2-1}$

(1)

$\therefore y = -x$, $x \neq 0$ as $t \neq 0$

(straight line)

i.e., locus of M is the equation $y = -x$ excluding the origin. (1

## Question 8

a) (i) $x + y^2 = 0$ $\quad r_1 = -x + 4$

$r_2 = 3$ ①

$$A.\ \text{annulus} = \pi[(4-x)^2 - 9]$$
$$= \pi[16 - 8x + x^2 - 9]$$
$$= \pi[7 - 8x + x^2] \quad ①$$
$$= \pi(7 + 8y^2 + y^4)$$
$$= \pi(y^4 + 8y^2 + 7) \quad ①$$

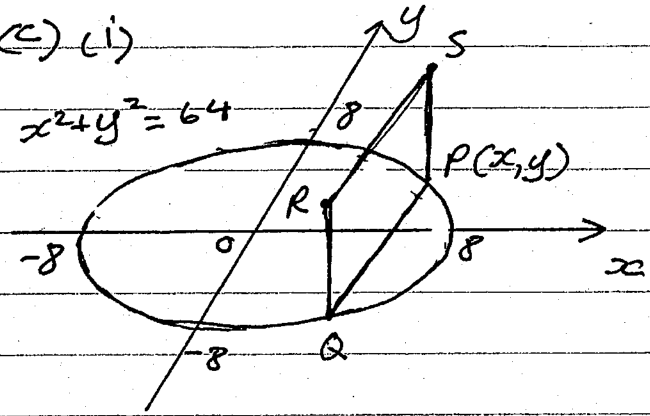
(ii) $$V = \int_{-1}^{1} y^4 + 8y^2 + 7 \; dy$$
$$= 2\pi\left[\frac{y^5}{5} + \frac{8y^3}{3} + 7y\right]_0^1 \quad ①$$
$$= 2\pi\left(\frac{1}{5} + \frac{8}{3} + 7 - 0\right)$$
$\therefore$ Volume $= 19\frac{11}{15}\pi$ units$^3$ ①

b) (i) $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{b^2x}{a^2y}$$
$\therefore$ gradient normal $= \dfrac{-a^2y_1}{b^2x_1}$ ①

at $(x_1, y_1)$
$$y - y_1 = \frac{-a^2y_1}{b^2x_1}(x - x_1) \quad ①$$
$$b^2x_1y - b^2x_1y_1 = -a^2xy_1 + a^2x_1y_1$$
$$a^2y_1x + b^2x_1y = x_1y_1(a^2+b^2)$$
$$\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2 \quad ①$$

(ii) let $y = 0$, $\quad x = \dfrac{x_1(a^2+b^2)}{a^2}$
$$NG = \frac{x_1(a^2+b^2)}{a^2} - x_1$$
$$= \frac{x_1b^2}{a^2} \quad ①$$
$$ON = x_1 \quad ①$$
$$NG : ON = \frac{x_1b^2}{a^2} : x_1 = b^2 : a^2$$

c) (i)



①

$A$ square $PQRS = (2y)^2$
$= 4y^2$
$= 4(64 - x^2)$ ①

(ii) $$V = \int_{-8}^{8} 4(64 - x^2)\, dx$$
$$= 8\int_0^8 (64 - x^2)$$
$$= 8\left[64x - \frac{x^3}{3}\right]_0^8 \quad ①$$
$$= 8\left(512 - \frac{512}{3}\right)$$
$\therefore$ Volume $= 2730\frac{2}{3}$ cm$^3$ ①

Mostly Well done

Reduction formula Some people tried pulling out $\sin^2 x$ instead of $\sin x$

Some Integrals not proplrly formed.

HAYES

2.

(a) i) & ii) very well done

(b) very well done

(c) i) well done, most students established the eqn of hyperbola using co-ord geometry techniques

ii) poorly done many failed to realise

$|2+8| - |2-8| = PS' - PS$
$= 2(PM' - PM)$
$= 2MM'$
$= 8$

iii) Well done → many did not equal ∧ segments

(d) i) well done ii) some success

iii) poorly done

FRASER

(a) (i) generally well executed

(ii) many students did not make the connection that $b^2 - 2 \geq 0$ for real roots.
As the question was only worth 1 mark, $b < -\sqrt{2}$, $b > \sqrt{2}$ was not penalised.

(iii) some minor errors here.

b) (i) well done

(ii) " "

(iii) some students did not take enough care with the last part of this proof and were penalised accordingly.

BEEVERS

4.

Not Well done

$S''(x) > 0$ is not an increasing function condition.

$y = \frac{1}{2}(e^x - e^{-x})$

$\therefore x = \frac{1}{2}(e^y - e^{-y})$

$2x = e^y - e^{-y} = e^y - \frac{1}{e^y}$

$\therefore e^{2y} - 2xe^y - 1 = 0$

$\therefore e^y = \frac{2x \pm \sqrt{4x^2+4}}{2}$

$\therefore y = \ln(x + \sqrt{x^2+1})$ as $e^y > 0$

or

$y = \int \frac{1}{\sqrt{1+x^2}}\,dx$

$= \ln(x + \sqrt{x^2+1})$

Standard integrals.

HAYES

a) very well done, some students got muddled setting up integral

b i) routine work well done

ii) some students quoted formula instead of using calculus

iii) Students either knew how to proceed or did not get started.

FRASER

6(a) Areas of concern parts (iii) & (v) mainly. In (v) some used $e^{f(x)}$ instead of $f(e^x)$.

b(i) well done

(ii) could be done algebraically or by adding the ordinates of the two graphs. In some cases errors were made for either method.

(iii) more errors here than expected

(c) question specifically mentioned equations of asymptotes - which was poorly done. 1 mark was awarded for correct shape for both parts or correct asymptotes for both parts.

BEEVERS

Fairly well done

Using the normals at P + Q to find the centre of the circle in a part (iii) confused many.

The explanation of why the locus of M excluded the origin was messy. It was based on the fact (which was given) that $t \neq 0$ and if you write x and y in terms of t

$$x = \frac{-4t}{t^2-1} \text{ and } y = \frac{4t}{t^2-1}$$

(you can also easily see $y = -x$ is locus)

then if $t \neq 0$ $x \neq 0$, $y \neq 0$

HAYES

8. a) i) Some students wrote $r_1 = x + 4$ rather than $-x + 4$.

ii) well done

b) i) often carelessness in determining gradient of normal.

iii) This provided difficulty for some students.

c) i) & ii) very well done

FRASER