Question 1 (15 Marks) **Marks**

a) Find the modulus and argument of the complex numbers w and z **4**

Where $z = \dfrac{1+i}{1-i}$ and $w = \dfrac{\sqrt{2}}{1-i}$

b) Plot the points $z,\ w,\ z+w$ from part a) on an accurate **3**

Argand diagram and hence find the exact value of $\tan(\frac{3\pi}{8})$.

c) The vertices of a square taken anticlockwise are P, Q, R and S. **2**
If the points P and Q are represented by the complex numbers
$z_P = -1 + 4i$ and $z_Q = -3$
Find the other corners of the square R and S and its centre
in the form a+ib .

d) Determine the greatest and least values of arg z, **2**
when $|z - 8i - 5| = 6$, answer to the nearest minute.

e) In the Argand plane **4**

(i) shade: $|z+3| + |z-3| \le 10$ *and* $3 \le |z - 3 + 2i| \le 4$

(ii) sketch: $\arg(z-5) - \arg(z+3) = \dfrac{\pi}{4}$

Question 2 (15 Marks) **Marks**

a) Show $\int_1^2 \frac{1}{x^2} \ln(x+1)\ dx = \frac{1}{2}\ln\frac{4}{3} + \int_1^2 \frac{1}{x(x+1)}\ dx$ **4**

and hence evaluate

$\int_1^2 \frac{1}{x^2} \ln(x+1)\, dx$ leaving answer in simplest exact form.

b) Simplify $\frac{1}{1-\sin x} - \frac{1}{1+\sin x}$ and

3

hence find

$$\int \frac{1}{1-\sin x} - \frac{1}{1+\sin x}\ dx$$

c) Find $\int \frac{1}{\sqrt{16-25x^2}}\ dx$ **2**

d) Evaluate $\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\ dx$ leave answer in exact form. **2**

e) Find $\int \frac{-2x+4}{(x^2+1)(x-1)^2}\ dx$ **4**

Question 3 (15 Marks) **Marks**

a) Let $t = \tan\frac{\theta}{2}$

(i) Find expressions for $\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2}$ in terms of t **2**

(ii) Hence show $\sin\theta = \frac{2t}{1+t^2}$ and $\cos\theta = \frac{1-t^2}{1+t^2}$ **2**

(iii) Show that $\frac{dt}{d\theta} = \frac{1}{2}(1+t^2)$ **1**

(iv) Hence evaluate $\int_0^{\frac{\pi}{2}} \frac{d\theta}{5+3\sin\theta+4\cos\theta}$ **4**

b) If $I_n = \int x^n(2x+c)^{-\frac{1}{2}}\,dx$, show that

(i) $I_n = \frac{x^n(2x+c)^{\frac{1}{2}}}{2n+1} - \frac{ncI_{n-1}}{2n+1}$ **3**

(ii) Hence evaluate $\int_0^1 x^3(2x+1)^{-\frac{1}{2}}\,dx$ **3**

Question 4 (15 Marks) **Marks**

a) The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse with equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with a>b and eccentricity e.

The foci of the ellipse are S and S' and M , M' are the feet of the perpendiculars from P onto the directrices corresponding to S and S'.

The Normal to the ellipse at P meets the major axis of the ellipse At H.

(i) Draw a sketch to illustrate the above information. **2**

(ii) Prove $SP+S'P=2a$. **1**

(iii) Show that the coordinates of H are $\left(\frac{(a^2-b^2)\cos\theta}{a}\ ,\ 0\right)$. **3**

(iv) Show that $\frac{HS}{HS'}=\frac{1-e\cos\theta}{1+e\cos\theta}=\frac{PS}{PS'}$ **2**

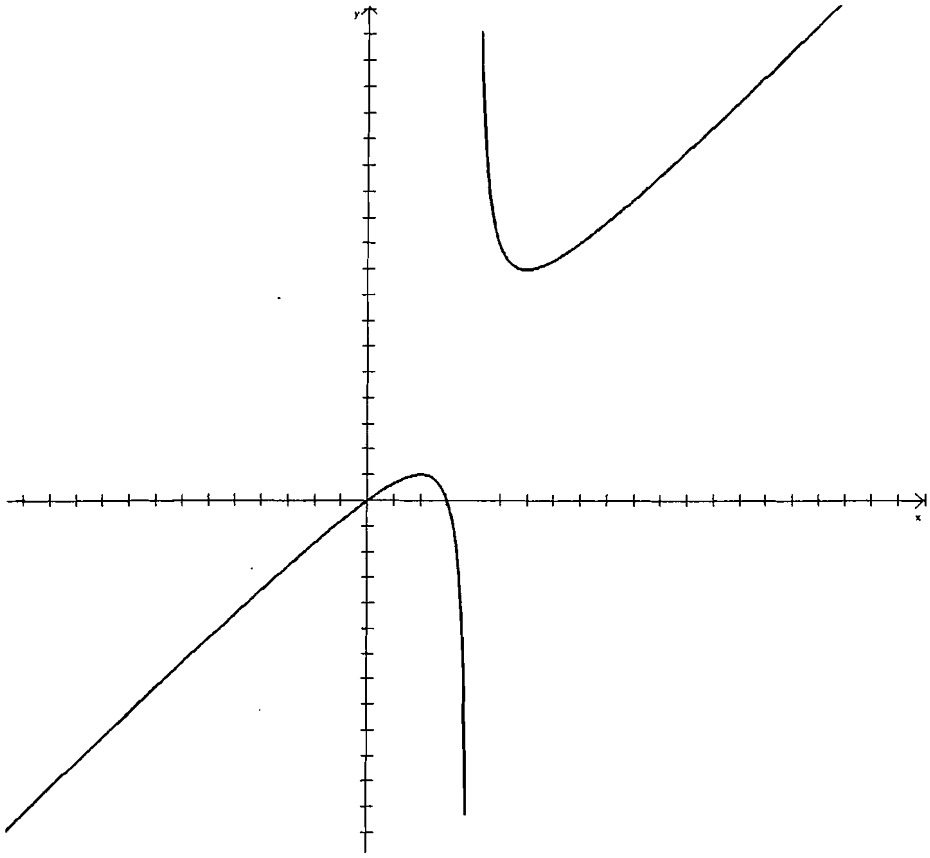
b) Show that the locus of the point $Q\left\{\frac{a}{2}(t+\frac{1}{t}),\frac{b}{2}(t-\frac{1}{t})\right\}$ for varying values of t is the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. **2**

(I) Show the gradient of the tangent at Q is $\frac{b}{a}\left(\frac{t^2+1}{t^2-1}\right)$ **2**

(II) Derive the equation of the tangent at Q **3**

Question 5 (15 Marks) **Marks**

a) The diagram shows the graph of $y = f(x)$. The graph has A vertical asymptote at $x = 4$.



Draw separate one third page sketches of the graphs of the following

(i) $y = \sqrt{f(x)}$ **2**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y^2 = f(x)$ **2**

(iv) $y = \cos(f(x))$ **2**

This question continues on the next page

Question 5 continued **Marks**

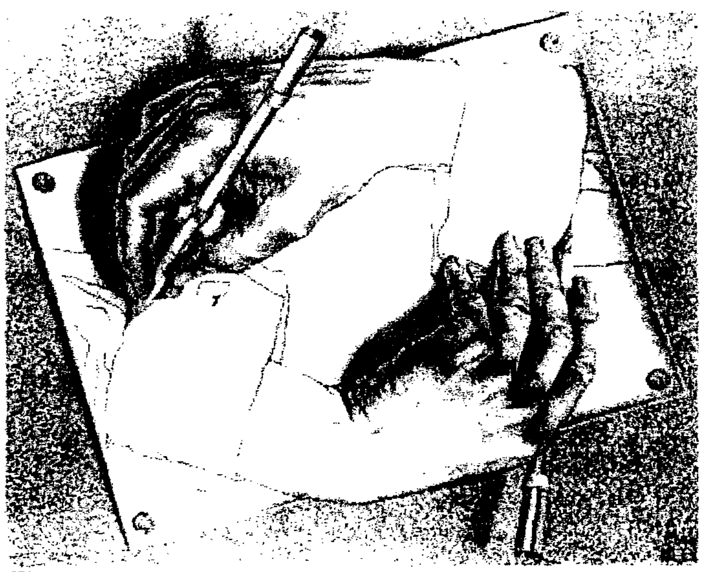
b) Sketch the graph of $y = x + \dfrac{x}{x^2 - 25}$ **4**

clearly indicating any asymptotes and any points where the graph meets the axes

c) Find the equation of the normal to the curve **3**

$x^3 y - 3xy^2 + 2y^3 = 6$ at $(1,2)$

End of Question 5



Question 6 (15 Marks) **Marks**

a) If $\frac{p}{q}$ is a zero of the polynomial (p and q are relatively prime)

$P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0 \ , a_n \neq 0$

and $a_0, a_1, a_2, a_3, ...a_n$ are integers,

(i) Show q / a_n (q divides a_n) and p / a_0 (p divides a_0) **2**

(ii) Given $P(x) = x^3 - 4x^2 - 3x - 10$ ha s a rational root, factor $P(x)$ over the complex field. **3**

b) Show that if the polynomial $P(x)$ has a root of α multiplicity m, then $P'(x)$ has a root of multiplicity $m-1$. **1**

Given that $P(x) = x^4 + x^3 - 3x^2 - 5x - 2$ has a three fold root, Find all the roots of $P(x)$. **2**

c) If α, β, δ are the roots of $p(x) = 2x^3 - 4x^2 - 3x - 1$ Find the values of $\alpha^3 + \beta^3 + \delta^3$. **2**

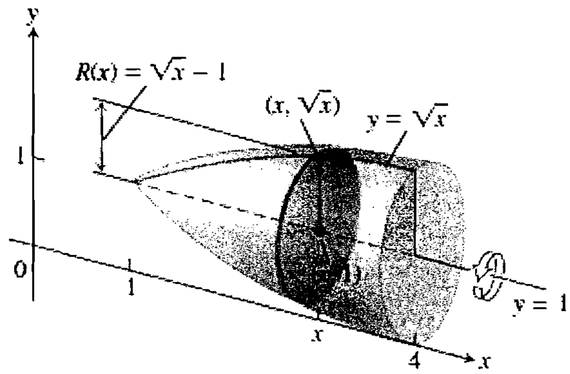
d) Let $f(t) = t^3 + ct + d$ where c and d are constants
Suppose that the equation $f(t) = 0$ has three distinct real roots t_1, t_2, and t_3.

(i) Show that $t_1^2 + t_2^2 + t_3^2 = -2c$ **2**

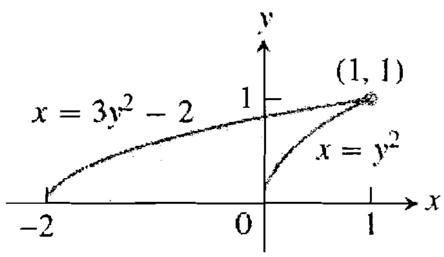
(ii) If the function $y = f(t)$ has two turning points at $t = u$ and $t = v$ and $f(u) \times f(v) < 0$
Show that $27d^2 + 4c^3 < 0$. **3**

Question 7 (15 Marks) **Marks**

a) Find the volume of the solid generated by revolving the region bounded By $y=\sqrt{x}$ and the lines $x=1,\ x=4$ about the line $y=1$. **2**
Use the slicing method.



b) The region shown here is revolved about the x-axis to generate a solid. **3**



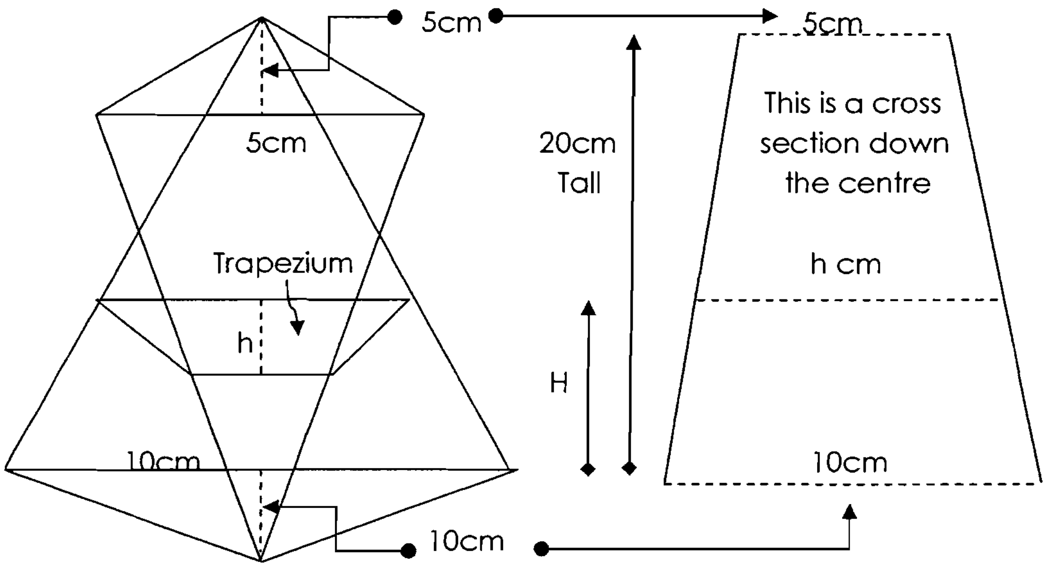
Use the method of cylindrical shells to find the volume

c) The circle $(x-6)^2+(y-4)^2=4$ is rotated around the line $x=2$. **5**
Calculate the exact volume generated.

This question is continued on the next page

Question 7 continued **Marks**

d) A Saltshaker 20 cm tall is made with isosceles triangular ends and a cross section which is an isosceles trapezium. **5**
Note top and bottom triangles have bases and perpendicular heights equal.



The Trapezium is located H cm above the base, show using similarity

that the trapezium has an area of $A = 50 - \frac{10H}{4} + \frac{H^2}{32}\ cm^2$

Hence find the volume of the saltshaker to the nearest millilitre.

Question 8 (15 Marks) **Marks**

a) Use De Moivre's theorem to express $\cos 5\theta,\ \sin 5\theta$ in terms of $\sin\theta$ and $\cos\theta$. **5**

Hence express $\tan 5\theta$ as a rational function of t where $t = \tan\theta$.

Deduce that $\tan\frac{\pi}{5}\tan\frac{2\pi}{5}\tan\frac{3\pi}{5}\tan\frac{4\pi}{5} = 5$.

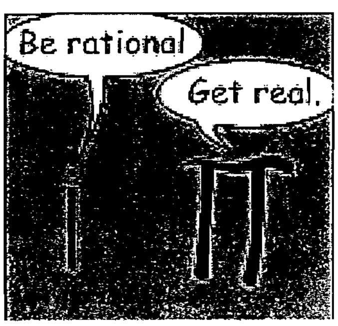
b) Find $\int \ln(\sqrt{x} + \sqrt{1+x})dx$ **3**

c) If $y = \frac{1}{2}(e^{ax} - e^{-ax})$

(i) Show that $x = \frac{1}{a}\ln(y + \sqrt{1+y^2})$ **2**

(ii) Show that $\left(\frac{dy}{dx}\right)^2 - a^2y^2 = a^2$ **2**

(iii) Hence deduce that $\int \frac{dy}{\sqrt{1+y^2}} = \log_e(y + \sqrt{1+y^2}) + c$ **3**



End of Examination

Question 1 Solutions Ext 2 Maths HSC 2008 Trial [M.]

a) $z = \frac{1+i}{1-i}$ $\quad |z| = \frac{|1+i|}{|1-i|} = \frac{\sqrt{2}}{\sqrt{2}} = 1$ ✓

$\arg z = \arg(1+i) - \arg(1-i)$
$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$ ✓

clearly show $|z|, |w|$ $\arg z$ $\arg w$

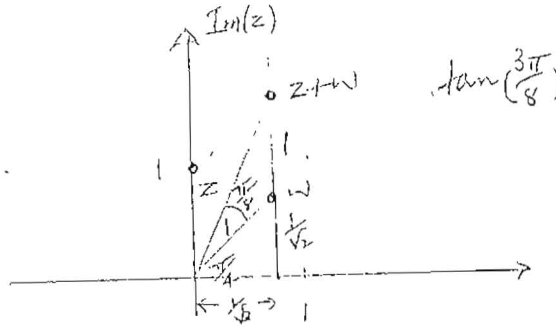
$w = \frac{\sqrt{2}}{1-i}$ $\quad |w| = \frac{\sqrt{2}}{|1-i|} = \frac{\sqrt{2}}{\sqrt{2}} = 1$ ✓

$\arg w = \arg\sqrt{2} - \arg(1-i)$
$= 0 - \left(-\frac{\pi}{4}\right) = \frac{\pi}{4}$ ✓

$\frac{T}{4}$

b) $z = 1\,\text{cis}\,\frac{\pi}{2} = i$
$w = 1\,\text{cis}\,\frac{\pi}{4} = \frac{1}{\sqrt{2}}(1+i)$ ✓

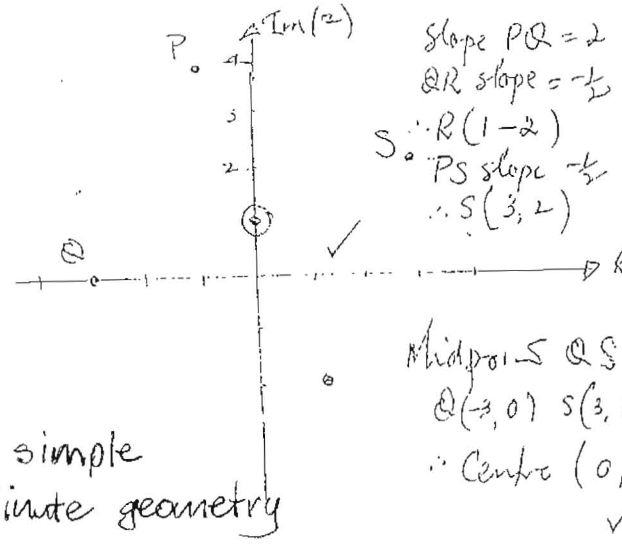
$z + w = \frac{1}{\sqrt{2}}(1+i) + i = \frac{1}{\sqrt{2}} + \left(\frac{\sqrt{2}+1}{\sqrt{2}}\right) i$ ✓



$\tan\left(\frac{3\pi}{8}\right) = \frac{1+\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \sqrt{2}+1$ ✓

$\frac{T}{3}$

c)



Slope $PQ = 2$
QR slope $= -\frac{1}{2}$
$\therefore R(1,-2)$ $\quad R: 1-2i$
PS slope $-\frac{1}{2}$
$\therefore S(3,2)$ $\quad S: 3+2i$

✓

Midpoint QS
$Q(-3,0)$ $\quad S(3,2)$
$\therefore$ Centre $(0,1)$ or i ✓

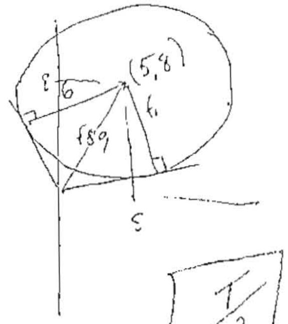
$\frac{T}{2}$

Use simple coordinate geometry

Question 1 cont

d) $|z-(5+8i)| = 6$

least $\arg z = \sin^{-1}\frac{8}{\sqrt{89}} - \sin^{-1}\frac{6}{\sqrt{89}}$
$\doteqdot 18°30'$ ✓

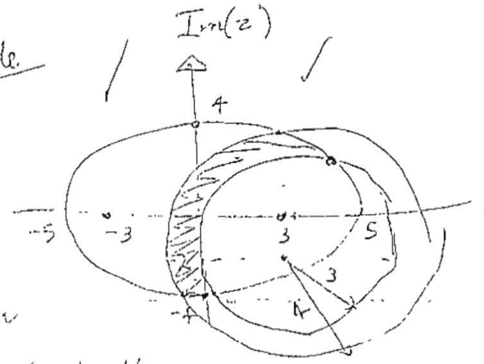


Max $\arg z = \sin^{-1}\frac{8}{\sqrt{89}} + \sin^{-1}\frac{6}{\sqrt{89}}$
$\doteqdot 97°29'$ ✓

$\frac{T}{2}$

e) Shade ✓

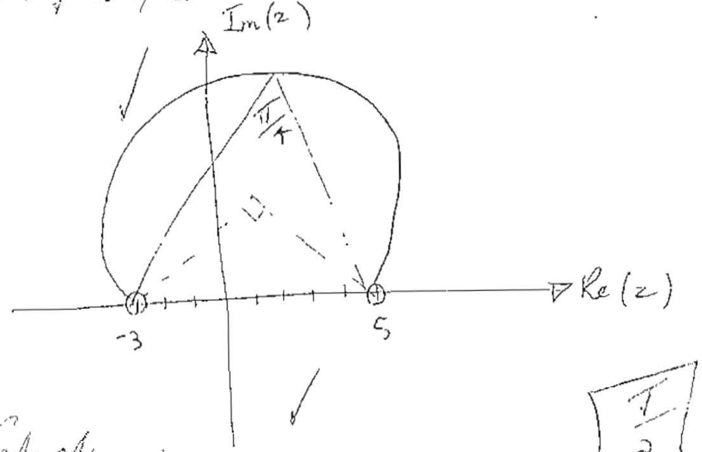
(i)



need to see radii and centre x y intercepts of ellipse

$\frac{T}{2}$

(ii)



open circles essential

$\frac{T}{2}$

$\frac{CT}{15}$

$du = \frac{1}{1+x}dx \qquad V = \int x^{-2}dx = \frac{-1}{x}$

$= \frac{-1}{x}\ln(x+1)\Big]_1^2 + \int_1^2 \frac{dx}{x(x+1)}$

$= -\frac{1}{2}\ln 3 + 1\ln 2 + \int_1^2 \frac{dx}{x(x+1)}$

Now

$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$

$\therefore 1 = A(x+1) + B(x)$

let $x = 0 \therefore A = 1$

$x = -1 \therefore B = -1$

$= -\frac{1}{2}\ln 3 + \frac{1}{2}\ln 4 + \int_1^2 \frac{1}{x} - \frac{1}{x+1}\,dx$

$= \frac{1}{2}\ln\frac{4}{3} + \ln\left(\frac{x}{x+1}\right)\Big]_1^2$

$= \frac{1}{2}\ln\frac{4}{3} + \ln\frac{2}{3} - \ln\frac{1}{2}$

$= \frac{1}{2}\ln\frac{4}{3} + \ln\frac{4}{3} = \frac{3}{2}\ln\frac{4}{3}$

$\frac{T}{4}$

b) $\frac{1}{1-\sin x} - \frac{1}{1+\sin x} = \frac{1+\sin x - (1-\sin x)}{1-\sin^2 x}$

$= \frac{2\sin x}{\cos^2 x} = 2\cdot(\cos x)^{-1}\frac{\sin x}{\cos x}$

$= 2\sec x \tan x$

$\therefore \int 2\sec x\tan x\,dx = 2\sec x + C$

$\frac{T}{3}$

c) $\int \frac{dx}{\sqrt{16-25x^2}} = \int \frac{dx}{\sqrt{4^2-(5x)^2}}$ let $u = 5x$, $du = 5dx$

$= \frac{1}{5}\frac{du}{\sqrt{4^2-u^2}}$

$= \frac{1}{5}\sin^{-1}\left(\frac{u}{4}\right) + C$

$= \frac{1}{5}\sin^{-1}\left(\frac{5x}{4}\right) + C$

$\frac{T}{2}$

d) $\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\,dx$

Note $1+\cos 4x = 2\cos^2 2x$

$\therefore \int_0^{\frac{\pi}{4}} \sqrt{2\cos^2 2x}\,dx = \sqrt{2}\int_0^{\frac{\pi}{4}} \cos 2x\,dx$

$= \frac{\sqrt{2}}{2}\left[\sin 2x\right]_0^{\frac{\pi}{4}} = \frac{\sqrt{2}}{2}[1-0]$

$= \frac{\sqrt{2}}{2}$

$\frac{T}{2}$

e) $\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}$ or $\frac{Cx+D}{(x-1)^2}$

$\therefore -2x+4 = (Ax+B)(x-1)^2 + C(x-1)(x^2+1)^2 + D(x^2+1)$

let $x = 1 \therefore 2 = 2D \therefore \boxed{D = 1}$

equate coeff x^3: $A + C = 0 \therefore A = -C$

x^2: $-2A + B - C + D = 0$ as $D = 1$

$\therefore -2A + B - C = -1$ if $A = -C$: $2C + B - C = -1$

$C + B = -1$

x: $\therefore -2 = A - 2B + C$

if $A = -C$

$-2B = -2$

$\therefore B = 1$

$\therefore C = -2$

$\therefore \int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx = \int \frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\,dx$

$= \int \frac{2x}{x^2+1} + \frac{1}{x^2+1} \ldots$

$= \ln(x^2+1) + \tan^{-1}x - 2\ln(x-1) - \frac{1}{x-1} + C$

$\frac{T}{4}$

$$\frac{1}{\sqrt{1+t^2}}$$

ii) $\sin\theta = \sin\left(\frac{\theta}{2}+\frac{\theta}{2}\right) = 2\cdot\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$= 2\cdot\frac{t}{\sqrt{1+t^2}}\cdot\frac{1}{\sqrt{1+t^2}} = \frac{2t}{1+t^2}$ ✓

$\cos\theta = \cos\left(\frac{\theta}{2}+\frac{\theta}{2}\right) = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$

$= \frac{1}{1+t^2} - \frac{t^2}{1+t^2} = \frac{1-t^2}{1+t^2}$ ✓

$\boxed{\frac{1}{2}}$

i) $t = \tan\frac{\theta}{2}$ $\quad \frac{dt}{d\theta} = \frac{1}{2}\sec^2\frac{\theta}{2} = \frac{1}{2}\left(1+\tan^2\frac{\theta}{2}\right)$

$\therefore \frac{dt}{d\theta} = \frac{1}{2}(1+t^2)$ ✓ $\quad\boxed{\frac{1}{1}}$

$\int_0^{\frac{\pi}{2}} \frac{dx}{5+3\sin x+4\cos x}$ $\quad$ let $t = \tan\frac{x}{2}$ $\quad\therefore dt = \sec^2\frac{x}{2}\cdot\frac{1}{2}\cdot dx$

$2dt = (1+t^2)\,dx$ ✓

$\sin x = \frac{2t}{1+t^2}$

$\cos x = \frac{1-t^2}{1+t^2}$

$$\int_0^{\frac{\pi}{2}} \frac{dx}{5+3\cdot\frac{2t}{1+t^2}+4\cdot\frac{(1-t^2)}{1+t^2}} = \int_0^1 \frac{2dt}{5(1+t^2)+6t+4(1-t^2)}$$

$$\int_0^1 \frac{2dt}{9+6t+t^2} = \int_0^1 \frac{2dt}{(t+3)^2} ✓ = 2\int_0^1 (t+3)^{-2}dt$$

$$-2\left[\frac{1}{t+3}\right]_0^1 = -2\left[\frac{1}{4}-\frac{1}{3}\right] = -2\cdot -\frac{1}{12}$$

$\frac{1}{6}$ ✓

$\boxed{\frac{1}{4}}$

$u = x^n \quad du = nx^{n-1}dx$

$dV = (2x+c)^{-\frac{1}{2}}dx$

$V = \frac{(2x+c)^{\frac{1}{2}}}{2\times\frac{1}{2}} = (2x+c)^{\frac{1}{2}}$

$$\therefore I_n = x^n(2x+c)^{\frac{1}{2}} - \int nx^{n-1}(2x+c)^{\frac{1}{2}}dx$$

$$= x^n(2x+c)^{\frac{1}{2}} - \int nx^{n-1}(2x+c)(2x+c)^{-\frac{1}{2}}dx$$

$$I_n = x^n(2x+c)^{\frac{1}{2}} - 2n\int x^n(2x+c)^{-\frac{1}{2}}dx - nc\,I_{n-1}$$

$$\therefore (2n+1)I_n = x^n(2x+c)^{\frac{1}{2}} - nc\,I_{n-1}$$

$$\therefore I_n = \frac{x^n(2x+c)^{\frac{1}{2}}}{2n+1} + \frac{nc\,I_{n-1}}{2n+1}$$

$\boxed{\frac{1}{3}}$

$$\therefore I_3 = \left.\frac{x^3(2x+1)^{\frac{1}{2}}}{7}\right]_0^1 - \frac{3I_2}{7}$$

$$= \frac{\sqrt{3}}{7} - \frac{3}{7}I_2$$ ✓

$$I_2 = \left.\frac{x^2(2x+1)^{\frac{1}{2}}}{5}\right]_0^1 - \frac{2I_1}{5}$$

$$= \frac{\sqrt{3}}{5} - \frac{2I_1}{5}$$

$$I_1 = \left.\frac{x(2x+1)^{\frac{1}{2}}}{3}\right]_0^1 - \frac{I_0}{3} = \frac{\sqrt{3}}{3} - \frac{I_0}{3}$$

$$I_0 = \int_0^1 (2x+1)^{-\frac{1}{2}}dx = \left.\frac{(2x+1)^{\frac{1}{2}}}{2\times\frac{1}{2}}\right]_0^1 = \sqrt{3}-1.$$

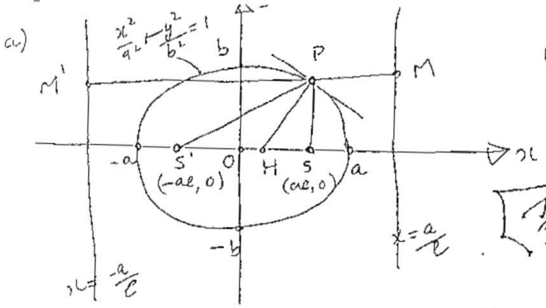
$$\therefore I_3 = \frac{\sqrt{3}}{7} - \frac{3}{7}\left[\frac{\sqrt{3}}{5} - \frac{2}{5}\left[\frac{\sqrt{3}}{3} - \frac{1}{3}(\sqrt{3}-1)\right]\right]$$

$$= \frac{\sqrt{3}}{7} - \frac{3}{7}\left[\frac{\sqrt{3}}{5} - \frac{2}{5}\cdot\frac{1}{3}\right]$$

$$= \frac{\sqrt{3}}{7} - \frac{3}{7}\left[\frac{3\sqrt{3}-2}{15}\right] = \frac{5\sqrt{3}-3\sqrt{3}+2}{35}$$

$$= 2\sqrt{3}+2$$

(a)



$\frac{1}{2}$

(ii) $PS = ePM$ (by defn)

$PS' = ePM'$

$\therefore PS + PS' = e(PM + PM')$

$= e \cdot \frac{2a}{e} = \underline{2a}$

$\frac{1}{1}$

(iii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \therefore \frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = \frac{-b^2x}{a^2y}$

at $P(a\cos\theta, b\sin\theta)$ $\quad \therefore \frac{dy}{dx} = \frac{-b^2}{a} \frac{a\cos\theta}{b\sin\theta} = -\frac{b}{a}\cot\theta$

$\therefore$ Gradient of normal $m = \frac{a}{b}\tan\theta$ using $m_1m_2 = -1$

Eqn of normal $\quad \frac{y - b\sin\theta}{x - a\cos\theta} = \frac{a}{b}\tan\theta$

$\therefore y = \frac{a}{b}\tan\theta\,(x - a\cos\theta) + b\sin\theta$

$y = \frac{a}{b}x\tan\theta - \frac{a^2}{b}\sin\theta + b\sin\theta$

when $y = 0$ $\quad \therefore \sin\theta\left(\frac{a^2-b^2}{b}\right) = \frac{ax}{b}\tan\theta$

$\therefore x = \frac{a^2-b^2}{a} \cdot \frac{\sin\theta\cos\theta}{\sin\theta} = \frac{a^2-b^2}{a}\cos\theta$

ie $\quad H\left(\frac{a^2-b^2}{a}\cos\theta,\ 0\right)$

$\frac{1}{3}$

(iv) $\frac{HS}{HS'} = \frac{OS - OH}{OS' + OH} = \frac{ae - \frac{a^2-b^2}{a}\cos\theta}{ae + \frac{a^2-b^2}{a}\cos\theta}$ $\quad$ using $b^2 = a^2(1-e^2)$

$= \frac{a^2e - (a^2-b^2)\cos\theta}{a^2e + (a^2-b^2)\cos\theta} = \frac{a^2e - (a^2 - a^2(1-e^2))\cos\theta}{a^2e + (a^2 - a^2(1-e^2))\cos\theta}$

$= \frac{a^2e - a^2e^2\cos\theta}{a^2e + a^2e^2\cos\theta} = \frac{a^2e(1 - e\cos\theta)}{a^2e(1 + e\cos\theta)}$

$= \frac{1 - e\cos\theta}{1 + e\cos\theta}$

$\frac{PS}{PS'} = \frac{ePM}{ePM'} = \frac{PM}{PM'} = \frac{\frac{a}{e} - a\cos\theta}{\frac{a}{e} + a\cos\theta} = \frac{a - ae\cos\theta}{a + ae\cos\theta}$

$= \frac{a(1 - e\cos\theta)}{a(1 + e\cos\theta)} = \frac{1 - e\cos\theta}{1 + e\cos\theta} = \frac{HS}{HS'}$

$\frac{1}{2}$

(b) $\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$ $\quad x^2 = \frac{a^2}{4}\left(t + \frac{1}{t}\right)^2 \quad y^2 = \frac{b^2}{4}\left(t - \frac{1}{t}\right)^2$

LHS

$\frac{1}{4}\left(t + \frac{1}{t}\right)^2 - \frac{1}{4}\left(t - \frac{1}{t}\right)^2 = \frac{1}{4}\left(t^2 + 2 + \frac{1}{t^2}\right) - \frac{1}{4}\left(t^2 - 2 + \frac{1}{t^2}\right)$

$= \frac{1}{2} + \frac{1}{2} = 1 = \text{RHS}$

$\frac{1}{2}$

(i) Q lies on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \therefore \frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 1$

$\therefore \frac{dy}{dx} = \frac{xb^2}{ya^2} = \frac{\frac{b^2a}{2}\left(t + \frac{1}{t}\right)}{\frac{a^2b}{2}\left(t - \frac{1}{t}\right)} = \frac{ab^2\left(\frac{t^2+1}{t}\right)}{a^2b\left(\frac{t^2-1}{t}\right)} = \frac{b}{a}\left(\frac{t^2+1}{t^2-1}\right)$

(ii) Equation of tangent at Q

$\frac{1}{2}$

$y - \frac{b}{2}\left(\frac{t^2-1}{t}\right) = \frac{b}{a}\left(\frac{t^2+1}{t^2-1}\right)\left(x - \frac{a}{2}\left(\frac{t^2+1}{t}\right)\right)$

$= \frac{bx}{a}\left(\frac{t^2+1}{t^2-1}\right) - \frac{b}{2}\left(\frac{(t^2+1)^2}{t(t^2-1)}\right)$

$\therefore y - \frac{bx}{a}\left(\frac{t^2+1}{t^2-1}\right) = \frac{b}{2}\left[\frac{t^2-1}{t} - \frac{(t^2+1)^2}{t(t^2-1)}\right]$

$= \frac{b}{2}\left[\frac{(t^2-1)^2 - (t^2+1)^2}{t(t^2-1)}\right]$

$= \frac{b}{2}\left[\frac{(t^2-1+t^2+1)(t^2-1-t^2-1)}{t(t^2-1)}\right]$

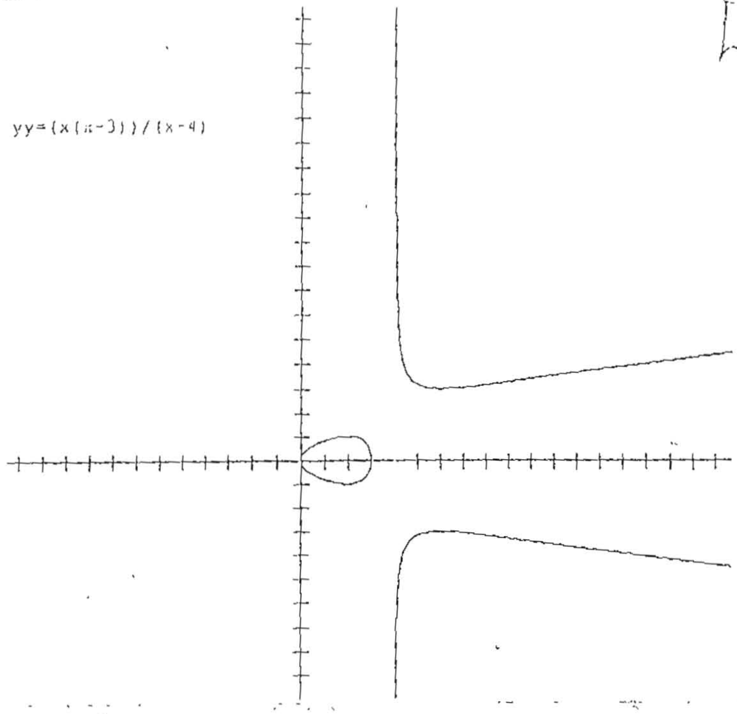
$\therefore y - \frac{bx}{a}\left(\frac{t^2+1}{t^2-1}\right) = \frac{b}{2} \cdot \frac{2t^2 \times -2}{t(t^2-1)} = \frac{-2bt}{t^2-1}$

$\therefore \underline{\frac{bx}{a}(t^2+1) - (t^2-1)y - 2bt = 0}$

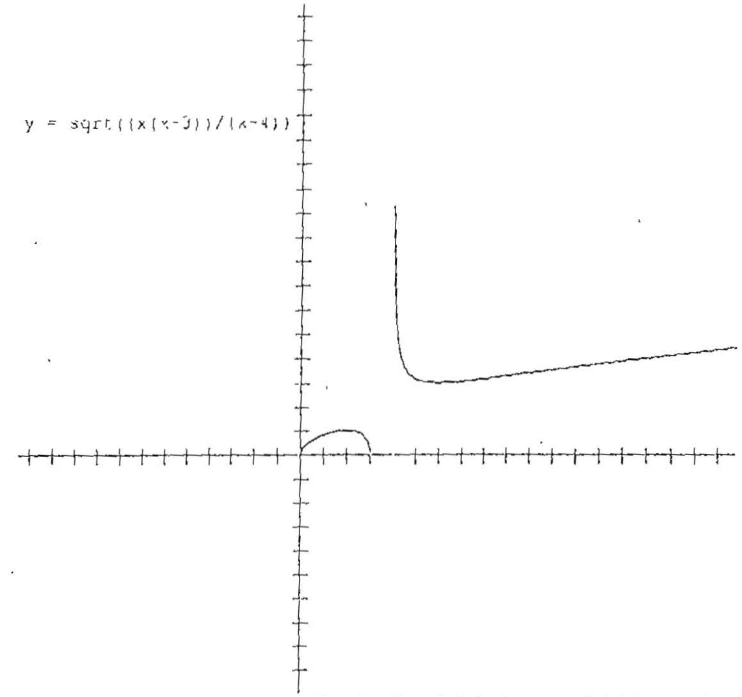
$\frac{1}{3}$

5. iii)
ii)

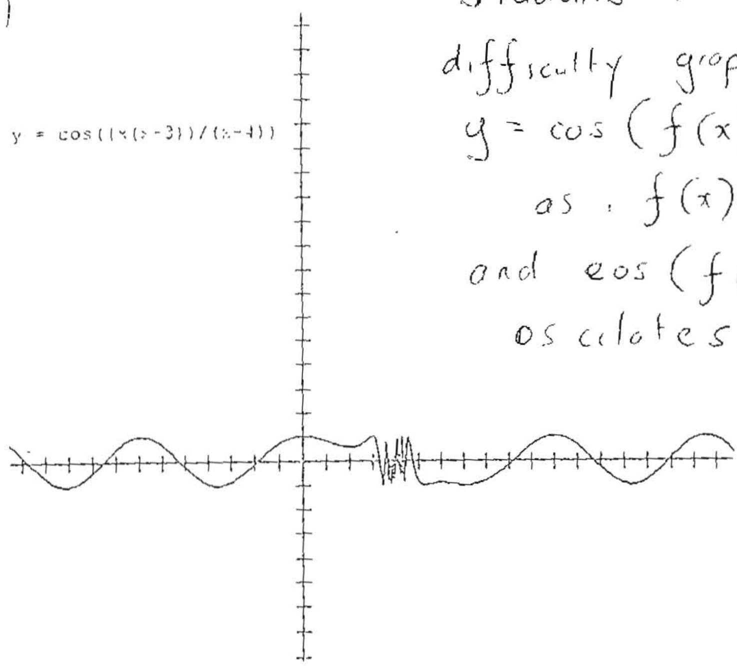
2 each



i)

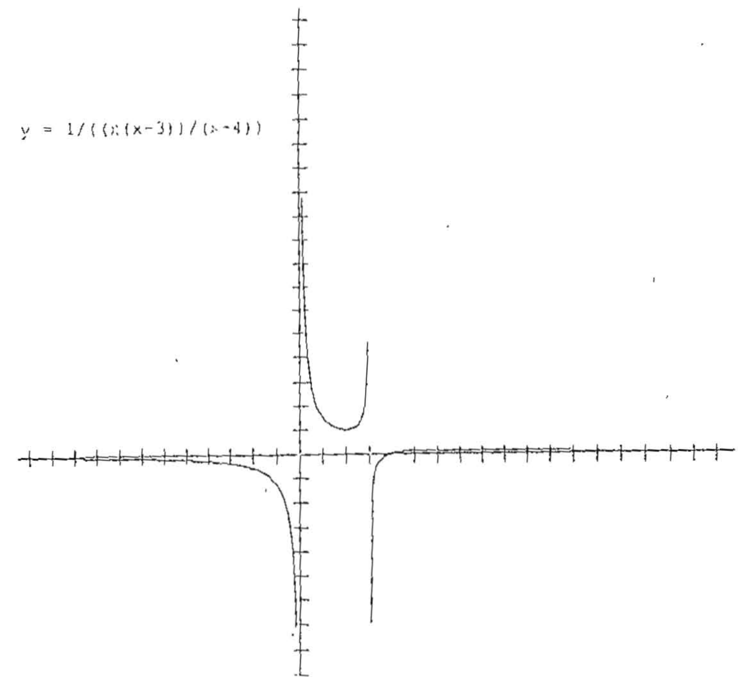


iv)



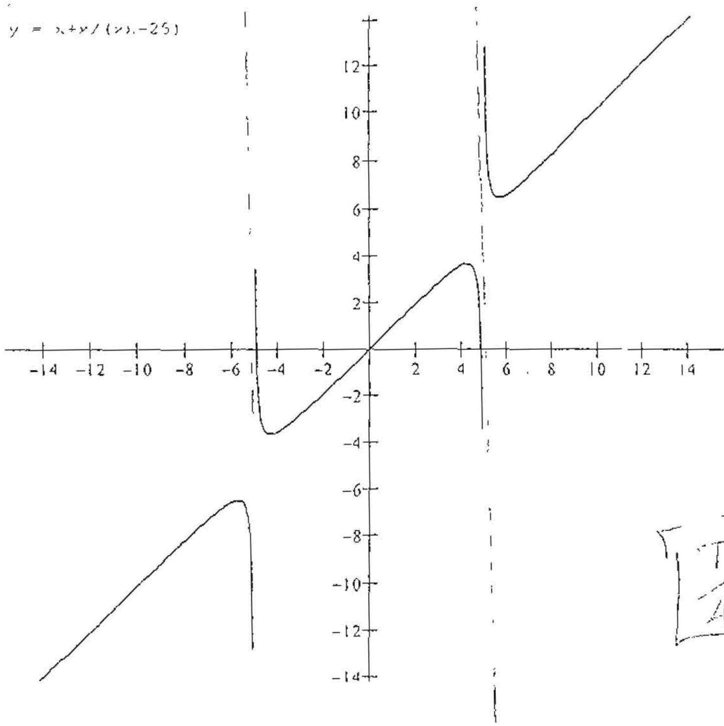
Students had difficulty graphing $y = \cos(f(x))$ as $f(x) \to \infty$ and $\cos(f(x))$ oscilates faster

ii)



Answer Q5 b)

$y = x + x/(x^2-25)$



$\frac{T}{4}$

Suggested Marking
asymptotes : $y = x$ $x = \pm 5$
analysis $\lim_{x \to \pm 5^{\pm}}$

Many students had difficulty determining the precise nature of the function for $-5 < x < 5$

Q5 a) $x^3y - 3xy^2 + 2y^3 = 6$

$$x^3 \frac{dy}{dx} + y\,3x^2 - 3\left[x . 2y\frac{dy}{dx} + y^2 . 1\right] + 2 . 3y^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left[x^3 - 6xy + 6y^2\right] + 3x^2y - 3y^2 = 0$$

$$\therefore \frac{dy}{dx} = \frac{3y^2 - 3x^2y}{x^3 - 6xy + 6y^2}$$

at $(1, 2)$

$$\frac{dy}{dx} = \frac{3.4 - 3.1.2}{1 - 6.2 + 24}$$

$$\frac{dy}{dx} = \frac{12-6}{1-12+24} = \frac{6}{13}$$ *

many careless errors in finding gradient of tangent

$\therefore$ eqn of normal $m = -\frac{13}{6}$

$$y - 2 = -\frac{13}{6}(x-1)$$

$$6y - 12 = -13x + 13$$

$$\boxed{13x + 6y - 25 = 0}$$

eqn of Normal.

$\frac{T}{3}$

$\frac{GT}{15}$

* sometimes gradient of tangent was used instead of gradient of normal

Question 6

(i) $\frac{p}{q}$ is zero of $P(x)$

$\therefore a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_1\frac{p}{q} + a_0 = 0 \qquad \times q^n$

$a_np^n + a_{n-1}p^{n-1}q + \cdots + a_1pq^{n-1} + a_0q^n = 0$

Note any expression with $\frac{p}{q}$ is fractional

$\therefore -a_np^n = a_{n-1}p^{n-1}q + \cdots + a_1pq^{n-1} + a_0q^n$

$= q(a_{n-1}p^{n-1} + \cdots + a_1pq^{n-2} + a_0q^{n-1})$

Since p, q relatively prime so q and p^n are also relatively prime

$\therefore q / a_n$ (q divides a_n)

Similarly. $-a_0q^n = a_np^n + a_{n-1}p^{n-1}q + \cdots + a_1pq^{n-1}$

$= p(a_np^{n-1} + a_{n-1}p^{n-2}q + \cdots + a_1q^{n-1})$

Since p, q rel prime p, q^n rel. prime

$\therefore p / a_0$.

T/2

(ii) $P(x) = x^3 - 4x^2 - 3x - 10$ monic $a_n = 1$

$p/-10 \rightarrow \pm1 \ \pm2 \ \pm5 \ \pm10$

$P(5) = 0 \qquad \therefore P(x) = (x-5)(x^2+x+2)$

$= (x-5)\left(x - \left(\frac{-1+\sqrt{7}i}{2}\right)\right)\left(x - \left(\frac{-1-\sqrt{7}i}{2}\right)\right)$

over $\mathbb{C}$

T/3

$P(x) = (x-\alpha)^m Q(x)$

$P'(x) = (x-\alpha)^m Q'(x) + Q(x) \cdot m(x-\alpha)^{m-1}$

$= (x-\alpha)^{m-1}\left((x-\alpha)Q'(x) + mQ(x)\right)$

ie $P'(x)$ has root α of multiplicity $(m-1)$.

T/1

Since $P(x) = x^4 + x^3 - 3x^2 - 5x - 2$ 3 fold

$P'(x) = 4x^3 + 3x^2 - 6x - 5$ 2 fold

$P''(x) = 12x^2 + 6x - 6$ 1 fold

$= 6(2x-1)(x+1)$

$P''(x) = 0$ for $x = \frac{1}{2}, -1$ but $P'(\frac{1}{2}) \neq 0$

$P'(-1) = 0$

$\therefore P(x) = (x+1)^3(x-2)$ by inspection

Hence roots $x = -1, -1, -1, 2$.

T/2

c) $2\alpha^3 - 4\alpha^2 - 3\alpha - 1 = 0$

$\therefore 2\beta^3 - 4\beta^2 - 3\beta - 1 = 0$

$2\delta^3 - 4\delta^2 - 3\delta - 1 = 0$

$\therefore 2(\alpha^3+\beta^3+\delta^3) = 4(\alpha^2+\beta^2+\delta^2) + 3(\alpha+\beta+\delta) + 3$

$\alpha^2+\beta^2+\delta^2 = (\alpha+\beta+\delta)^2 - 2(\alpha\beta+\beta\delta+\delta\alpha) = 2^2 - 2(-\frac{3}{2}) = 7$

$\alpha^3+\beta^3+\delta^3 = 2\cdot7 + \frac{3}{2}\cdot2 + \frac{3}{2} = \boxed{18\frac{1}{2}}$

$= 14 + 3 + 1\frac{1}{2}$

T/2

d) $t_1^2 + t_2^2 + t_3^2 = (t_1+t_2+t_3)^2 - 2(t_1t_2 + t_2t_3 + t_1t_3)$

but $t_1t_2 + t_2t_3 + t_1t_3 = c \qquad \Sigma\alpha_i = -\frac{b}{a} = 0$

(i) $\therefore t_1^2 + t_2^2 + t_3^2 = 0^2 - 2c = -2c$

T/2

(ii) $f(t) = t^3 + ct + d$

$f'(t) = 3t^2 + c = 0 \quad \therefore t = \pm\sqrt{\frac{-c}{3}}$

let $u = \sqrt{\frac{-c}{3}}$, $v = -\sqrt{\frac{-c}{3}} \qquad f(u)f(v) = (u^3 + cu + d)(v^3 + cv + d)$

$\therefore f(u)f(v) = \left(-\frac{c}{3}\sqrt{\frac{-c}{3}} + c\sqrt{\frac{-c}{3}} + d\right)\left(\frac{c}{3}\sqrt{\frac{-c}{3}} - c\sqrt{\frac{-c}{3}} + d\right)$

simplify first

$= \left(\frac{2c}{3}\sqrt{\frac{-c}{3}} + d\right)\left(-\frac{2c}{3}\sqrt{\frac{-c}{3}} + d\right)$ diff of 2 squares

$= d^2 - \frac{4c^2}{9}\cdot\frac{-c}{3} < 0$

$= d^2 + \frac{4c^3}{27} < 0$

ie $\therefore 27d^2 + 4c^3 < 0$

T/3

a) $\delta V = \pi r^2 \delta x \qquad r(x) = \sqrt{x} - 1$

$\therefore \delta V = \pi(\sqrt{x}-1)^2 \delta x \qquad \therefore V = \lim_{\delta x \to 0} \sum_{x=1}^{4} \pi(\sqrt{x}-1)^2 \delta x$ ✓

$\therefore V = \pi \int_1^4 x - 2\sqrt{x} + 1 \, dx$

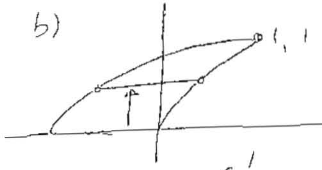
$= \pi \left[\frac{x^2}{2} - \frac{4}{3}x^{\frac{3}{2}} + x \right]_1^4 = \pi \left[8 - \frac{32}{3} + 4 - \left(\frac{1}{2} - \frac{4}{3} + 1 \right) \right]$

✓ $= \pi \left[12 - 10\frac{2}{3} + \left(-\frac{1}{6} \right) \right] = \pi \times 1\frac{1}{6} = \frac{7\pi}{6} u^3$

≈ 5 mins

$\frac{\mathrm{I}}{2}$

b)



$\delta V = 2\pi r h \, \delta y \qquad h = y^2 - (3y^2 - 2)$

✓ $\delta V = 2\pi y (2 - 2y^2) \delta y \qquad = 2 - 2y^2$ Care needed

$r = y$ to be taken

✓ $\therefore V = \lim_{\delta y \to 0} \sum_{y=0}^{1} 2\pi(2y - 2y^3) \, \delta y$ in determining volume.

$\therefore V = 2\pi \int_0^1 2y - 2y^3 \, dy$

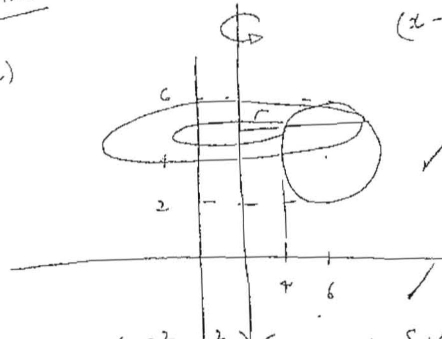
$= 4\pi \int_0^1 y - y^3 \, dy = 4\pi \left[\frac{y^2}{2} - \frac{y^4}{4} \right]_0^1 = 4\pi \left[\frac{1}{2} - \frac{1}{4} \right]$ ✓

$\therefore = \pi \; u^3$

< 5 min

$\frac{\mathrm{I}}{3}$

c)



$(x-6)^2 + (y-4)^2 = 4.$

$x - 6 = \pm\sqrt{4 - (y-4)^2}$

✓ $\therefore x = 6 \pm \sqrt{4 - (y-4)^2}$

$R = 6 \pm \sqrt{4-(y-4)^2} - 2$

$= 4 + \sqrt{4-(y-4)^2}$

$r = 4 - \sqrt{4-(y-4)^2}$ ✓

$\delta V = \pi(R^2 - r^2)\delta y \qquad \therefore \delta V = \pi(R+r)(R-r)\delta y$

✓ $\delta V = \pi \; 8 \cdot 2\sqrt{4-(y-4)^2} \; \delta y$

$\therefore V = \lim_{\delta y \to 0} \sum_{y=2}^{6} 16\pi\sqrt{4-(y-4)^2} \; \delta y$

$\therefore V = 16\pi \int_2^6 \sqrt{4-(y-4)^2} \, dy$ ✓ or $\frac{1}{2}$ area of circle radius 2

$\frac{1}{2}\pi 2^2 = 2\pi$

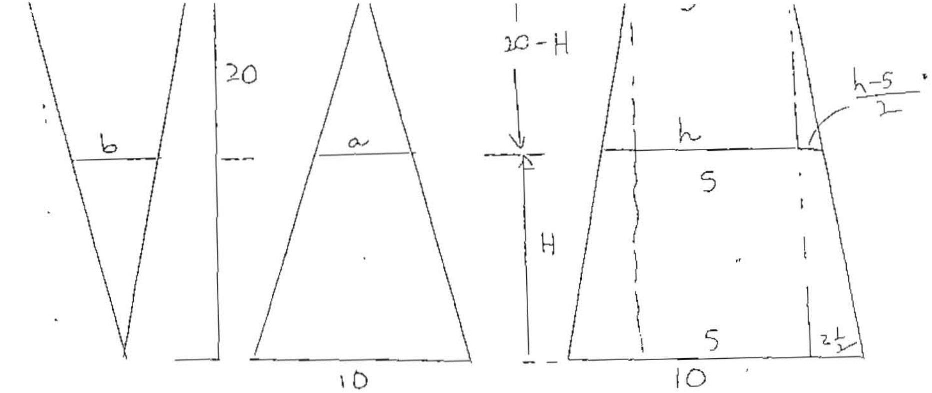
$= 16\pi \cdot 2\pi$

$= 32\pi^2 \, u^3$ ✓

$\frac{\mathrm{I}}{5}$

many approaches were taken, often gaps were found in the set up leading to an incomplete soln.

(d)



for b

$\frac{b}{5} = \frac{H}{20}$

$b = \frac{H}{4}$ ✓

for a

$\frac{a}{10} = \frac{20-H}{20}$

$20a = 10(20-H)$

$a = 10 - \frac{H}{2}$ ✓

for h

$\frac{\frac{h-5}{2}}{\frac{5}{2}} = \frac{20-H}{20}$

$\frac{h-5}{5} = \frac{20-H}{20}$

$h - 5 = \frac{20-H}{4}$

✓ $h - 5 = 5 - \frac{H}{4}$

$h = 10 - \frac{H}{4}$

$\therefore A = \frac{1}{2}h(a+b)$

$A = \frac{1}{2}\left(10 - \frac{H}{4}\right)\left[10 - \frac{H}{2} + \frac{H}{4}\right]$

$A = \frac{1}{2}\left(10 - \frac{H}{4}\right)\left(10 - \frac{H}{4}\right)$

$= \frac{1}{2}\left(100 - 2 \times 10 \times \frac{H}{4} + \frac{H^2}{16}\right)$

$= 50 - \frac{10H}{4} + \frac{H^2}{32}$

$\therefore V = \int_0^{20} A \, dH = \int_0^{20} 50 - \frac{10H}{4} + \frac{H^2}{32} \, dH.$

✓ $= \left[50H - \frac{10H^2}{8} + \frac{H^3}{96} \right]_0^{20}$

$= 50 \cdot 20 - \frac{10 \cdot 20^2}{8} + \frac{20^3}{96} - 0$

✓ $= 1000 - 500 + 83\frac{1}{3}$

$\boxed{V = 583\frac{1}{3} \text{ cm}^3}$

usually well done if attempted

$\frac{\mathrm{I}}{5}$

Qn 8

$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$ (by De Moivres theorem)

expand LHS using pascals $\triangle$

$(\cos\theta)^5 + 5(\cos\theta)^4(i\sin\theta) + 10\cos^3\theta(i\sin\theta)^2$
$+ 10\cos^2\theta(i\sin\theta)^3 + 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5$

equating real and imaginary parts.

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

$\therefore \tan 5\theta = \dfrac{\sin 5\theta}{\cos 5\theta} = \dfrac{5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta}{\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta}$

$\div \cos^5\theta \qquad = \dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$

$t = \tan\theta \qquad \therefore \tan 5\theta = \dfrac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$ as a rational function of t

If $\tan 5\theta = 0 \qquad 5\theta = n\pi$ for $n = 0, 1, 2, 3, 4$ distinct

$\therefore \theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$

$5t - 10t^3 + t^5 = 0$ $\triangleleft$ need this eqn.

$t(5 - 10t^2 + t^4) = 0$

$\therefore t^4 - 10t^2 + 5 = 0$ has roots $t = \tan\frac{\pi}{5}, \tan\frac{2\pi}{5}, \ldots$

The product of roots $= \frac{5}{1} = 5$

$\tan\frac{\pi}{5}\tan\frac{2\pi}{5}\tan\frac{3\pi}{5}\tan\frac{4\pi}{5} = 5.$

$\boxed{T/5}$

b) $\int \ln(\sqrt{x} + \sqrt{1+x})\,dx$. let $u = \ln(\sqrt{x} + \sqrt{1+x})$

$\dfrac{du}{dx} = \dfrac{1}{\sqrt{x} + \sqrt{1+x}}\left(\dfrac{1}{2\sqrt{x}} + \dfrac{1}{2\sqrt{1+x}}\right)$

$= \dfrac{1}{\sqrt{x} + \sqrt{1+x}}\left[\dfrac{2\sqrt{1+x} + 2\sqrt{x}}{4\sqrt{1+x}\cdot\sqrt{x}}\right]$

$= \dfrac{1}{2\sqrt{x}\sqrt{1+x}}$

$dV = dx \quad \therefore V = x$

By Parts

$\therefore = x\ln(\sqrt{x} + \sqrt{1+x}) - \frac{1}{2}\int \dfrac{x\,dx}{\sqrt{x}\sqrt{1+x}}$

$= x\ln(\sqrt{x} + \sqrt{1+x}) - \frac{1}{2}\int \dfrac{x\,dx}{\sqrt{x^2+x}}$ $\qquad x^2 + x = (x + \frac{1}{2})^2 - \frac{1}{4}$

$= x\ln(\sqrt{x} + \sqrt{1+x}) - \frac{1}{2}\int \dfrac{x}{\sqrt{(x+\frac{1}{2})^2 - \frac{1}{4}}}\,dx$

let $x + \frac{1}{2} = \frac{1}{2}\sec\theta \longrightarrow x = \frac{1}{2}\sec\theta - \frac{1}{2}$

$dx = \frac{1}{2}\sec\theta\tan\theta\,d\theta \qquad x = \frac{1}{2}(\sec\theta - 1)$

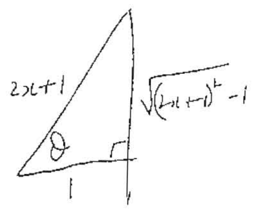
$\therefore = x\ln(\sqrt{x} + \sqrt{1+x}) - \frac{1}{4}\cdot\frac{1}{2}\int \dfrac{(\sec\theta - 1)\sec\theta\tan\theta\,d\theta}{\sqrt{\frac{1}{4}\sec^2\theta - \frac{1}{4}}}$ $\qquad = \frac{1}{2}\tan\theta$

$= x\ln(\sqrt{x} + \sqrt{1+x}) - \frac{1}{4}\int(\sec\theta - 1)\sec\theta\,d\theta$

$= x\ln(\sqrt{x} + \sqrt{1+x}) - \frac{1}{4}\int \sec^2\theta - \sec\theta\,d\theta$

$- \frac{1}{4}\left[\tan\theta - \ln(\sec\theta + \tan\theta)\right] + c$

Now $\sec\theta = \dfrac{x + \frac{1}{2}}{\frac{1}{2}} = 2x + 1 \qquad -\frac{1}{4}\left[\sqrt{(2x+1)^2 - 1} - \ln\left(2x + 1 + \sqrt{(2x+1)^2 - 1}\right)\right]$



many other forms $+c$

$\boxed{1/3}$

$y = \frac{1}{2}(e^{ax} - e^{-ax})$

$y^2 = \frac{1}{4}(e^{2ax} - 2 + e^{-2ax})$

$y^2 + 1 = \frac{1}{4}(e^{2ax} - 2 + 4 + e^{-2ax})$

$y^2 + 1 = \frac{1}{4}(e^{ax} + e^{-ax})^2$

$\therefore \sqrt{y^2+1} = \frac{1}{2}(e^{ax} + e^{-ax})$

$\frac{dy}{dx} = \frac{1}{2}(ae^{ax} + ae^{-ax})$

$= \frac{a}{2}(e^{ax} + e^{-ax})$.

$y + \sqrt{y^2+1} = e^{ax}$

$\therefore \ln(y + \sqrt{y^2+1}) = ax$

$\therefore x = \frac{1}{a}\ln(y + \sqrt{y^2+1})$

$\boxed{\tfrac{T}{2}}$

(ii) $\left(\frac{dy}{dx}\right)^2 = \frac{a^2}{4}(e^{ax} + e^{-ax})^2$

$+ \; a^2y^2 = -\frac{a^2}{4}(e^{ax} - e^{-ax})^2$

$\therefore \left(\frac{dy}{dx}\right)^2 - a^2y^2 = \frac{a^2}{4}(e^{2ax} + 2 + e^{-2ax})$

$- \frac{a^2}{4}(e^{2ax} - 2 + e^{-2ax})$

$= \frac{a^2}{4}2 + \frac{2a^2}{4} = a^2$

$\boxed{\tfrac{T}{2}}$

(iii) $\left(\frac{dy}{dx}\right)^2 = a^2y^2 + a^2$

$= a^2(y^2+1)$

Note Hence (use (ii))

$\therefore \frac{dy}{dx} = a\sqrt{y^2+1}$

$\frac{dx}{dy} = \frac{1}{a\sqrt{y^2+1}}$

$x = \int \frac{1}{a} \frac{dy}{\sqrt{y^2+1}}$

but $x = \frac{1}{a}\ln(y + \sqrt{y^2+1})$

$\int \frac{dy}{\sqrt{y^2+1}} = \ln(y + \sqrt{y^2+1}) + C$

$\boxed{\tfrac{T}{3}}$

STANDARD INTEGRALS

$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$

$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$

$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$

$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$

$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$

$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$

$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$

$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$

$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x + \sqrt{x^2-a^2}\right), \quad x > a > 0$

$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x + \sqrt{x^2+a^2}\right)$

NOTE : $\ln x = \log_e x, \quad x > 0$