FABER EST SUAE QUISQUE FORTUNAE

FORT STREET HIGH SCHOOL

Name:

Teacher: Please tick

- Mr Bayas
- Mr Fraser
- Mr Hayes

2010

HIGHER SCHOOL CERTIFICATE COURSE

ASSESSMENT TASK 4: TRIAL HSC

Mathematics Extension 2

TIME ALLOWED: 3 HOURS
(PLUS 5 MINUTES READING TIME)

Outcomes Assessed	Questions	Marks
Determines the important features of graphs of a wide variety of functions, including conic sections	2, 4	
Applies appropriate algebraic techniques to complex numbers and polynomials	1, 3	
Applies further techniques of integration, such as slicing and cylindrical shells, integration by parts and recurrence formulae, to problems	5, 6	
Synthesises mathematical solutions to harder problems and communicates them in an appropriate form, resisted motion	8,7	

Question	1	2	3	4	5	6	7	8	Total	%
Marks	/15	/15	/15	/15	/15	/15	/15	/15	/120	

Directions to candidates:

- Attempt all questions
- The marks allocated for each question are indicated
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board – approved calculators may be used
- Each new question is to be started in a new booklet

Question 1 (15 Marks) **Start a new booklet** **Marks**

a) Let $z = 5 - 6i$ and $w = 3 + 4i$. Express the following in the form a + ib where a and b are real numbers.

(i) z^2 1

(ii) $\frac{z}{w}$ 2

b) (i) Express $w = 8 + 8i$ in modulus-argument form 1

(ii) Hence, or otherwise find all numbers z such that $z^5 = 8 + 8i$ giving your answer in modulus-argument form. 3

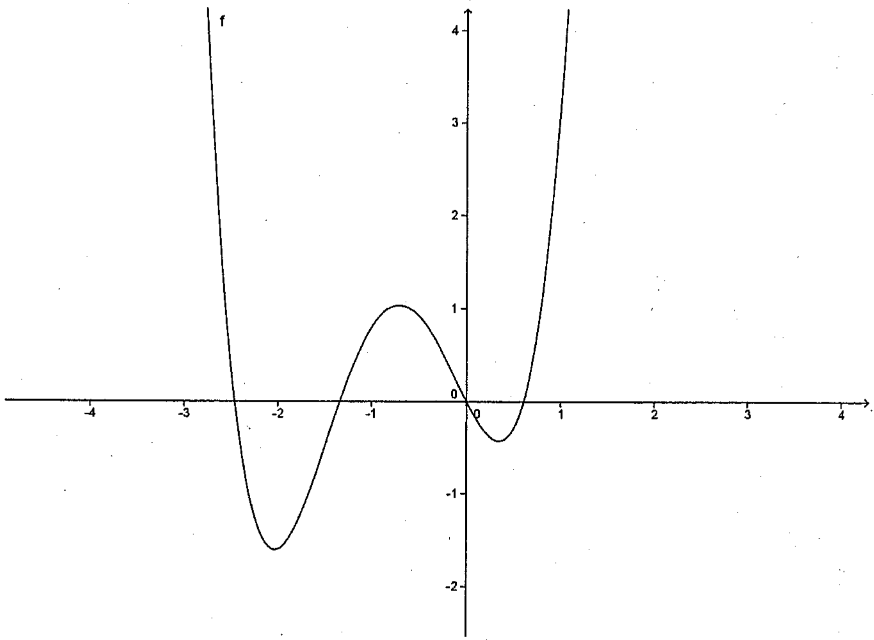
c) Sketch the region in the Argand diagram defined by 3
$|z - 2 + i| < 3$ *and* $-\frac{\pi}{3} \leq \arg(z - 2 + i) \leq \frac{\pi}{3}$
Indicate whether corner points are included or excluded. You do not need to find coordinates of the corner points or intercepts.

d) Find $\sqrt{1+i}$ in the form a + ib where a and b are real numbers. Hence find an exact value for tan $(\frac{\pi}{8})$. 3

e) Given Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$. The complex number z can be expressed in polar form as $z = re^{i\theta}$ where $r = |z|$ *and* $\theta = \arg(z)$. Use the polar form of z to find $\ln(z)$ and hence find $\ln(1 + i)$ in the form a + ib where a and b are real numbers. 2

Question 2 (15 Marks) **Start a new booklet** **Marks**

a) The diagram shows the graph of $y = f(x)$



Draw separate one third page sketches of the graphs of the following

(i) $y = \dfrac{1}{f(x)}$ 2

(ii) $y^2 = f(x)$ 2

(iii) $y = 2^{f(x)}$ 2

(iv) $y = f(\frac{1}{x})$ 2

Question 2 continued **Marks**

b) Consider the function $f(x) = \ln(2 + 2\cos(2x))$, $-2\pi \le x \le 2\pi$

(i) Show that the function f is even and the curve $y = f(x)$ is concave down for all values of x in its domain, except where its not defined. 3

(ii) Sketch using a third of a page, the graph of the curve $= f(x)$. 2

c) Find the coordinates of the points where the tangent to the curve $x^2 + 2xy + 3y^2 = 18$ is horizontal. 2

End of Question 2

Next question, Question 3 on the next page , page 4

Question 3 (15 Marks) **Start a new booklet** **Marks**

a) (i) Prove the theorem 2

If α is a zero of multiplicity r of the real polynomial equation $P(x) = 0$, then α is a zero of multiplicity $r-1$ of $P'(x) = 0$.

(ii) The polynomial equation $3x^5 - ax^2 + b = 0$ has a multiple root. 3

Show that $8788a^5 = 28125b^3$

b) The polynomial $P(z)$ is defined by 3

$$P(z) = z^4 - 2z^3 - z^2 + 2z + 10$$

Given that $z - 2 + i$ is a factor of $P(z)$, express $P(z)$ as a product of real quadratic factors.

c) (i) Show that $\cos(P+Q) + \cos(P-Q) = 2cosPcosQ$. 1

(ii) Let α and β be the roots of the equation $z^2\,sin^2\emptyset - zsin2\emptyset + 1 = 0$.

1. Show that $\alpha + \beta = 2\,cos\emptyset\;cosec\emptyset$ 1

2. Show that $\alpha^2 + \beta^2 = 2\cos 2\emptyset cosec^2\,\emptyset$ 1

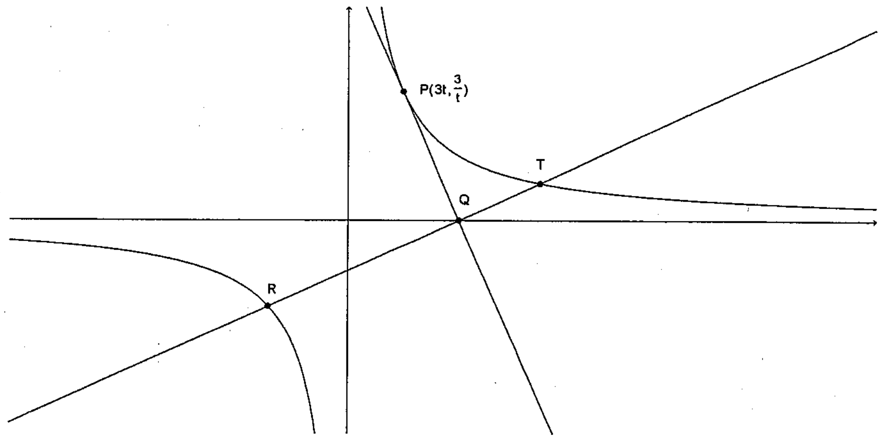
3. Hence by mathematical induction, 4

prove that if n is a positive integer then

$\alpha^n + \beta^n = 2cosn\emptyset cosec^n\emptyset$

Question 4 (15 Marks) **Start a new booklet** **Marks**

a) $P(3t, \frac{3}{t})$ is a point on the rectangular hyperbola xy = 9. The tangent at P cuts the x axis at Q and the line through Q, perpendicular to the tangent at P, cuts the hyperbola at the points R and T as shown



(i) Show that the equation of the tangent at P is $x + t^2y = 6t$. 2

(ii) Show that the line through Q, perpendicular to the tangent at P, has equation $t^2x - y = 6t^3$ 3

(iii) If M is the midpoint of RT, show M has coordinates $(3t, -3t^3)$. 3

(iv) Find the equation of the locus of M, as P moves on the curve xy = 9 . 1

b) The Hyperbola H has equation $x^2 - 3y^2 = 6$

Show that the equation of the normal to H at $P(2\sqrt{2}, \sqrt{2})$ is $3x + 2y = 8\sqrt{2}$. 2

c) The Points $M(acos\propto, bsin\propto)$ and $N(-asin\propto, bcos\propto)$ lie on the ellipse E: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Find the equations of the tangents at M and N and show these tangents intersect at the point $P\big(a(cos\propto - sin\propto), b(sin\propto + cos\propto)\big)$. 4

Question 5 (15 Marks) **Start a new booklet** **Marks**

a) Evaluate correct to 3 decimal places

$$\int_0^1 \frac{e^{2x}dx}{e^{4x}+1}$$ 2

b) Find $$\int \frac{dp}{\sqrt{9+8p-p^2}}$$ 2

c) Using the substitution $t = tan\frac{\theta}{2}$, find

$$\int \frac{2d\theta}{5-4sin\theta}$$ 3

d) Find $$\int \frac{x^5-7x^2+8}{x^3-8}\,dx$$ 4

e) If $I_n = \int_0^{\frac{\pi}{4}} sec^n x dx$ for n ≥ 0 4

(integral from zero to pi over 4 of secx to the power n dx)

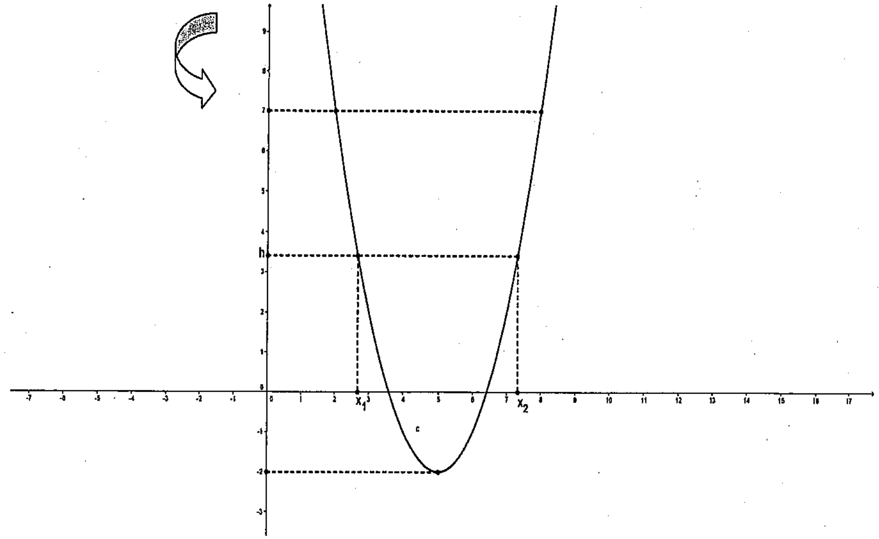
show that

$$I_n = \frac{(\sqrt{2})^{n-2}}{n-1} + \frac{n-2}{n-1} I_{n-2} \quad \text{for } n \geq 2$$

and deduce $I_6 = \frac{28}{15}$

Question 6 (15 Marks) **Start a new booklet** **Marks**

a) A flat top parabolic torus is formed by rotating the area inside the parabola $y = x^2 - 10x + 23$ between the lines $y = -2$ and $y = 7$ around the y axis.



The cross section at $y = h$ $where$ $-2 \leq h \leq 7$, is an annulus. The annulus has inner radius x_1 and outer radius x_2 where x_1 and x_2 are the solutions to $x^2 - 10x + 23$ = h

(i) Find x_1 and x_2 in terms of h 1

(ii) Find the area of the cross-section at height h, in terms of h. 2

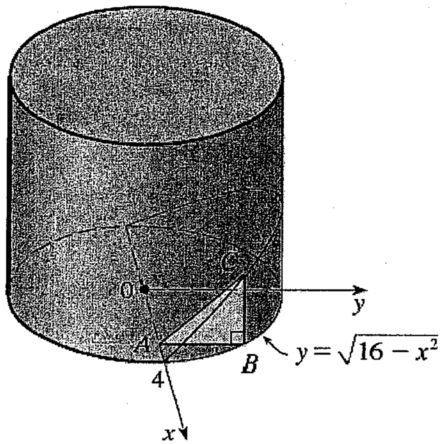
(iii) Find the volume of the flat top parabolic torus. 2
Leave answer in exact form.

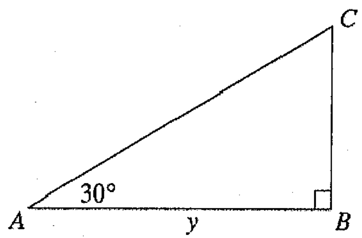
Question 6. Continued **Marks**

b) A wedge is cut out of a circular cylinder of radius 4 by two planes. One plane is perpendicular to the axis of the cylinder.
The other intersects the first at an angle of 30 along a diameter of the cylinder.

(i) Show the cross sectional area is $A(x) = \frac{16-x^2}{2\sqrt{3}}$ 2

(ii) Hence find the volume of the wedge. 3

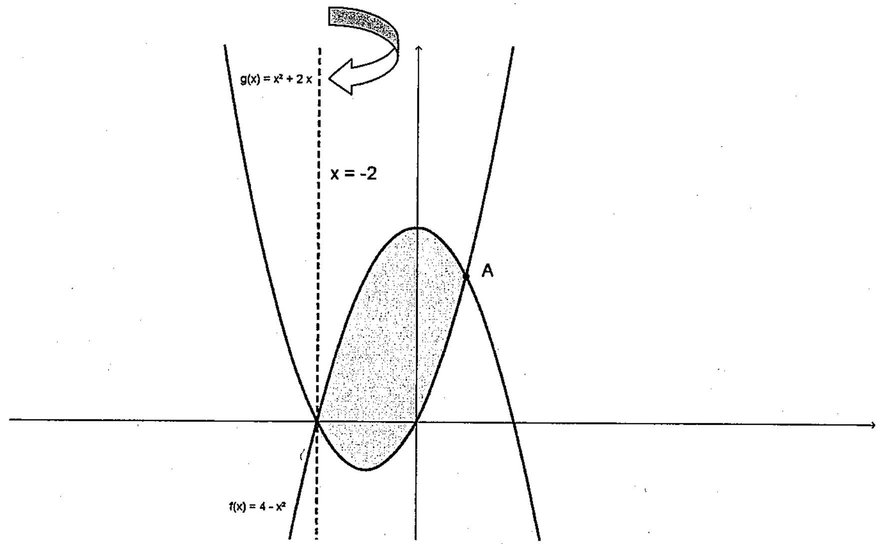




Question 6. continued **Marks**

c)

The lightly shaded region bounded by $y = 4 - x^2, y = x^2 + 2x$ is rotated about the line $x = -2$. The point A is the intersection of $y = 4 - x^2$ and $y = x^2 + 2x$ in the first quadrant.



(i) Find the coordinate of A 1

(ii) Use the method of cylindrical shells to express the volume of the resulting solid of revolution as an integral. 2

(iii) Evaluate the integral in part (ii), leave answer in exact form. 2

Question 7. (15 Marks) **Start a new booklet** **Marks**

a) A cannon ball of mass 1 kilogram is projected vertically upward from the origin with an initial speed of $20m/s$. The cannon ball is subjected to gravity $10ms^{-2}$ and air resistance $\frac{v^2}{20}$.
The upward equation of motion is
$\ddot{y} = -\frac{v^2}{20} - 10$

(i) Using $\ddot{y} = v\frac{dv}{dy}$ show that while the cannon ball Is rising $v^2 = 600e^{-\frac{y}{10}} - 200$ 3

(ii) Hence find the maximum height reached by the cannon ball correct to 2 decimal places. 1

(iii) Using $\ddot{y} = \frac{dv}{dt}$ find how long the cannon ball takes to reach this maximum height correct to 2 decimal places? 2

(iv) How fast is the cannon ball travelling when it returns to the origin correct to 2 decimal places? 3

b) A cylindrical water tank has a constant cross-sectional area A.
Water drains through a hole at the bottom of the tank.
The Volume of water decreases at a rate (- k times the cube root of h)
$\frac{dV}{dt} = -k\sqrt[3]{h}$ Where k is a positive constant and h is the depth of water.
Initially the tank is full and it takes T seconds to drain. Thus $h = h_0$ $when$ $t = 0$
And $h = 0$ $when$ $t = T$.

(i) Show that $\frac{dh}{dt} = -\frac{k}{A}\sqrt[3]{h}$ 2

(ii) By considering the equation for $\frac{dt}{dh}$ or otherwise
Show $h^2 = {h_0}^2\left(1 - \frac{t}{T}\right)^3$. 3

(iii) Suppose it takes 12 seconds for half the water to drain. How long does it take to empty the full tank? 1
to nearest second.

Question 8. (15 Marks) **Start a new booklet** **Marks**

a) Let α be a real number and suppose z is a complex number such that
$$z + \frac{1}{z} = 2\cos\alpha$$

(i) By reducing the above equation to a quadratic equation in z, solve for z and use de Moivre's theorem to show that 3
$z^n + \frac{1}{z^n} = 2\cos n\alpha$.

(ii) Let $w = z + \frac{1}{z}$. Prove that 2
$w^3 + w^2 - 2w - 2 = z + \frac{1}{z} + \left(z^2 + \frac{1}{z^2}\right) + \left(z^3 + \frac{1}{z^3}\right)$.

(iii) Hence, or otherwise , find all solutions of 3
$\cos\alpha + \cos 2\alpha + \cos 3\alpha = 0$, in the range $0 \le \alpha \le 2\pi$.

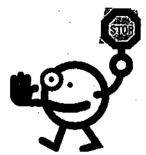
b) Prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$, 1

Hence evaluate $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$. 3

c) The area A of the surface of revolution generated by rotating a smooth arc $y = f(x), a \le x \le b$ around the x axis, is given by the integral formula 3

$$A = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx$$

Rotate the circle $x^2 + y^2 = r^2$ around the x axis and show that the surface Area of the sphere generated is $4\pi r^2$.



End of Examination

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1 $\boxed{15 \text{ Marks}}$

a) (i) $z^2 = (5-6i)(5-6i) = 25 - 60i + 36i^2 = -11 - 60i$ $\boxed{1}$

(ii) $\dfrac{z}{w} = \dfrac{5-6i}{3-4i} \times \dfrac{3+4i}{3+4i}$ $\boxed{1}$ $= \dfrac{15+20i-18i-24i^2}{9+16} = \dfrac{39+2i}{25}$ $\boxed{1}$

b) (i) $8+8i = \sqrt{8^2+8^2}\,\text{cis}\,(\tan^{-1}(1)) = 8\sqrt{2}\,\text{cis}\,\frac{\pi}{4}$ $\boxed{1}$

$|w| = 8\sqrt{2}$, $\arg(w) = \tan^{-1}(1) = \frac{\pi}{4}$

(ii) $z^5 = 8+8i = 8\sqrt{2}\,\text{cis}\left(\frac{\pi}{4} + 2k\pi\right)$ $k = 0,1,2,3,4$ unique

$z = (8\sqrt{2})^{\frac{1}{5}}\,\text{cis}\left(\frac{1}{5}\left(\frac{\pi}{4} + 2k\pi\right)\right)$.

$\boxed{1}$ $2k\pi$
$\boxed{1}$ $2^{\frac{7}{10}}$
$\boxed{1}$ all correct arguements.

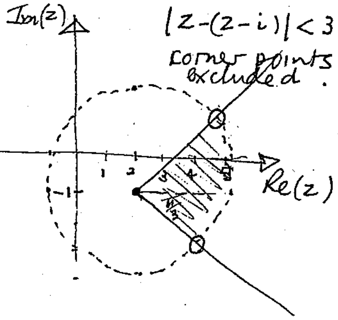
$z_0 = 2^{\frac{7}{10}}\,\text{cis}\,\frac{\pi}{20}$, $z_1 = 2^{\frac{7}{10}}\,\text{cis}\,\frac{9\pi}{20}$ $z_2 = 2^{\frac{7}{10}}\,\text{cis}\,\frac{17\pi}{20}$

$z_3 = 2^{\frac{7}{10}}\,\text{cis}\,\frac{5\pi}{4}$, $z_4 = 2^{\frac{7}{10}}\,\text{cis}\,\cancel{\frac{8\pi}{5}}\ \frac{33\pi}{20}$

c) $|z-(2-i)| < 3$

corner points excluded.

$\boxed{1}$ circle
$\boxed{1}$ arg
$\boxed{1}$ region and corner points



d) $\sqrt{1+i} = a+ib$ $a,b>0$

① $\therefore a^2 - b^2 = 1$ $2ab = 1$ $\therefore b = \frac{1}{2a}$

$\therefore a^2 - \left(\frac{1}{2a}\right)^2 = 1 \Rightarrow 4a^4 - 1 = 4a^2$

$\therefore 4a^4 - 4a^2 - 1 = 0$ let $p = a^2$

$\therefore 4p^2 - 4p - 1 = 0$ $\leftarrow \boxed{1}$

$\therefore p = \frac{4 \pm \sqrt{16+16}}{8} = \frac{1 \pm \sqrt{2}}{2}$ $p > 0$

$\therefore p = \frac{1+\sqrt{2}}{2}$ $\therefore a = \sqrt{\frac{1+\sqrt{2}}{2}}$, $a > 0$

Now $a^2 - b^2 = 1$ from ①

$\therefore \frac{1+\sqrt{2}}{2} - b^2 = 1$ $\therefore b^2 = \frac{\sqrt{2}-1}{2}$

$\therefore a = \sqrt{\frac{1+\sqrt{2}}{2}}$, $b = \sqrt{\frac{\sqrt{2}-1}{2}}$ $\leftarrow \boxed{1}$

$\sqrt{1+i}$ has arg of $\frac{\pi}{8}$ $\boxed{1}$

$\therefore \tan\frac{\pi}{8} = \frac{b}{a} = \frac{\sqrt{\sqrt{2}-1}}{\sqrt{1+\sqrt{2}}} = \sqrt{2} - 1$

Simplify

$\sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1}} = \sqrt{2} - 1$

e) Euler

$e^{i\theta} = \text{cis}\,\theta$

$\therefore z = re^{i\theta}$

if $z = 1+i$ $|z| = \sqrt{2}$ $\arg z = \frac{\pi}{4}$

$\boxed{1}$ $\therefore z = \sqrt{2}\,\text{cis}\,\frac{\pi}{4} = \sqrt{2}e^{i\frac{\pi}{4}}$

$\ln z = \ln\sqrt{2}e^{i\frac{\pi}{4}}$

$= \ln\sqrt{2} + \ln e^{i\frac{\pi}{4}}$

$= \frac{1}{2}\ln 2 + i\frac{\pi}{4}$

$\therefore a = \ln\sqrt{2}$ or $\frac{1}{2}\ln 2$

$b = \frac{\pi}{4}$.

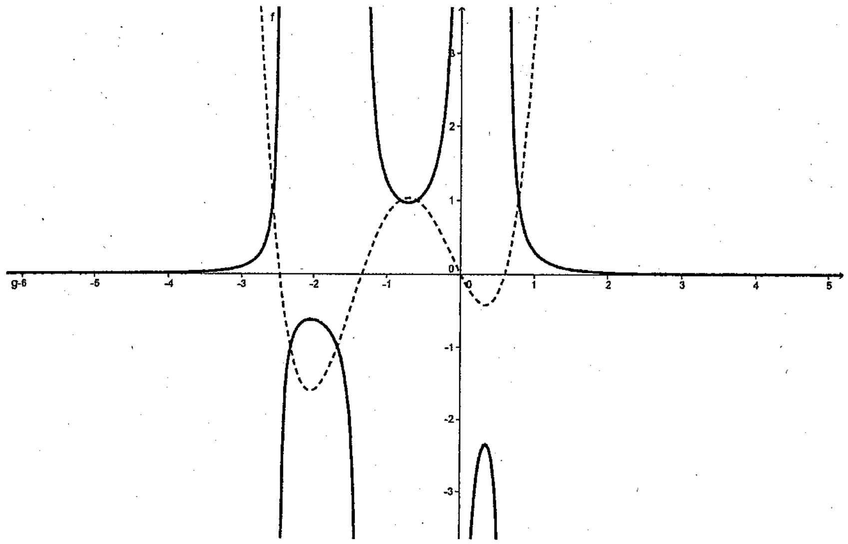
$\boxed{1}$

ie $\ln z = \frac{1}{2}\ln 2 + \frac{\pi}{4}i$

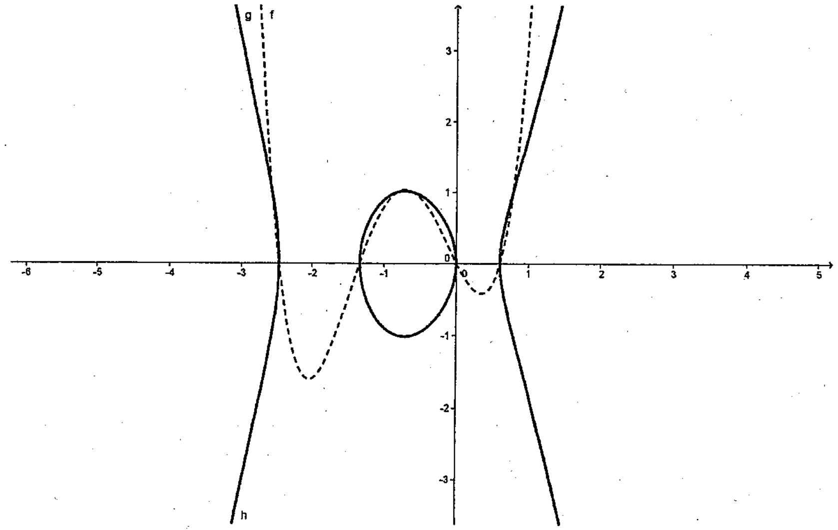
where $z = 1+i$

Q2 a)

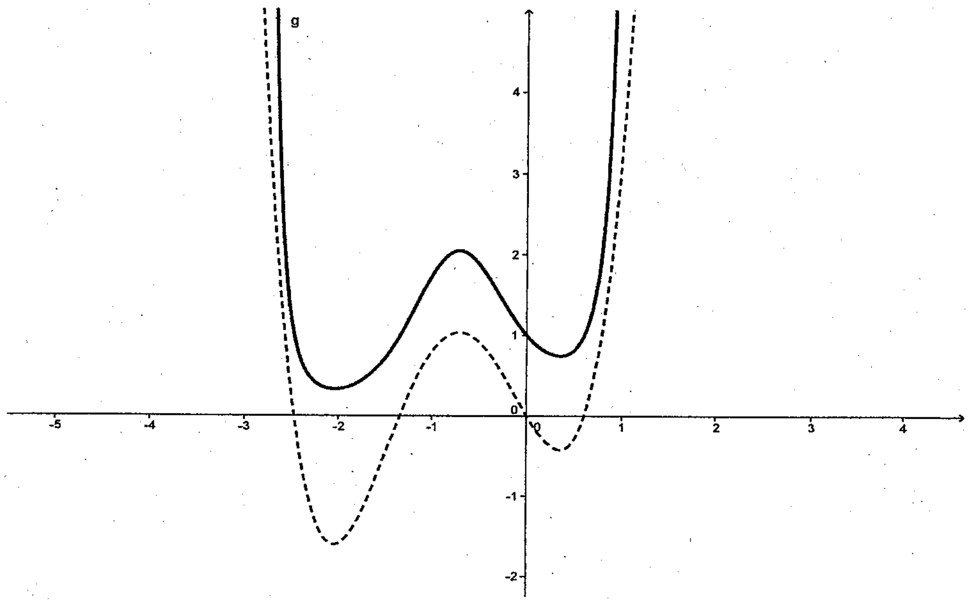
(i)



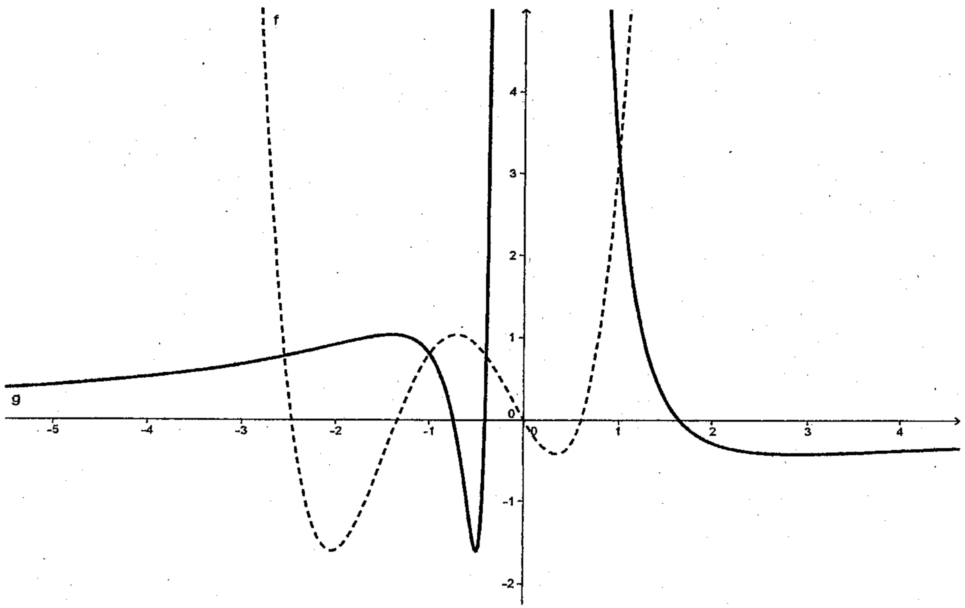
(ii)



(iii)



iv)



Question 2

b) $f(x) = \ln(2+2\cos(2x))$

(i) $f(-x) = \ln(2+2\cos(-2x))$ as $\cos(-2x) = \cos 2x$

$= \ln(2+2\cos(2x))$

$\boxed{1}$ $= f(x)$ ∴ $f(x)$ is even

$f'(x) = \frac{-4\sin 2x}{2+2\cos 2x} = \frac{-2\sin(2x)}{1+\cos 2x}$

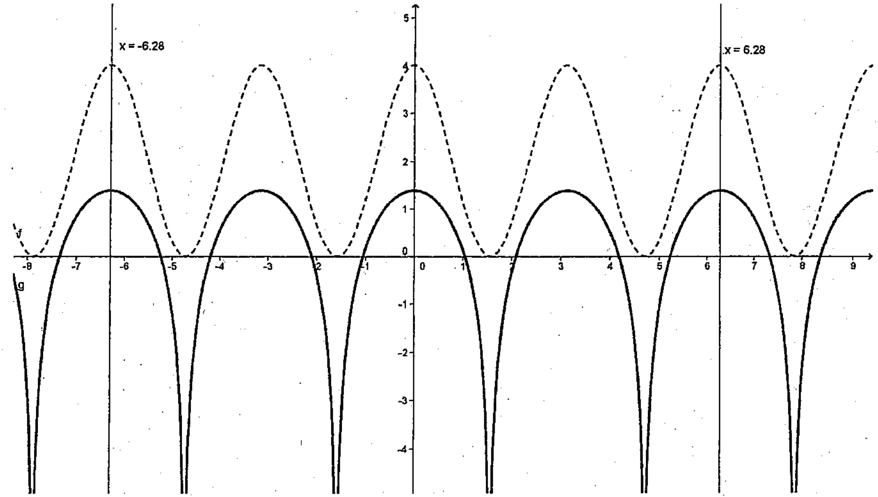
$f''(x) = \frac{(1+\cos 2x).-4\cos 2x + (2\sin 2x).-2\sin 2x}{(1+\cos 2x)^2}$

$= \frac{-4\cos 2x - 4\cos^2 2x - 4\sin^2 2x}{(1+\cos 2x)^2}$ $\quad -1 \le \cos 2x \le 1$

$= \frac{-4\cancel{(1+\cos 2x)}}{(1+\cos 2x)^2} = \frac{-4}{1+\cos 2x}$

$\boxed{2}$ ∴ $f''(x) < 0$ except where $\cos 2x = -1$ where not defined.

(ii) Sketch. $\boxed{2}$



Q2 c) $x^2+2xy+3y^2=18$

∴ $2x + 2x\frac{dy}{dx} + 2y + 6y^2\frac{dy}{dx} = 0$ ∴ $\frac{dy}{dx} = \frac{-(2x+2y)}{2x+6y^2}$ $\boxed{1}$ $\frac{dy}{dx}$ implicit correct

If tangent horizontal $\frac{dy}{dx}=0$ ∴ $2x+2y=0$ ∴ $x=-y$

Sub into original eqn

∴ $y^2-2y^2+3y^2=18$ ∴ $2y^2=18$ $y^2=9$ ∴ $y=\pm3$.

∴ points $(3,-3)$ and $(-3,3)$. $\boxed{1}$ correct points.

Question 3 [15 Marks]

a) (i) $P(x) = (x-\alpha)^r Q(x)$ $\therefore P'(x) = r(x-\alpha)^{r-1}Q(x) + (x-\alpha)^r Q'(x)$

[1] $\therefore P'(x) = (x-\alpha)^{r-1}[rQ(x) + (x-\alpha)Q'(x)]$

[1] $\therefore \alpha$ is a root of multiplicity $r-1$ of $P'(x)=0$.

(ii) $P(x) = 3x^5 - ax^2 + b = 0$

$\therefore P'(x) = 15x^4 - 2ax = 0$ $\therefore x(15x^3 - 2a) = 0$

[1] $\therefore x = 0$ or $15x^3 = 2a$ $\therefore x = \left(\frac{2a}{15}\right)^{\frac{1}{3}}$

[1] Sub into $P(x) = 3\left(\frac{2a}{15}\right)^{\frac{5}{3}} - a\left(\frac{2a}{15}\right)^{\frac{2}{3}} + b = 0$

$\therefore 3a^{\frac{5}{3}}\left(\frac{2}{15}\right)^{\frac{5}{3}} - a^{\frac{5}{3}}\left(\frac{2}{15}\right)^{\frac{2}{3}} + b = 0$

$\therefore a^{\frac{5}{3}}\left[3\left(\frac{2}{15}\right)^{\frac{5}{3}} - \left(\frac{2}{15}\right)^{\frac{2}{3}}\right] = -b$

$a^{\frac{5}{3}}\left[3\left(\frac{2}{15}\right)^{\frac{2}{3}}\left[\frac{2}{15} - \frac{1}{3}\right]\right] = -b$

$a^{\frac{5}{3}}\left[3\cdot\left(\frac{2}{15}\right)^{\frac{2}{3}}\cdot\frac{-3}{15}\right] = -b$ Cube both sides

$\therefore a^5 \cdot 3^3\left(\frac{2}{15}\right)^2\left(\frac{-3}{15}\right)^3 = (-b)^3$

[1] $\therefore a^5 \cdot 3^3 \cdot 2^2 \cdot (-3)^3 = 15^2 \cdot 15^3 \cdot -b^3$

$-\cancel{237276}\,2916\,a^5 = -759375\,b^3$

$\therefore \cancel{8788}\,108\,a^5 = 28125\,b^3 \Rightarrow 12a^5 = 3125\,b^3$

b) $z-2+i$ factor $\therefore z-(2-i) \to 2-i$ zero. Real coeff.

[1] $\therefore 2+i$ is also a zero, Hence $(z-(2+i))(z-(2-i))$ is a factor

[1] ie $z^2 - 4z + 5$ is a factor

$$\begin{array}{r l}
 & z^2 + 2z + 2 \\
z^2 - 4z + 5 \,) & z^4 - 2z^3 - z^2 + 2z + 10 \\
 & z^4 - 4z^3 + 5z^2 \\
\hline
 & 2z^3 - 6z^2 + 2z + 10 \\
 & 2z^3 - 8z^2 + 10z \\
\hline
 & 2z^2 - 8z + 10 \\
 & 2z^2 - 8z + 10 \\
\hline
 & 0
\end{array}$$

[1] $\therefore P(z) = (z^2 - 4z + 5)(z^2 + 2z + 2)$

(product of real quadratic factors)

Question 3 (cont)

c) (i) $\cos(P+Q) + \cos(P-Q)$

[1] $= \cos P\cos Q - \cancel{\sin P\sin Q} + \cos P\cos Q + \cancel{\sin P\sin Q}$

$= 2\cos P\cos Q$.

(ii) $z^2\sin^2\phi - z\sin 2\phi + 1 = 0$ $\alpha\beta = \frac{1}{\sin^2\phi} = \operatorname{cosec}^2\phi$.

[1] 1. $\alpha+\beta = \frac{\sin 2\phi}{\sin^2\phi} = \frac{2\cancel{\sin\phi}\cos\phi}{\sin^{\cancel{2}}\phi} = \frac{2\cos\phi}{\sin\phi}$

$= 2\cos\phi\operatorname{cosec}\phi$,

[1] 2. $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= (2\cos\phi\operatorname{cosec}\phi)^2 - 2\operatorname{cosec}^2\phi$

$= (2\cos^2\phi - 1)\operatorname{cosec}^2\phi$

$= \cos 2\phi\operatorname{cosec}^2\phi$.

3. From 1. and 2. the formula is true for $n=1$ and $n=2$

Assume true for $n=k,\ k-1$ (for all n $2<n\leq k$)

ie $\alpha^k+\beta^k = 2\cos k\phi\operatorname{cosec}^k\phi$, $\alpha^{k-1}+\beta^{k-1} = 2\cos(k-1)\phi\operatorname{cosec}^{k-1}\phi$

Now prove true for $n=k+1$.

[1] ie $\alpha^{k+1}+\beta^{k+1} = 2\cos(k+1)\phi\operatorname{cosec}^{k+1}\phi$,

Multiply original equation by z^{k-1} (ii).

$\therefore z^{k+1}\sin^2\phi - z^k\sin 2\phi + z^{k-1} = 0$

Sub in α, β $\therefore \alpha^{k+1}\sin^2\phi - \alpha^k\sin 2\phi + \alpha^{k-1} = 0$

[1] $\beta^{k+1}\sin^2\phi - \beta^k\sin 2\phi + \beta^{k-1} = 0$

add (rearrange)

$(\alpha^{k+1}+\beta^{k+1})(\sin^2\phi) = (\alpha^k+\beta^k)\sin 2\phi - \alpha^{k-1} - \beta^{k-1}$

using assumption $= (2\cos k\phi\operatorname{cosec}^k\phi)\sin 2\phi - (\alpha^{k-1}+\beta^{k-1})$

[1] divide by $\sin^2\phi$

$\therefore \alpha^{k+1}+\beta^{k+1} = 2\cos k\phi\operatorname{cosec}^k\phi\cdot\frac{\sin 2\phi}{\sin^2\phi} - \frac{2\cos(k-1)\phi\operatorname{cosec}^{k-1}\phi}{\sin^2\phi}$

$= 4\cos k\phi\operatorname{cosec}^{k+1}\phi\cos\phi - 2\cos(k-1)\phi\operatorname{cosec}^{k+1}\phi$

$= 2\operatorname{cosec}^{k+1}\phi\,[2\cos k\phi\cos\phi - \cos(k-1)\phi]$

[1] $= 2\operatorname{cosec}^{k+1}\phi\,[\cos(k\phi+\phi) + \cancel{\cos(k\phi-\phi)} - \cancel{\cos(k-1)\phi}]$

$= 2\operatorname{cosec}^{k+1}\phi\cos(k+1)\phi = RHS$

Hence since formula is true for $n=1, 2$ and with our assumptions on interval $2<n\leq k$ true for $n=k+1$, so by the principle of mathematical induction $\alpha^n+\beta^n = 2\operatorname{cosec}^n\phi\cos n\phi$ for integers

Question 5 $\boxed{15 \text{ Marks}}$

a) $\int_0^1 \frac{e^{2x}\,dx}{e^{4x}+1}$ let $u = e^{2x}$ $du = 2e^{2x}dx$ $\therefore \frac{1}{2}du = e^{2x}dx$ $x=0$ $u=1$, $x=1$ $u=e^2$ $\boxed{1}$ Sub + bounds

$\therefore \frac{1}{2}\int_1^{e^2} \frac{du}{u^2+1} = \frac{1}{2}\tan^{-1}u\Big]_1^{e^2} = \frac{1}{2}\left[\tan^{-1}(e^2) - \tan^{-1}(1)\right] \doteqdot 0.325$ $\boxed{1}$ - answer

b) $\int \frac{dp}{\sqrt{9+8p-p^2}} = \int \frac{dp}{\sqrt{-(p^2-8p-9)}} = \int \frac{dp}{\sqrt{-[(p-4)^2-25]}}$ $\boxed{1}$ compl. square

$\boxed{SI}$ $= \int \frac{dp}{\sqrt{25-(p-4)^2}} = \sin^{-1}\left(\frac{p-4}{5}\right) + c$ $\boxed{1}$ use SI correc

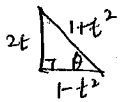
c) $\int \frac{2\,d\theta}{5-4\sin\theta} = \int \frac{\frac{2.2dt}{1+t^2}}{\frac{5(1+t^2)-4.2t}{1+t^2}} = \int \frac{4dt}{5t^2-8t+5}$ $\boxed{1}$ Sub

$t = \tan\frac{\theta}{2}$

$\frac{dt}{d\theta} = \frac{1}{2}\sec^2\frac{\theta}{2} = \frac{1}{2}(1+\tan^2\frac{\theta}{2})$

$\therefore d\theta = \frac{2dt}{1+t^2}$ $\boxed{1}$ $d\theta/dt$

$= \frac{4}{5}\int \frac{dt}{t^2-\frac{8}{5}t+1} = \frac{4}{5}\int \frac{dt}{(t-\frac{4}{5})^2-(\frac{4}{5})^2+1}$



$\sin\theta = \frac{2t}{1+t^2}$

$= \frac{4}{5}\int \frac{dt}{(t-\frac{4}{5})^2+\frac{9}{25}} = \frac{4}{5}\cdot\frac{5}{3}\tan^{-1}\left(\frac{t-\frac{4}{5}}{\frac{3}{5}}\right)$

$= \frac{4}{3}\tan^{-1}\left(\frac{5t-4}{3}\right) + c$ $\boxed{1}$ Answer

d) $\int \frac{x^5-7x^2+8}{x^3-8}dx = \int x^2 dx + \int \frac{x^2+8}{x^3-8}dx$ $\boxed{1}$ division

$$\begin{array}{r} x^2 \\ x^3-8 \enclose{longdiv}{x^5-7x^2+8} \\ \underline{x^5-8x^2} \\ x^2+8 \end{array}$$

$= \frac{x^3}{3} + \int \frac{(x^2+8)\,dx}{(x-2)(x^2+2x+4)}$ ← PF

$= \frac{x^3}{3} + \int \frac{1}{x-2} - \frac{2}{x^2+2x+4}\,dx$ $\boxed{1}$ $\boxed{1}$ PF

PF $\frac{x^2+8}{(x-2)(x^2+2x+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4}$

$\therefore x^2+8 = A(x^2+2x+4) + (Bx+C)(x-2)$

let $x=2$

$\therefore 12 = 12A$ $\therefore \boxed{A=1}$

Equate Coeff x^2

$A+B=1$ $\therefore \boxed{B=0}$

Consts.

$8 = 4A-2C$

$4 = -2C$ $\therefore \boxed{C=-2}$

$= \frac{x^3}{3} + \ln(x-2) - 2\int \frac{dx}{(x+1)^2+3}$

$= \frac{x^3}{3} + \ln(x-2) - \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + C$ $\boxed{1}$

Question 4 $\boxed{15 \text{ Marks}}$

a) (i) $y = 9x^{-1}$ $y' = \frac{-9}{x^2}$ at $x = 3t$, $y' = \frac{-9}{9t^2} = -\frac{1}{t^2}$ $\boxed{1}$ slope

Equation of tangent $y - \frac{3}{t} = -\frac{1}{t^2}(x-3t)$ $\boxed{1}$ eqn

$\therefore t^2y - 3t = -x + 3t$ ie $x + t^2y = 6t$.

(ii) At Q $y=0$ $\therefore x = 6t$ ie $Q(6t, 0)$ $\boxed{1}$ Q

perpendicular slope $m = t^2$ $\therefore y - 0 = t^2(x-6t)$ $\boxed{1}$ slope $\boxed{1}$ eqn.

$\therefore t^2x - y = 6t^3$.

(iii) Solving $t^2x - y = 6t^3$ and $xy = 9$ for R, T $\boxed{1}$ solve

$\therefore t^2x - \frac{9}{x} = 6t^3$ ie $t^2x^2 - 6t^3x - 9 = 0$ $\boxed{1}$ roots

Sum of roots $\alpha+\beta = \frac{-b}{a} = \frac{6t^3}{t^2} = 6t$ ie $\frac{\alpha+\beta}{2} = 3t$

Sub $x = 3t$ into line $t^2x - y = 6t^3$ $\therefore y = t^2.3t - 6t^3 = -3t^3$

$\therefore$ Midpoint $(3t, -3t^3)$. $\boxed{1}$ Midpt.

(iv) Locus of $M(3t, -3t^3)$ $\therefore x = 3t \to t = \frac{x}{3}$

$y = -3t^3 = -3\left(\frac{x}{3}\right)^3$ $\therefore \boxed{y = \frac{-x^3}{9}}$ ← locus $\boxed{1}$ locus

b) $x^2 - 3y^2 = 6$ $\therefore 2x - 6y\frac{dy}{dx} = 0$ $\therefore \frac{dy}{dx} = \frac{2x}{6y} = \frac{x}{3y}$

$\frac{dy}{dx}\Big]_P = \frac{2\sqrt{2}}{3\sqrt{2}} = \frac{2}{3}$ $\therefore$ slope of normal $= -\frac{3}{2}$ as $m_1m_2 = -1$ $\boxed{1}$ slope N

$\therefore y - \sqrt{2} = -\frac{3}{2}(x - 2\sqrt{2})$ is eqn of normal $\boxed{1}$ eqn

$2y - 2\sqrt{2} = -3x + 6\sqrt{2}$

ie $3x + 2y = 8\sqrt{2}$.

c) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\therefore \frac{2x}{a^2} + \frac{2y}{b^2}\cdot\frac{dy}{dx} = 0$ $\therefore \frac{dy}{dx} = \frac{-b^2x}{a^2y}$

$\boxed{1} \to ①$ At $M(a\cos\alpha, b\sin\alpha)$ $\therefore m = \frac{-b^2}{a^2}\cdot\frac{a\cos\alpha}{b\sin\alpha} = \frac{-b\cos\alpha}{a\sin\alpha}$

$\boxed{1} \to ②$ At $N(-a\sin\alpha, b\cos\alpha)$ $\therefore m = \frac{-b^2}{a^2}\cdot\frac{-a\sin\alpha}{b\cos\alpha} = \frac{b\sin\alpha}{a\cos\alpha}$.

$\boxed{1}$ P_x $\therefore$ Eqn of Tangent at M

$\boxed{1}$ P_y $y - b\sin\alpha = -\frac{b\cos\alpha}{a\sin\alpha}(x - a\cos\alpha)$ $\therefore ay\sin\alpha - ab\sin^2\alpha = -xb\cos\alpha + ab\cos^2\alpha$

$\therefore ay\sin\alpha + xb\cos\alpha = ab$ ie $\frac{y\sin\alpha}{b} + \frac{x\cos\alpha}{a} = 1$ ← ①

Eqn of Tangent at N

$y - b\cos\alpha = \frac{b\sin\alpha}{a\cos\alpha}(x + b\cos\alpha)$ $\therefore ay\cos\alpha - ab\cos^2\alpha = xb\sin\alpha + ab\sin^2\alpha$

$\therefore ay\cos\alpha - xb\sin\alpha = ab$ ie $\frac{y\cos\alpha}{b} - \frac{x\sin\alpha}{a} = 1$ ← ②

① × $\sin\alpha$ $\frac{y\sin^2\alpha}{b} + \frac{x\cos\alpha\sin\alpha}{a} = \sin\alpha$

② × $\cos\alpha$ $\frac{y\cos^2\alpha}{b} - \frac{x\sin\alpha\cos\alpha}{a} = \cos\alpha$ $\Big]$ $\therefore \frac{y}{b} = \sin\alpha + \cos\alpha$ ie $y = (\sin\alpha + \cos\alpha)b$.

Sub $y = b(\sin\alpha + \cos\alpha)$ into either $\frac{x\cos\alpha}{a} + \frac{b(\sin\alpha+\cos\alpha)\sin\alpha}{b} = 1$

e) $I_n = \int_0^{\frac{\pi}{4}} \sec^n x\,dx = \int_0^{\frac{\pi}{4}} \sec^{n-2} x \sec^2 x\,dx$

$u = \sec^{n-2} x \qquad dV = \sec^2 x\,dx \qquad \boxed{1}\ u = \sec^{n-2}x$

$du = (n-2)\sec^{n-3} x \sec x \tan x\,dx \qquad \therefore V = \tan x$

$\therefore \int_0^{\frac{\pi}{4}} \sec^n x\,dx = \left. \tan x \sec^{n-2} x \right]_0^{\frac{\pi}{4}} - (n-2)\int \sec^{n-2} x \tan^2 x\,dx$

$\therefore \quad I_n = (\sec \frac{\pi}{4})^{n-2} - 0 - (n-2)\int \sec^{n-2} x (\sec^2 x - 1)\,dx \qquad \boxed{1}$ rearr

$= (\sqrt{2})^{n-2} - (n-2)\int_0^{\frac{\pi}{4}} \sec^n x - \sec^{n-2} x\,dx$

$\therefore (\overset{(n-1)}{n-2+1}) I_n = (\sqrt{2})^{n-2} + (n-2) I_{n-2}$

$\therefore \quad I_n = \frac{(\sqrt{2})^{n-2}}{n-1} + \left(\frac{n-2}{n-1}\right) I_{n-2} \qquad \boxed{1}\ I_n$

$\therefore I_6 = \frac{\sqrt{2}^4}{5} + \frac{4}{5} I_4$

$= \frac{(\sqrt{2})^4}{5} + \frac{4}{5}\left[\frac{(\sqrt{2})^2}{3} + \frac{2}{3} I_2\right] = \frac{4}{5} + \frac{4}{5}\left[\frac{2}{3} + \frac{2}{3}\left[\frac{\sqrt{2}^0}{1} + \frac{0}{1} I_0\right]\right.$

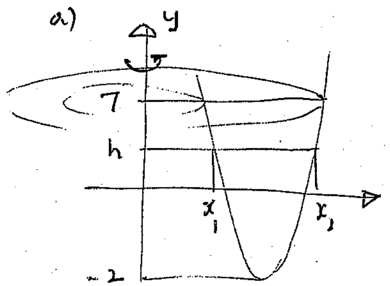
$= \frac{4}{5} + \frac{4}{5}\left[\frac{2}{3} + \frac{2}{3}\right] = \frac{4}{5} + \frac{4}{5}\left[\frac{4}{3}\right]$

$= \frac{4}{5} + \frac{16}{15}$

$= \frac{28}{15} \qquad \boxed{1}$ sub

Question 6 [15 Marks]

a)



$y = (x-5)^2 - 2$

$h = (x-5)^2 - 2$

$\therefore x = 5 \pm \sqrt{h+2}$

(i) $x_1 = 5 - \sqrt{h+2}$

$x_2 = 5 + \sqrt{h+2} \qquad \boxed{1}\ x_1, x_2$

(ii) $A = \pi(R^2 - r^2) = \pi[(R+r)(R-r)]$, $R+r = x_2 + x_1$, $R-r = x_2 - x_1$ $\qquad \boxed{1}$ simpl

$= \pi[(10)(2\sqrt{h+2})]$

$A = 20\pi\sqrt{h+2} \qquad \boxed{1}\ A(h)$

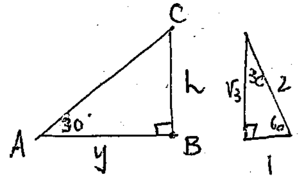
(iii) $\Delta V = A(h)\Delta h = 20\pi\sqrt{h+2}\,\Delta h$

$\therefore V = \lim_{\Delta h \to 0} \sum_{h=-2}^{7} 20\pi\sqrt{h+2}\,\Delta h = 20\pi\int_{-2}^{7} \sqrt{h+2}\,dh \qquad \boxed{1}$ Develop formula

$= 20\pi \cdot \frac{2}{3}\left[(h+2)^{\frac{3}{2}}\right]_{-2}^{7} = \frac{40\pi}{3}\left[9^{\frac{3}{2}} - 0\right] \qquad \boxed{1}$ Answer

$= \frac{\overset{9}{\cancel{27}} \times 40\pi}{\cancel{3}} = 360\pi\ u^3$

b) $A = \frac{1}{2}bh$



$\tan 30 = \frac{h}{y}$

$h = y\tan 30 = \frac{y}{\sqrt{3}}$

$y = \sqrt{16 - x^2}$

(i) $\therefore A = \frac{1}{2} y \cdot \frac{y}{\sqrt{3}} = \frac{1}{2}\sqrt{16-x^2} \cdot \frac{1}{\sqrt{3}}\sqrt{16-x^2} \qquad \boxed{1}\ \frac{y}{\sqrt{3}}$

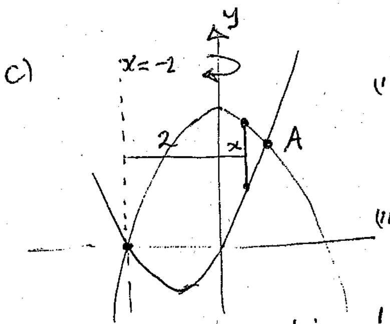
$\therefore A(x) = \frac{16 - x^2}{2\sqrt{3}} \qquad \boxed{1}\ A(x)$

(ii) $V = \int_{-4}^{4} A(x)\,dx = \int_{-4}^{4} \frac{16-x^2}{2\sqrt{3}}\,dx \qquad \boxed{1}$ Volume

$= \cancel{2}\int_0^4 \frac{16-x^2}{\cancel{2}\sqrt{3}}\,dx = \frac{1}{\sqrt{3}}\left[16x - \frac{x^3}{3}\right]_0^4 \qquad \boxed{1}\ I$

$= \frac{128}{3\sqrt{3}} = \frac{128\sqrt{3}}{9}\ u^3 \qquad \boxed{1}$ Answer

c)



(i) $4 - x^2 = x^2 + 2x \quad \therefore 2x^2 + 2x - 4 = 0$

$\therefore 2(x^2 + x - 2) = 0 \quad 2(x-1)(x+2) = 0$

$\therefore x = 1, -2 \quad \therefore A(1, 3) \qquad \boxed{1}\ A$

(ii) $\Delta V = 2\pi r h\,\Delta x$

$\Delta V = 2\pi(2+x)\left[4 - x^2 - (x^2 + 2x)\right]\Delta x$, with $4 - 2x^2 - 2x$ $\qquad \boxed{1}\ \Delta$

$= 4\pi(2+x)(2 - x - x^2)\,\Delta x$

$= 4\pi\left[4 - 3x^2 - x^3\right]\Delta x \qquad \boxed{1}\ V$

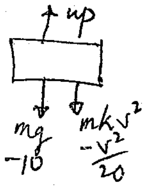
(ii) cont. $\therefore V = \lim_{\Delta x \to 0} \sum_{x=-2}^{1} 4\pi(4 - 3x^2 - x^3)\,\Delta x = 4\pi\int_{-2}^{1} 4 - 3x^2 - x^3\,dx$

(iii) $V = 4\pi\left[4x - x^3 - \frac{x^4}{4}\right]_{-2}^{1} = \left[4 - 1 - \frac{1}{4} - \left[\cancel{-8+8}-4\right]\right] \qquad \boxed{1}$ Eval. I

$= 4\pi \cdot \left[2\frac{3}{4} + 4\right] = 6\frac{3}{4} \cdot 4\pi = 27\pi\ u^3 \qquad \boxed{1}\ A$

Question 7 | 15 Marks

a)



$\frac{4}{4}$

$m\ddot{y} = -mg - \frac{mv^2}{20} \quad \therefore \ddot{y} = -10 - \frac{v^2}{20}$

$\therefore v\frac{dv}{dy} = \frac{-200-v^2}{20} \quad \therefore \frac{dv}{dy} = \frac{-200-v^2}{20v}$

$\therefore \frac{dy}{dv} = \frac{-20v}{200+v^2} \quad \therefore y\Big]_0^y = -10\int_{20}^{v} \frac{2v}{200+v^2}\,dv$

$\therefore y = -10\ln(200+v^2) + 10\ln 600 = 10\ln\left(\frac{600}{200+v^2}\right)$

$\therefore -\frac{y}{10} = \ln\left(\frac{200+v^2}{600}\right) \Rightarrow e^{-\frac{y}{10}} = \frac{200+v^2}{600}$

ie $600e^{-\frac{y}{10}} = 200+v^2 \Rightarrow v^2 = 600e^{-\frac{y}{10}} - 200$

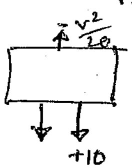
(ii) Max height $v=0 \quad \therefore 200 = 600e^{-\frac{y}{10}} \Rightarrow \frac{1}{3} = e^{-\frac{y}{10}}$

$\ln\frac{1}{3} = \frac{-y}{10} \quad \therefore y = -10\ln\frac{1}{3} \doteqdot 10.99\text{m} \quad (10\ln 3)$ $\boxed{1}$

(iii) $\frac{dv}{dt} = -\frac{v^2-200}{20} \quad \therefore dt = \frac{-20\,dv}{v^2+200} \quad dt\Big]_0^t = \frac{-20}{\sqrt{200}}\tan^{-1}\frac{v}{\sqrt{200}}\Big]_{20}^{0}$ $\boxed{1}$

$\therefore t = \frac{-20}{\sqrt{200}}\tan^{-1}0 + \frac{20}{\sqrt{200}}\tan^{-1}\left(\frac{20}{\sqrt{200}}\right) \approx 1.35\text{ secs}$ $\boxed{1}$ Answer

(iv)



$\ddot{y} = 10 - \frac{v^2}{20} \quad \therefore v\frac{dv}{dy} = \frac{200-v^2}{20}$

$\therefore \frac{dy}{dv} = \frac{20v}{200-v^2} \quad dy\Big]_0^{10\ln 3} = -10\int_0^v \frac{-2v\,dv}{200-v^2}$

$\therefore 10\ln 3 = -10\ln(200-v^2)\Big]_0^v = -10\ln(200-v^2) + 10\ln 200$

$\therefore \ln 3 = -\ln(200-v^2) + \ln 200 \Rightarrow \ln\frac{3}{200} = -\ln(200-v^2)$

ie $\frac{200}{3} = 200 - v^2 \quad v^2 = 200 - \frac{200}{3} = \frac{400}{3} \quad \therefore v = \frac{20}{\sqrt{3}} \approx 11.55\text{m/s}$

b) (i) $V = Ah \quad \therefore \frac{dV}{dt} = A\cdot\frac{dh}{dt}$ given $\frac{dV}{dt} = -k\sqrt[3]{h}$ $\boxed{1}$ Rate from $V=Ah$

$A\frac{dh}{dt} = -k\sqrt[3]{h} \Rightarrow \frac{dh}{dt} = -\frac{k}{A}\sqrt[3]{h}$ $\boxed{1}$ Rearrange.

(ii) $\frac{dh}{dt} = -\frac{k}{A}h^{\frac{1}{3}}$ separate $h^{-\frac{1}{3}}dh = \frac{-k}{A}dt$ integrate $\boxed{1}$ Separate

$\int_{h_0}^{h} h^{-\frac{1}{3}}dh = \int_0^t -\frac{k}{A}dt \Rightarrow \frac{3}{2}h^{\frac{2}{3}}\Big]_{h_0}^{h} = -\frac{k}{A}t$ $\boxed{1}$ Integrate bounds const

$\therefore \frac{3}{2}\left[h^{\frac{2}{3}} - h_0^{\frac{2}{3}}\right] = -\frac{k}{A}t \quad \therefore \frac{3}{2}h^{\frac{2}{3}} = \frac{3}{2}h_0^{\frac{2}{3}} - \frac{k}{A}t$

It takes T secs to drain $\therefore t=T,\ h=0 \quad \therefore \frac{k}{A} = \frac{3}{2}h_0^{\frac{2}{3}}T^{-1}$

$\therefore h^{\frac{2}{3}} = h_0^{\frac{2}{3}} - \frac{h_0^{\frac{2}{3}}}{T}t \rightarrow h^{\frac{2}{3}} = h_0^{\frac{2}{3}}\left(1-\frac{t}{T}\right) \quad \therefore h^2 = h_0^2\left(1-\frac{t}{T}\right)^3$ $\boxed{1}$

(iii) $h = \frac{h_0}{2} \quad t = 12\text{secs} \quad \left(\frac{h_0}{2}\right)^2 = h_0^2\left(1-\frac{12}{T}\right)^3 \quad \therefore \frac{1}{4} = \left(1-\frac{12}{T}\right)^3$

$1-\frac{12}{T} = \sqrt[3]{\frac{1}{4}} \quad \therefore \frac{12}{T} = 1-\sqrt[3]{0.25} \quad \therefore \frac{T}{12} = \frac{1}{1-\sqrt[3]{0.25}}$ $\boxed{1}$

$\therefore T = \frac{12}{1-\sqrt[3]{0.25}} \approx 32.428.. \doteqdot 32\text{ secs}$ (to nearest sec.)

Question 8 | 15 Marks

a) (i) $z + \frac{1}{z} = 2\cos\alpha \quad \therefore z^2 - 2\cos\alpha\, z + 1 = 0$ $\boxed{1}$ quad.

$\therefore (z-\cos\alpha)^2 - (\cos\alpha)^2 + 1 = 0 \quad \therefore (z-\cos\alpha)^2 = -1+\cos^2\alpha = -\sin^2\alpha$

ie $z - \cos\alpha = \pm i\sin\alpha \quad \therefore z = \cos\alpha \pm i\sin\alpha$ $\boxed{1}$ $z = \text{cis}(\alpha)$, $\text{cis}(-\alpha)$

$\therefore z = \text{cis}\,\alpha$ or $\text{cis}(-\alpha)$

If $z = \text{cis}\,\alpha$ then by de Moivre's theorem

$z^n = \text{cis}\,n\alpha$ and $z^{-n} = \text{cis}(-n\alpha) \quad \cos(-n\alpha) = \cos n\alpha$

If $z = \text{cis}\,\alpha$ or $z = \text{cis}(-\alpha)$ $\quad z^n + z^{-n} = \cos n\alpha + i\sin n\alpha + \cos n\alpha - i\sin n\alpha = 2\cos n\alpha$ $\boxed{1}$ result.

(ii) Let $w = z + \frac{1}{z} \quad w^2 = \left(z+\frac{1}{z}\right)^2 = z^2 + \frac{1}{z^2} + 2$ $\boxed{1}$

Now $w^3 + w^2 - 2w - 2 = w^2(w+1) - 2(w+1)$

$= (w+1)(w^2-2) = \left(z+\frac{1}{z}+1\right)\left(z^2+\frac{1}{z^2}\right)$

$= z^3 + \frac{1}{z} + z + \frac{1}{z^3} + z^2 + \frac{1}{z^2}$

$= z + \frac{1}{z} + z^2 + \frac{1}{z^2} + z^3 + \frac{1}{z^3}$ $\boxed{1}$

(iii) $z + \frac{1}{z} + z^2 + \frac{1}{z^2} + z^3 + \frac{1}{z^3} = 2\cos\alpha + 2\cos 2\alpha + 2\cos 3\alpha = 0$

ie $\cos\alpha + \cos 2\alpha + \cos 3\alpha = (w+1)(w^2-2),\ w = z + \frac{1}{z}$ $\boxed{1}$

$\therefore w = -1, \sqrt{2}$ or $-\sqrt{2} \quad \therefore 2\cos\alpha = -1, \sqrt{2}$ or $-\sqrt{2}$

If $\cos\alpha = -\frac{1}{2},\ \alpha = \frac{2\pi}{3}, \frac{4\pi}{3}, \quad \cos\alpha = \frac{1}{\sqrt{2}},\ \alpha = \frac{\pi}{4}$ or $\frac{7\pi}{4}$ $\boxed{1}$

$\cos\alpha = \frac{-1}{\sqrt{2}},\ \alpha = \frac{3\pi}{4}, \frac{5\pi}{4}$

$\therefore$ Six solutions $\alpha = \frac{\pi}{4}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{4\pi}{3}$ and $\frac{7\pi}{4}$. $\boxed{1}$

b) $\int_0^a f(x)\,dx = \int_0^a f(a-u)\,dx$ let $u = a - x$, $x=0\ u=a$, $x=a\ u=0$, $du = -dx$ $\boxed{1}$

$\therefore RHS = \int_a^0 f(u)\cdot -du = \int_0^a f(u)\,du = \int_0^a f(x)\,dx = LHS$

$\therefore \int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx = \int_0^\pi \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}dx = \int_0^\pi \frac{(\pi-x)\sin x}{1+\cos^2 x}dx$ $\boxed{1}$

$\cos(\pi-x) = -\cos x$

$\therefore \cos^2(\pi-x) = \cos^2 x \quad \therefore I = \int_0^\pi \frac{\pi\sin x}{1+\cos^2 x}dx - \int_0^\pi \frac{x\sin x\,dx}{1+\cos^2 x}$ $\boxed{1}$

$\sin(\pi-x) = \sin x$

$\therefore 2I = \pi\int_0^\pi \frac{\sin x}{1+\cos^2 x}dx = -\pi\left[\tan^{-1}(\cos x)\right]_0^\pi$

$\therefore 2I = -\pi\left[-\frac{\pi}{4} - \frac{\pi}{4}\right] = \frac{\pi^2}{2} \quad \therefore \boxed{I = \frac{\pi^2}{4}}$ $\boxed{1}$

c) $x^2 + y^2 = r^2$

$\therefore f(x) = \sqrt{r^2-x^2} \quad \therefore A = \int_{-r}^{r} 2\pi\sqrt{r^2-x^2}\sqrt{1+\left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}\,dx$ $\boxed{1}$

$2x + 2y\frac{dy}{dx} = 0 \quad = \int_{-r}^{r} 2\pi\sqrt{r^2-x^2}\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx$

$\therefore \frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y} \quad = \int_{-r}^{r} 2\pi\sqrt{r^2-x^2}\sqrt{\frac{r^2-x^2+x^2}{r^2-x^2}}\,dx$

$f'(x) = \frac{-x}{\sqrt{r^2-x^2}}$ $\boxed{1}$ $\quad = \int_{-r}^{r} 2\pi\sqrt{r^2-x^2}\cdot\frac{r}{\sqrt{r^2-x^2}}\,dx = 2\pi r x\Big]_{-r}^{r}$ $\boxed{1}$

$= 2\pi r^2 - (2\pi r(-r)) = \boxed{4\pi r^2}$