Name: ______________________________

Teacher: ____________________________

Class: ______________________________

FORT STREET HIGH SCHOOL

2014

HIGHER SCHOOL CERTIFICATE COURSE

ASSESSMENT TASK 3: TRIAL HSC

Mathematics Extension 2

Time allowed: 3 hours
(plus 5 minutes reading time)

Syllabus Outcomes	Assessment Area Description and Marking Guidelines	Questions
	Chooses and applies appropriate mathematical techniques in order to solve problems effectively	1-11
E2, E3	Applies appropriate strategies to construct arguments and proofs in the areas of complex numbers and polynomials	12
E4, E6	Uses efficient techniques for the algebraic manipulation of conic sections and determining features of a wide variety of graphs	13
E7, E8	Applies further techniques of integration, such as slicing and cylindrical shells, integration by parts and recurrence formulae, to problems	14
E5	Uses ideas and techniques of calculus to solve problems in mechanics involving resolution of forces, resisted motion and circular motion	15
E2-E8	Synthesises mathematical processes to solve harder problems and communicates solutions in an appropriate form	16

Total Marks 100

Section I 10 marks
Multiple Choice, attempt all questions.
Allow about 15 minutes for this section.

Section II 90 Marks
Attempt Questions 11-16.
Allow about 2 hours 45 minutes for this section.

Section I	Total 10	Marks
Q1-Q10		
Section II	Total 90	Marks
Q11	/15	
Q12	/15	
Q13	/15	
Q14	/15	
Q15	/15	
Q16	/15	
	Percent	

General Instructions:

- Questions 11-16 are to be started in a new booklet.
- The marks allocated for each question are indicated.
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations.
- Marks may be deducted for careless or badly arranged work.
- Board – approved calculators may be used.

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Section 1: Multiple Choice: Circle the correct answer on the paper below.

1 Which of the following complex numbers equals $(\sqrt{3}+i)^4$?

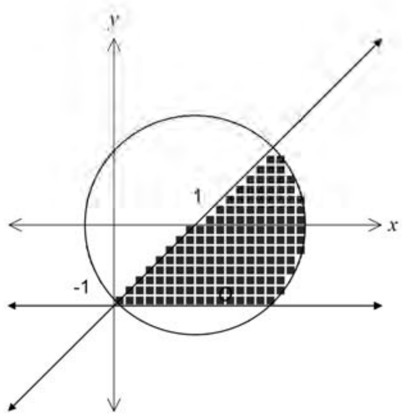
(A) $-2+\dfrac{2}{\sqrt{3}}i$

(B) $-8+\dfrac{8}{\sqrt{3}}i$

(C) $-2+2\sqrt{3}i$

(D) $-8+8\sqrt{3}i$

2 Consider the Argand diagram below.



Which inequality could define the shaded area?

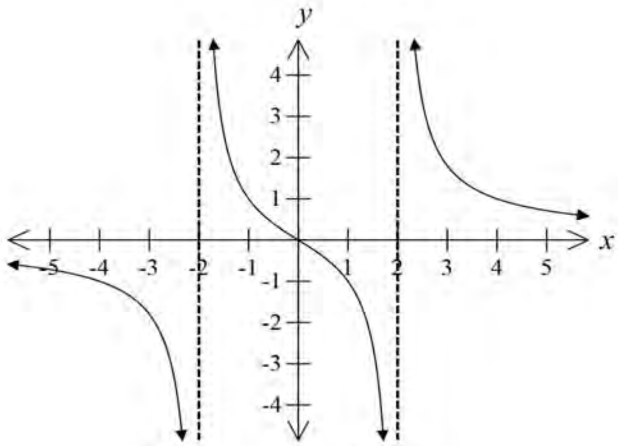
(A) $|z-1|\le\sqrt{2}$ and $0\le\arg(z-i)\le\dfrac{\pi}{4}$

(B) $|z-1|\le\sqrt{2}$ and $0\le\arg(z+i)\le\dfrac{\pi}{4}$

(C) $|z-1|\le 1$ and $0\le\arg(z-i)\le\dfrac{\pi}{4}$

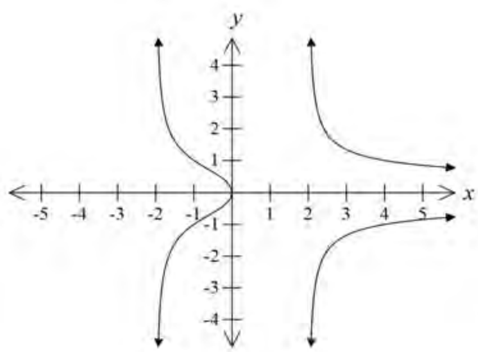
(D) $|z-1|\le 1$ and $0\le\arg(z+i)\le\dfrac{\pi}{4}$

3 The diagram shows the graph of the function $y = f(x)$.

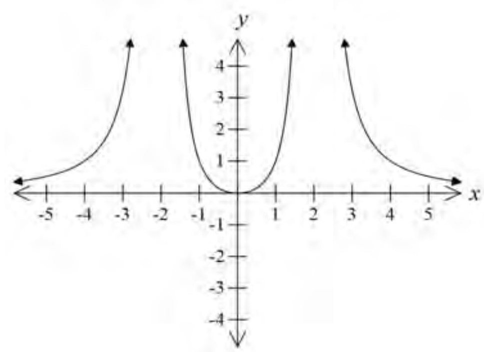


Which of the following is the graph of $y^2 = f(x)$?

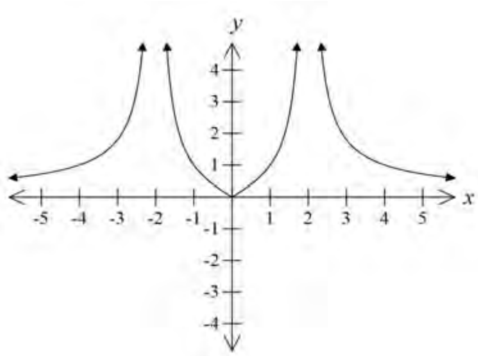
(A)



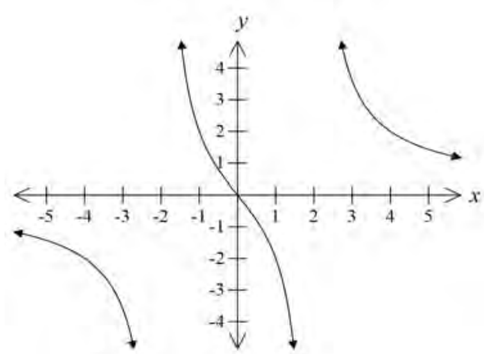
(B)



(C)



(D)



4 The points $P(a\cos\theta,\ b\sin\theta)$ and $Q(a\cos\phi,\ b\sin\phi)$ lie on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the chord PQ subtends a right angle at $(0,0)$. Which of the following is the correct expression?

(A) $\tan\theta\tan\phi=-\frac{b^2}{a^2}$

(B) $\tan\theta\tan\phi=-\frac{a^2}{b^2}$

(C) $\tan\theta\tan\phi=\frac{b^2}{a^2}$

(D) $\tan\theta\tan\phi=\frac{a^2}{b^2}$

5 The points $P(cp,\ \frac{c}{p})$ and $Q(cq,\ \frac{c}{q})$ lie on the same branch of the hyperbola $xy=c^2$ ($p \neq q$). The tangents at P and Q meet at the point T. What is the equation of the normal to the hyperbola at P?

(A) $p^2x-py+c-cp^4=0$

(B) $p^3x-py+c-cp^4=0$

(C) $x+p^2y-2c=0$

(D) $x+p^2y-2cp=0$

6 Which of the following is an expression for $\int\frac{x}{\sqrt{16-x^2}}dx$?

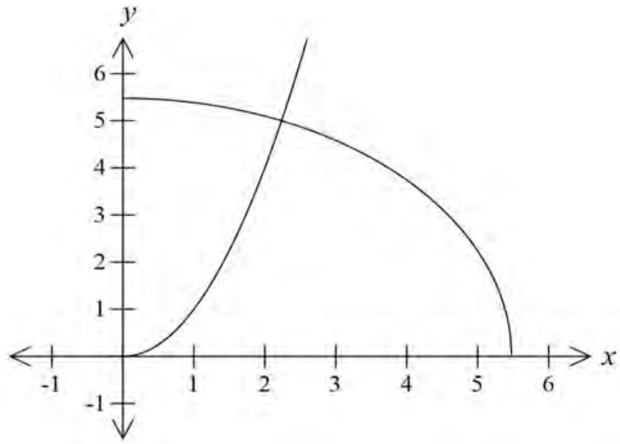
(A) $-2\sqrt{16-x^2}+c$

(B) $-\sqrt{16-x^2}+c$

(C) $\frac{1}{2}\sqrt{16-x^2}+c$

(D) $-\frac{1}{2}\sqrt{16-x^2}+c$

7 What is the volume of the solid formed when the region bounded by the curves $y = x^2$, $y = \sqrt{30 - x^2}$ and the y-axis is rotated about the y-axis? Use the method of slicing.



What is the correct expression for volume of this solid using the method of cylindrical shells?

(A) $V = \int_0^{\sqrt{5}} 2\pi \left(x^2 - \sqrt{30 - x^2} \right) dx$

(B) $V = \int_0^{\sqrt{5}} 2\pi x \left(x^2 - \sqrt{30 - x^2} \right) dx$

(C) $V = \int_0^{\sqrt{5}} 2\pi \left(\sqrt{30 - x^2} - x^2 \right) dx$

(D) $V = \int_0^{\sqrt{5}} 2\pi x \left(\sqrt{30 - x^2} - x^2 \right) dx$

8 Let α, β and γ be roots of the equation $x^3 + 3x^2 + 4 = 0$. Which of the following polynomial equations have roots α^2, β^2 and γ^2?

(A) $x^3 - 9x^2 - 24x - 4 = 0$

(B) $x^3 - 9x^2 - 12x - 4 = 0$

(C) $x^3 - 9x^2 - 24x - 16 = 0$

(D) $x^3 - 9x^2 - 12x - 16 = 0$

9 What is the solution to the equation $z^2 = i\,\overline{z}$?

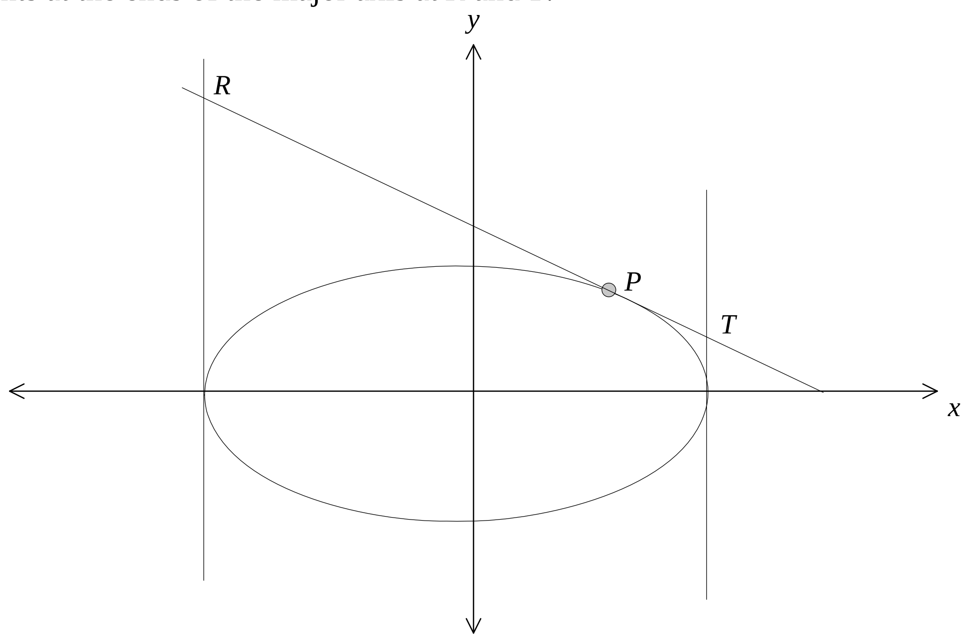
(A) $(0,0)$ and $(0,1)$

(B) $(0,0)$ and $(0,-1)$

(C) $(0,0)$, $(0,-1)$, $(\frac{\sqrt{3}}{2}, \frac{1}{2})$ and $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$

(D) $(0,0)$, $(0,1)$, $(\frac{\sqrt{3}}{2}, \frac{1}{2})$ and $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$

10 The point P lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where $a>b>0$. The tangent at P meets the tangents at the ends of the major axis at R and T.



What is the equation of the tangent at P?

(A) $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$

(B) $\frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2+b^2$

(C) $\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$

(D) $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$

Section Two: Free Response

Question 11 **Start a NEW Booklet**

a. If $z = 1 + i$ and $w = 1 - 3i$ find, in the form $x + iy$,

i. $\bar{z} - w$ **2**

ii. $\frac{z}{w}$ **2**

b. Find:

i. $\int \frac{x}{\sqrt{9 - 4x^2}}\,dx$ **2**

ii. $\int \frac{x^2}{x + 1}\,dx$ **2**

c. Solve $x^5 + 2x^4 - 2x^3 - 8x^2 - 7x - 2 = 0$ given that it has a root of multiplicity 4. **3**

d.

i. Write down the six complex sixth roots of unity in modulus/argument form. **1**

ii. Explain why the roots form a hexagon when placed on the Argand Diagram. **1**

iii. Factorize $z^6 - 1$ completely into factors over $\mathbb{R}$. **2**

Question 12 **Start a NEW Booklet**

a. In an Argand Diagram the points P, Q and R represent the complex numbers z_1, z_2 and $z_2 + i(z_2 - z_1)$ respectively.

i. Show that PQR is a right angled triangle. **2**

ii. Find, in terms of z_1 and z_2, the complex number represented by the point S such that $PQRS$ is a rectangle. **2**

b. If $z = r(\cos\theta + i\sin\theta)$, find r and the smallest positive value of θ which satisfies the equation $2z^2 = 9 + 3i\sqrt{3}$. **3**

c. The equation $2x^3 + 5x + 1 = 0$ has roots α, β and γ. Evaluate $\alpha^3 + \beta^3 + \gamma^3$. **2**

d. Find real numbers a, b and c such that $\dfrac{5x^2 - 5x + 14}{(x^2+4)(x-2)} = \dfrac{ax+b}{x^2+4} + \dfrac{c}{x-2}$. **2**

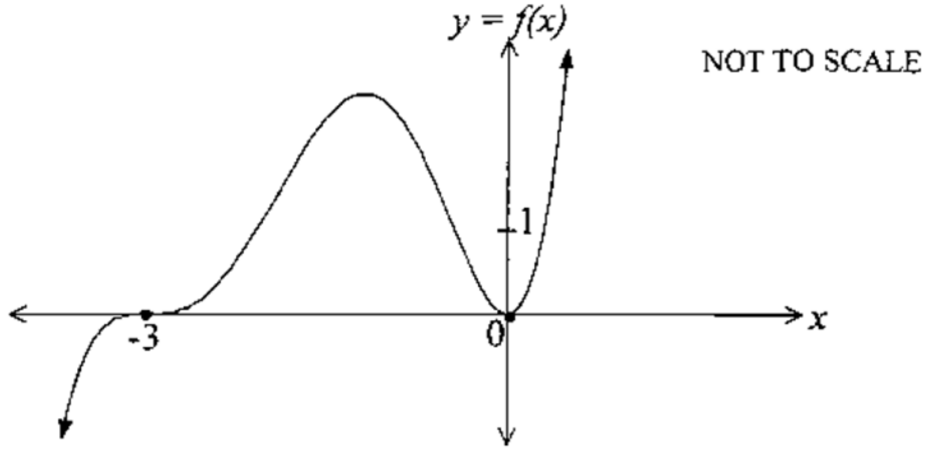
e. Let $P(x)$ and $Q(x)$ be distinct polynomials with a common factor $(x-a)$.

i. Show that $R(x) = P(x) - Q(x)$ will have the same common factor. **2**

ii. Hence if $P(x) = 6x^3 + 7x^2 - x - 2$ and $Q(x) = 6x^3 - 5x^2 - 3x + 2$, find the two zeros that $P(x)$ and $Q(x)$ have in common. **2**

Question 13 **Start a NEW Booklet**

a. Consider the graph of $y = f(x)$ shown below.



Draw a neat $\frac{1}{3}$ page sketch of the following, showing all behaviour near roots, turning points and $y = 1$.

i. $y = \dfrac{1}{\sqrt{f(x)}}$ **2**

ii. $y^2 = f(x)$ **2**

iii. $y = e^{f(x)}$ **2**

b. An ellipse has the equation $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$.

i. Find the eccentricity, the co-ordinates of the foci S and S', and the equations of the directricies. **3**

ii. Derive the equation of the tangent at the point $P(3\cos\theta, 2\sin\theta)$ on the ellipse, where θ is the auxiliary angle. **2**

iii. The ellipse meets the *y*-axis at the points *A* and *B*. The tangents to the ellipse at *A* and *B* meet the tangent at *P* at the points *C* and *D* respectively. Prove that $AC.BD = 9$. **4**

Question 14 **Start a NEW Booklet**

a. Find $\int \frac{\cos 2x}{\cos^2 x}\,dx$. **2**

b. Use integration by parts to evaluate $\int_0^1 \sin^{-1} x\,dx$. **3**

c. A solid has as its base the ellipse $\frac{x^2}{36}+\frac{y^2}{16}=1$. If each cross-section perpendicular to the major axis of the base is an equilateral triangle, show that the volume of the solid formed is $128\sqrt{3}$ cubic units. **4**

d. For the inequalities $x^2+y^2\le 1$ and $x^2 \le \frac{8}{3}y$:

i. On the same set of axes, sketch and shade the region satisfying both inequalities. **2**

ii. The area in part (i) is rotated about the *y*-axis through one complete revolution. Using the method of cylindrical shells, find the volume of the solid generated. **4**

Question 15 **Start a NEW Booklet**

a. A body of unit mass is projected vertically upwards in a medium that has a constant gravitational force g and a resistance $\frac{v}{10}$, where v is the velocity of the projectile at a given time t. The initial velocity is $10(20-g)$.

i. Show that the equation of motion of the projectile is $\frac{dv}{dt} = -g - \frac{v}{10}$. **1**

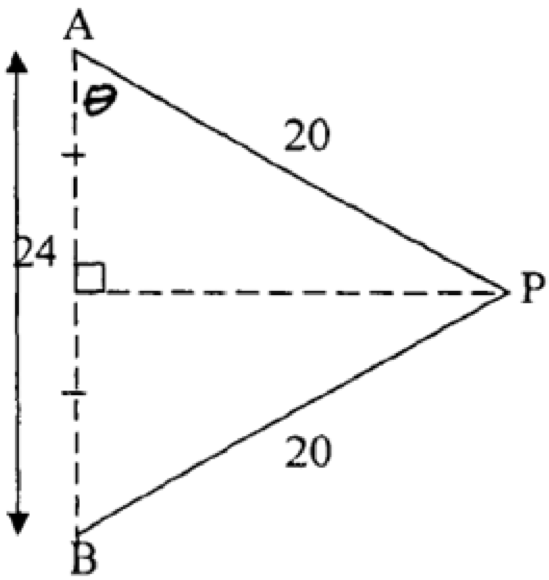
ii. Show that the time T for the particle to reach its greatest height is given by $T = 10\ln\left(\frac{20}{g}\right)$. **2**

iii. Show that the maximum height H is given by $H = 2000 - 10g\left[10+T\right]$ **3**

iv. If the particle then falls from this height, find the terminal velocity in this medium. **2**

Q15 continues over

b. A particle of mass *m* kg is attached at *P* by two strings, each of length 20cm, to two fixed points *A* and *B*, which are 24cm apart and lie on a vertical line as shown in the diagram. The particle moves with a constant speed *v* m/s in a horizontal circle about the midpoint of *AB* so that both pieces of string experience tension. The tension in *AP* is T_1 and the tension in *BP* is T_2. The acceleration due to gravity is g m/s^2.



i. Copy the diagram and show all forces acting on *P*. **1**

ii. Resolve the forces on *P* in both horizontal and vertical directions. **2**

iii. Find the tension in each string in terms of *m*, *v* and *g*. **2**

iv. Show that $v \geq 4\frac{\sqrt{3g}}{15}$. **2**

Question 16 **Start a NEW Booklet**

a.

i. Show that $\frac{d}{dx}\left(\ln\left(\sec x+\tan x\right)\right)=\sec x$. **1**

ii. Hence or otherwise, show that $\int_0^{\frac{\pi}{4}}\sec x\,dx=\ln\left(1+\sqrt{2}\right)$. **2**

iii. Let $I_n=\int_0^{\frac{\pi}{4}}\sec^n x\,dx$. Use integration by parts to show that, for $n\geq 2$,

$I_n=\frac{1}{n-1}\left(\left(\sqrt{2}\right)^{n-2}+\left(n-2\right)I_{n-2}\right)$. **3**

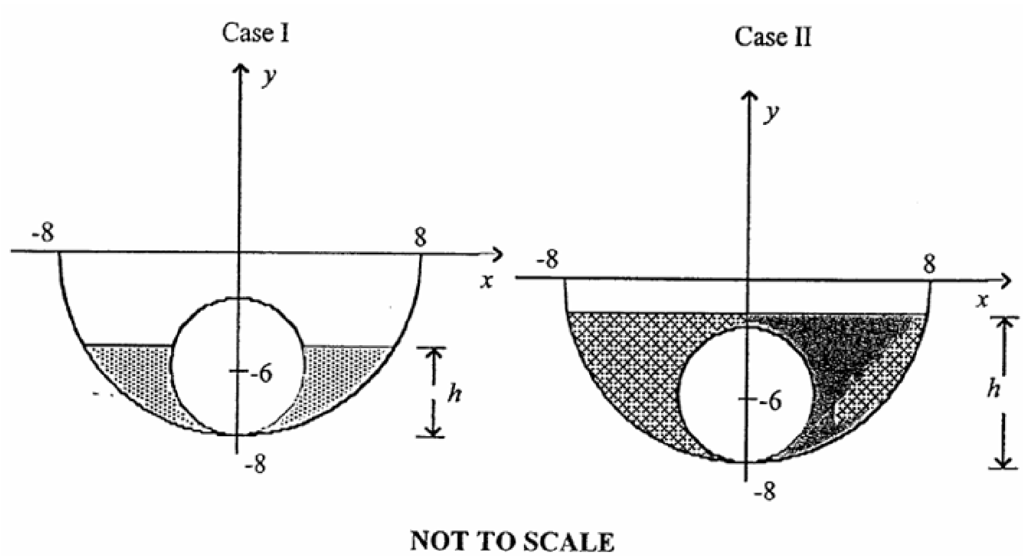
iv. Hence find I_3. **1**

Q16 continues over

b. A bowl shaped as a hemisphere of radius 8cm is filled with milk to a depth of *h* cm.

i. By taking slices perpendicular to the *y*-axis, deduce that the volume of milk is given by $V = \pi\left(8h^2 - \frac{h^3}{3}\right)$ cubic units. **3**

A (spherical) scoop of ice cream of radius 2cm is placed in the bowl. Assume that the ice cream does not melt or float. There are two possible cases as shown below.

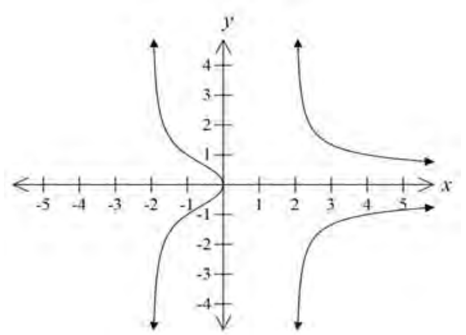
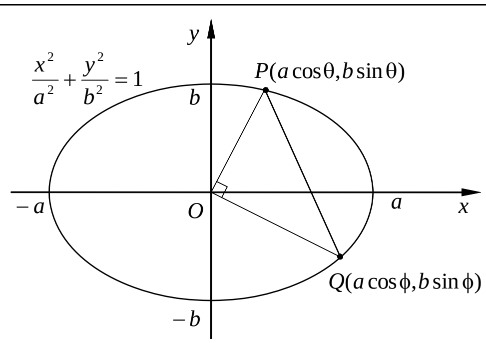


ii. Deduce that for Case I the volume of milk is $6\pi h^2$ cubic units. **4**

iii. Hence or otherwise, find the volume of milk in Case II. **1**

2014 Fort Street High Mathematics Extension 2 Trial – Solutions

Section I: Multiple Choice

1	$\sqrt{3}+i = 2(\frac{\sqrt{3}}{2}+\frac{1}{2}i)$ $= 2(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})$ $(\sqrt{3}+i)^4 = 2^4(\cos 4\times\frac{\pi}{6}+i\sin 4\times\frac{\pi}{6})$ $=16(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3})$ $=-8+8\sqrt{3}i$	1 Mark: D
2	$\lvert z-1\rvert \le \sqrt{2}$ represents a region with a centre is (1, 0) and radius is less than or equal to $\sqrt{2}$. $0 \le \arg(z+i) \le \frac{\pi}{4}$ represents a region between angle 0 and $\frac{\pi}{4}$ whose vertex is $(-1,0)$ not including the vertex $\lvert z-1\rvert \le \sqrt{2}$ and $0 \le \arg(z+i) \le \frac{\pi}{4}$	1 Mark: B
3		1 Mark: A
4	 POQ is a right-angled triangle. Therefore $OP^2+OQ^2=PQ^2$. $a^2\cos^2\theta+b^2\sin^2\theta+a^2\cos^2\varphi+b^2\sin^2\varphi$ $=a^2(\cos\theta-\cos\varphi)^2+b^2(\sin\theta-\sin\varphi)^2$	1 Mark: B

	$a^2(\cos^2\theta+\cos^2\varphi)+b^2(\sin^2\theta+\sin^2\varphi)$ $=a^2(\cos\theta-\cos\varphi)^2+b^2(\sin\theta-\sin\varphi)^2$ Hence $0=-2a^2\cos\theta\cos\varphi-2b^2\sin\theta\sin\varphi$ $2b^2\sin\theta\sin\varphi=-2a^2\cos\theta\cos\varphi$ $\frac{\sin\theta\sin\varphi}{\cos\theta\cos\varphi}=\frac{-2a^2}{2b^2}$ or $\tan\theta\tan\phi=-\frac{a^2}{b^2}$	
5	To find the gradient of the tangent. $xy=c^2$ $x\frac{dy}{dx}+y=0$ $\frac{dy}{dx}=-\frac{y}{x}$ At P $(cp,\ \frac{c}{p})$ $\frac{dy}{dx}=-\frac{\frac{c}{p}}{cp}=-\frac{1}{p^2}$ Gradient of the normal is p^2 $(m_1m_2=-1)$ Equation of the normal at P $(cp,\ \frac{c}{p})$ $y-\frac{c}{p}=p^2(x-cp)$ $py-c=p^3x-cp^4$ $p^3x-py+c-cp^4=0$	1 Mark: B
6	Let $u=16-x^2$ then $\frac{du}{dx}=-2x$ $xdx=-\frac{1}{2}du$ $\int\frac{x}{\sqrt{16-x^2}}dx=-\frac{1}{2}\int u^{-\frac{1}{2}}du$ $=-\frac{1}{2}\times 2u^{\frac{1}{2}}$ $=-\sqrt{16-x^2}$	1 Mark: B
7	$\sqrt{30-x^2}=x^2$ $30-x^2=x^4$ $x^4+x^2-30=0$ $\left(x^2+6\right)\left(x^2-5\right)=0$	1 Mark: D

	$x = \pm\sqrt{5}$ Cylindrical shells radius is x and height y $V = \lim_{\delta x \to 0} \sum_{x=0}^{\sqrt{5}} 2\pi xy\delta x$ $= \int_0^{\sqrt{5}} 2\pi x\left(\sqrt{30-x^2} - x^2\right)dx$	
8	If α, β and γ are zeros of $x^3 + 3x^2 + 4 = 0$ then Polynomial equation is $(\sqrt{x})^3 + 3(\sqrt{x})^2 + 4 = 0$ $(\sqrt{x})^3 = -(3x+4)$ $x^3 = 9x^2 + 24x + 16$ $x^3 - 9x^2 - 24x - 16 = 0$	1 Mark: C
9	Let $z = x + iy$ and $\overline{z} = x - iy$ $z^2 = i\overline{z}$ $(x+iy)^2 = i(x-iy)$ $x^2 - y^2 + 2xyi = y + ix$ Equating the real and imaginary parts $x^2 - y^2 = y$ (1) $2xy = x$ (2) Rearranging eqn (2) $x(2y-1) = 0$ $x = 0$ or $y = \frac{1}{2}$ Substitute $x = 0$ into eqn (1) $-y^2 = y$ $y(y+1) = 0$ $y = 0$ or $y = -1$ Substitute $y = \frac{1}{2}$ into eqn (1) $x^2 - \frac{1}{4} = \frac{1}{2}$ $x^2 = \frac{3}{4}$, $x = \pm\frac{\sqrt{3}}{2}$ Solution is $(0,0)$, $(0,-1)$, $(\frac{\sqrt{3}}{2}, \frac{1}{2})$ and $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$	1 Mark: C

10	To find the equation of tangent through P $x = a\cos\theta \qquad y = b\sin\theta$ $\frac{dx}{d\theta} = -a\sin\theta \qquad \frac{dy}{d\theta} = b\cos\theta$ $\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$ $= b\cos\theta \times \frac{1}{-a\sin\theta} = \frac{-b\cos\theta}{a\sin\theta}$ Equation of the tangent $y - y_1 = m(x - x_1)$ $y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$ $ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$ $bx\cos\theta + ay\sin\theta = ab(\sin^2\theta + \cos^2\theta)$ $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$	1 Mark: D

Section II:

Q11

Comments

a)

i. $\bar{z}-w=\overline{(1+i)}-(1-3i)$ | Both parts well done.

$=1-i-1+3i$ ❶

$=2i$ ❶

ii.

$\frac{z}{w}=\frac{1+i}{1-3i}$

$=\frac{1+i}{1-3i}\times\frac{1+3i}{1+3i}$ ❶

$=\frac{1+4i-3}{1-9i^2}$

$=\frac{-2-4i}{10}$

$=\frac{-1}{5}+\frac{2i}{5}$ ❶

b)

i. $\int\frac{x}{\sqrt{9-4x^2}}dx=\frac{-1}{4}\sqrt{9-4x^2}$ ❶ $+c$ ❶ | Both parts well done.

ii.

$\int\frac{x^2}{x+1}dx$

$=\int\frac{x^2-1+1}{x+1}dx$ ❶

$=\int\frac{(x+1)(x-1)+1}{x+1}dx$

$=\int(x-1)+\frac{1}{x+1}dx$

$=\frac{1}{2}x^2-x+\ln|x+1|+c$ ❶

c) Let $P(x)=x^5+2x^4-2x^3-8x^2-7x-2$. Then, with root of multiplicity 4: | First part well done.

$P'(x)=5x^4+8x^3-6x^2-16x-7$

$P''(x)=20x^3+24x^2-12x-16$

$P'''(x)=60x^2+48x-12$

$=12(5x^2+4x-1)$

$=12(x+1)(5x-1)$

$P'''(x)=0\Rightarrow x=-1,\frac{1}{5}$ ❶ for possible multiple roots. Considering iv.

$P(-1)$:

$$P(-1) = -1+2+2-8+7-2$$
$$= 0$$ ❶

Hence $P(x) = (x+1)^4(x-2)$ (by inspection considering constant term).

Thus the roots are -1 and 2. ❶

Some students factored only without explicitly stating the roots as required.

d)

i. $\pm 1, cis\frac{\pi}{3}, cis\frac{2\pi}{3}, cis\frac{4\pi}{3}, cis\frac{5\pi}{3}$ ❶

ii. Since their moduli is one, and their arguments differ by $\frac{\pi}{3}$, they form the vertices of a regular hexagon on the unit circle. ❶

iii. $z^6 - 1 = \left((z^3)^2 - 1\right)$ ❶

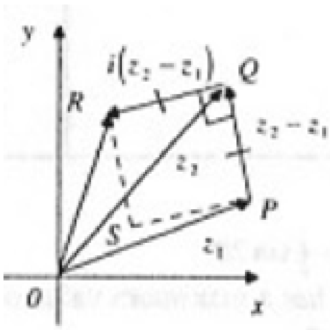
$$= (z^3-1)(z^3+1)$$
$$= (z-1)(z^2+z+1)(z+1)(z^2-z+1)$$ ❶

Some students did not factorise fully to produce the required result.

Q12

Comments

a)



(diagram ❶)

i. The vector $\overrightarrow{PQ}$ represents the complex number $z_2 - z_1$. $\overrightarrow{PQ}$ rotated anti-clockwise by $\frac{\pi}{2}$ represents $i(z_2 - z_1)$.Now vector $\overrightarrow{OR}$ is the sum of the vectors z_2 and $i(z_2 - z_1)$, as shown in the diagram. Clearly, $\angle PQR = \frac{\pi}{2}$ and hence ΔPQR is right-angled at Q. ❶

ii. For $PQRS$ a rectangle, $\overrightarrow{PS} \parallel \overrightarrow{QR}$ and $|PS| = |QR|$. Thus $i(z_2 - z_1)$ also represents $\overrightarrow{PS}$. ❶ Hence $\overrightarrow{OS}$ is represented by $z_1 + i(z_2 - z_1)$. ❶

Not very well done at all. Very few had a decent diagram to start with and failed to explain properly that $\angle PQR = \frac{\pi}{2}$.

Obvious difficulty in understand summation of vectors.

b) $z = r(\cos\theta + i\sin\theta)$ into $2z^2 = 9 + 3i\sqrt{3}$ gives:

$$2(rcis\theta)^2 = 9 + 3i\sqrt{3}$$

$$2r^2 cis(2\theta) = 6\sqrt{3}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

$$= 6\sqrt{3}cis\left(\frac{\pi}{6}\right) ❶$$

Then: $2r^2 = 6\sqrt{3}$ and $2\theta = \frac{\pi}{6}$

$r^2 = 3\sqrt{3}$

$= \sqrt{27}$

$\theta = \frac{\pi}{12}$ ❶

$r = \sqrt[4]{27}$ ❶

Reasonably well done.

c) From $2x^3 + 5x + 1 = 0$ with roots $\alpha, \beta \,\&\, \gamma$:

$\alpha + \beta + \gamma = \frac{-b}{a} = 0$ ①, $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = \frac{5}{2}$ ②, $\alpha\beta\gamma = \frac{-d}{a} = \frac{-1}{2}$ ③

Re-arranging $2x^3 + 5x + 1 = 0$ gives $2x^3 = -5x - 1$. Substituting the roots $\alpha, \beta \,\&\, \gamma$:

$2\alpha^3 = -5\alpha - 1$

$2\beta^3 = -5\beta - 1$

$2\gamma^3 = -5\gamma - 1$, then adding:

Many thought $\alpha + \beta + \gamma = \frac{-5}{2}$. Need to be more careful.

$$2\alpha^3+2\beta^3+2\gamma^3=-5\alpha-1-5\beta-1-5\gamma-1$$ ❶

$$2(\alpha^3+\beta^3+\gamma^3)=-5(\alpha+\beta+\gamma)-3$$

$$\alpha^3+\beta^3+\gamma^3=\frac{-5}{2}(\alpha+\beta+\gamma)-\frac{3}{2}$$

$$=\frac{-3}{2}$$ ❶ from ①.

d) $\frac{5x^2-5x+14}{(x^2+4)(x-2)}=\frac{ax+b}{x^2+4}+\frac{c}{x-2}$ leads to

$$5x^2-5x+14\equiv(ax+b)(x-2)+c(x^2+4)$$

$$x=2:\ 5.2^2-5.2+14\equiv(ax+b)(2-2)+c(2^2+4)$$

$$24=8c$$

$$c=3$$

$$x=0:\ 14\equiv(b)(-2)+3(+4)$$

$$14=-2b+12$$

$$2b=-2$$

$$b=-1$$ ❶

$$x=1:\ 5-5+14\equiv(a-1)(1-2)+3(1^2+4)$$

$$14=1-a+15$$

$$a=2$$

Hence $a=2, b=-1\ \&\ c=3$. ❶

Well done.

e) $P(x)$ and $Q(x)$ distinct with common factor $(x-a)$ means:

i. $P(x)=(x-a).A(x)$ and $Q(x)=(x-a).B(x)$. ❶ Hence:

$$R(x)=P(x)-Q(x)$$

$$=(x-a).A(x)-(x-a).B(x)$$

$$=(x-a)[A(x)-B(x)]$$ ❶ which clearly has a factor $(x-a)$.

Accepted alternative:

$$P(\alpha)=0$$

$$Q(\alpha)=0$$

$$R(\alpha)=P(\alpha)-Q(\alpha)$$

$$=0$$

$\therefore (x-\alpha)$ is a factor.

ii. With $P(x)=6x^3+7x^2-x-2$ and $Q(x)=6x^3-5x^2-3x+2$:

$$R(x)=6x^3+7x^2-x-2-(6x^3-5x^2-3x+2)$$

$$=12x^2+2x-4$$

$$=2(6x^2+x-2)$$ ❶

$$=2(6x^2+4x-3x-2)$$

$$=2(2x(3x+2)-(3x+2))$$

$$=2(2x-1)(3x+2)$$

Hence $x=\frac{-2}{3},\frac{1}{2}$ are the two common zeros. ❶

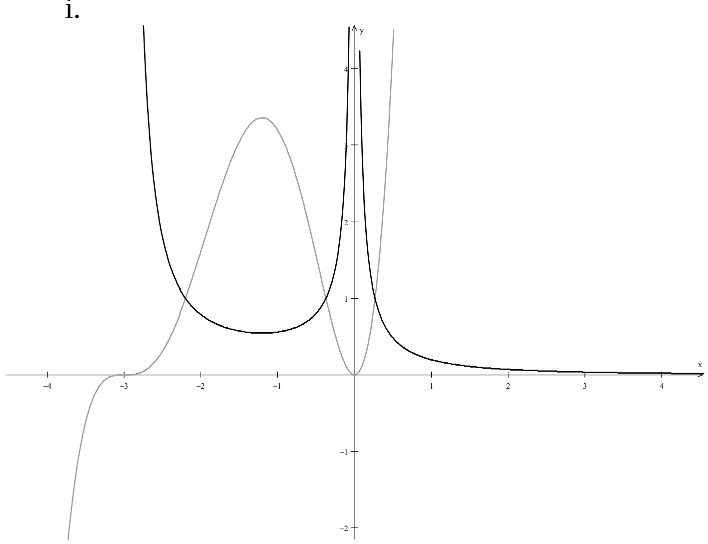
Well done.

Q13

Comments

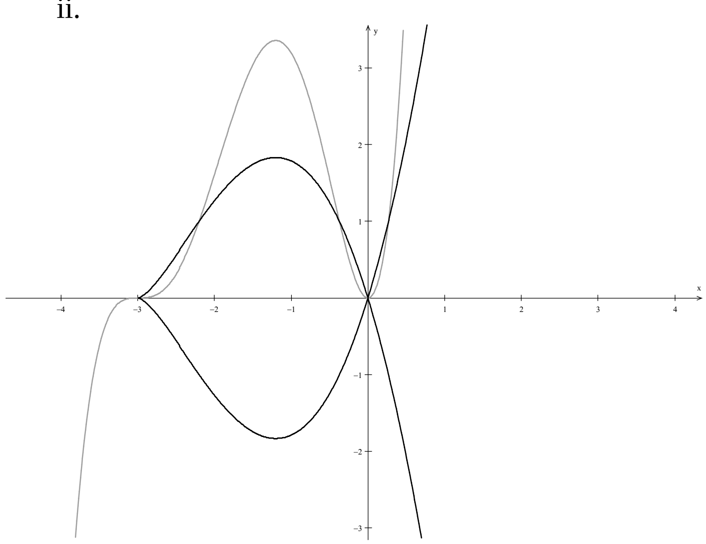
a)

i.



Generally well done. The main point of issue (in all graphing questions) is what happens when $y = 0, \pm 1$. Many did not make this clear or omitted showing this. This graph has 3 clear points where $y = 1$, so show what happens at these points. Many had the graph where $x < 0$ above the line $y = 1$.

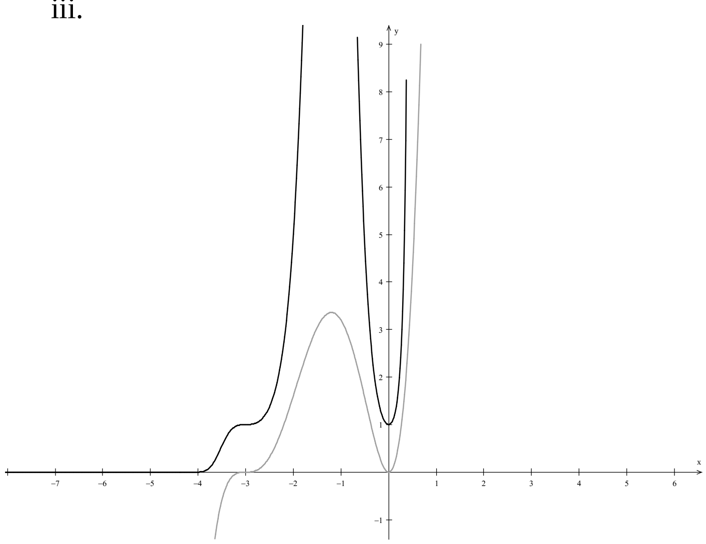
ii.



Many did not understand the effect of the $\sqrt{\ }$ for a cubic root, and the cusp it generates.

Some did not realise that $y^2 \geq 0 \Rightarrow f(x) \geq 0$ for graph to exist.

iii.



Mostly well done, but again the cubic effect at $x = -3$ was not well understood.
Also note $e^x \to \infty \, as \, x \to \infty$ <u>very rapidly</u> compared to non-exponential graphs.

b)

i. Eccentricity is $e^2 = 1 - \frac{b^2}{a^2}$ (ellipse has $0 \le e \le 1$). Thus

$$e^2 = 1 - \frac{4}{9}$$
$$= \frac{5}{9}$$

Hence $e = \frac{\sqrt{5}}{3}$; ❶

foci are $(\pm ae, 0)$ or $\left(\pm 3.\frac{\sqrt{5}}{3}, 0\right)$;

$$= \left(\pm\sqrt{5}, 0\right)$$ ❶

directrices are $y = \pm\frac{a}{e}$

$$= \pm 3.\frac{3}{\sqrt{5}}$$ ❶

$$= \pm\frac{9\sqrt{5}}{5}$$

Generally well done.

ii. differentiating $\frac{x^2}{9} + \frac{y^2}{4} = 1$:

$$\frac{2x}{9} + \frac{2y}{4}\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{-2x}{9}.\frac{2}{y}$$
$$= \frac{-4x}{9y}$$

At $P(3\cos\theta, 2\sin\theta)$:

$$\frac{dy}{dx} = \frac{-4.3\cos\theta}{9.2\sin\theta}$$
$$= \frac{-2\cos\theta}{3\sin\theta}$$ ❶

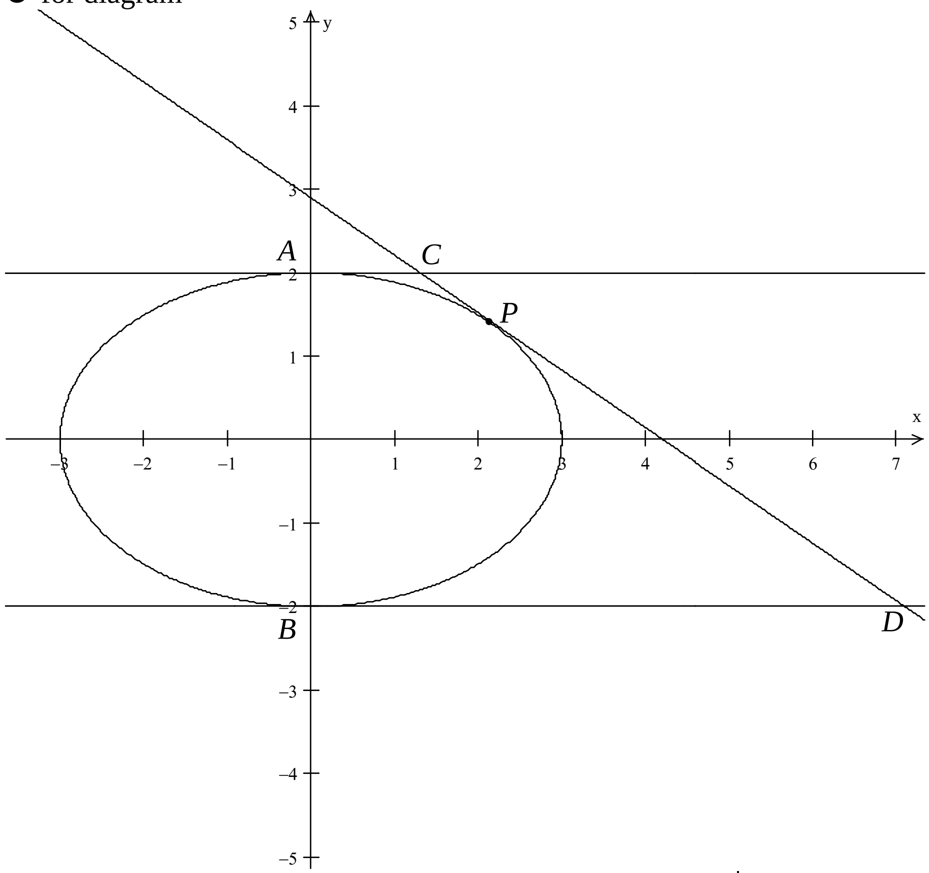
Thus the equation of the tangent is

$$y - y_1 = m(x - x_1)$$
$$y - 2\sin\theta = \frac{-2\cos\theta}{3\sin\theta}(x - 3\cos\theta)$$
$$3y\sin\theta - 6\sin^2\theta = -2x\cos\theta + 6\cos^2\theta$$
$$2x\cos\theta + 3y\sin\theta = 6\sin^2\theta + 6\cos^2\theta$$
$$\frac{x\cos\theta}{3} + \frac{y\sin\theta}{2} = \sin^2\theta + \cos^2\theta$$ ❶
$$\frac{x\cos\theta}{3} + \frac{y\sin\theta}{2} = 1$$

Generally well done.

$2x\cos\theta + 6y\sin\theta = 6$ was accepted, but note this is not the usual form for the tangent equation.

iii. ❶ for diagram



Working	Comments
Tangents at A and B are $y=\pm 2$; subst. in equation of tangent:	Generally well done.
$y=2:\Rightarrow \dfrac{x\cos\theta}{3}+\sin\theta=1$	
$x=\dfrac{3(1-\sin\theta)}{\cos\theta}$	
Thus $AC=\dfrac{3(1-\sin\theta)}{\cos\theta}$ ❶.	
$y=-2:\Rightarrow \dfrac{x\cos\theta}{3}-\sin\theta=1$	Many were not explicit about the distances – an x co-ordinate is **not** a distance!
$x=\dfrac{3(1+\sin\theta)}{\cos\theta}$	
Thus $BD=\dfrac{3(1+\sin\theta)}{\cos\theta}$ ❶. Then	
$AC.BD=\dfrac{3(1-\sin\theta)}{\cos\theta}.\dfrac{3(1+\sin\theta)}{\cos\theta}$	
$=\dfrac{9(1-\sin^2\theta)}{\cos^2\theta}$	
$=\dfrac{9\cos^2\theta}{\cos^2\theta}$ ❶	
$=9$ as reqd.	

Q14

Comments

a) $\int \frac{\cos 2x}{\cos^2 x} dx$

$= \int \frac{2\cos^2 x - 1}{\cos^2 x} dx$ ❶

$= \int 2 - \sec^2 x \, dx$

$= 2x - \tan x + c$ ❶

Some students unable to successfully integrate at the last step.

b) $\int_0^1 \sin^{-1} x dx$ with parts: $u = \sin^{-1} x$ $\quad dv = 1$ ❶

$du = \frac{1}{\sqrt{1-x^2}}$ $\quad v = x$

$= \left[x \sin^{-1} x \right]_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}} dx$ ❶

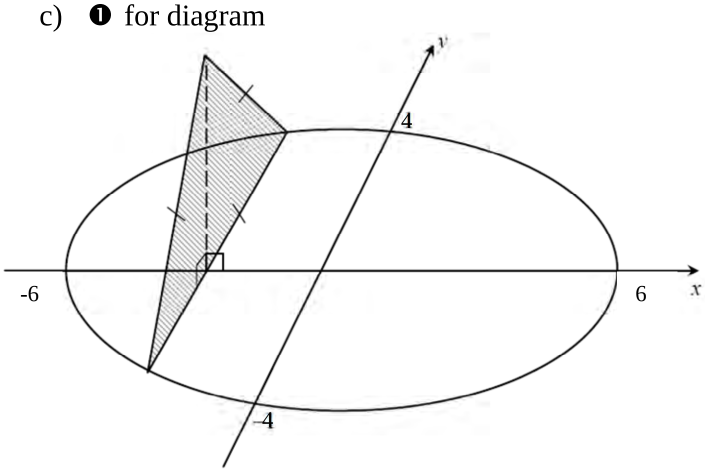
$= \left(1.\frac{\pi}{2} - 0 \right) + \frac{1}{2} \int_0^1 \frac{-2x}{\sqrt{1-x^2}} dx$

$= \frac{\pi}{2} + \frac{1}{2} \left[2\sqrt{1-x^2} \right]_0^1$

$= \frac{\pi}{2} - 1$ ❶

Some students failed to produce $2\sqrt{1-x^2}$ at the final step of the integration.

c) ❶ for diagram



For the triangle: $A = \frac{1}{2} ab \sin C$

$= \frac{1}{2} 2y.2y.\sin \frac{\pi}{3}$

$= \sqrt{3} y^2$ ❶

Then a slice of volume $\delta V = \sqrt{3} y^2 \delta x$

Re-arranging the equation: $y^2 = \frac{576 - 16x^2}{36}$, so

Well done, although a few students incorrectly attempted to work backwards from the result, rather than forward.

$$\delta V = \sqrt{3}\left(\frac{576-16x^2}{36}\right)\delta x$$ ❶

Then

$$V \approx \sum \delta V$$

$$= \lim_{\delta x \to 0} \sum_{-6}^{6} \frac{\sqrt{3}}{36}\left(576-16x^2\right)\delta x$$ ❶

$$= \frac{\sqrt{3}}{36}\int_{-6}^{6} 576-16x^2\,dx$$

$$= \frac{\sqrt{3}}{18}\int_{0}^{6} 576-16x^2\,dx$$

$$= \frac{\sqrt{3}}{18}\left[576x-\frac{16x^3}{3}\right]_0^6$$

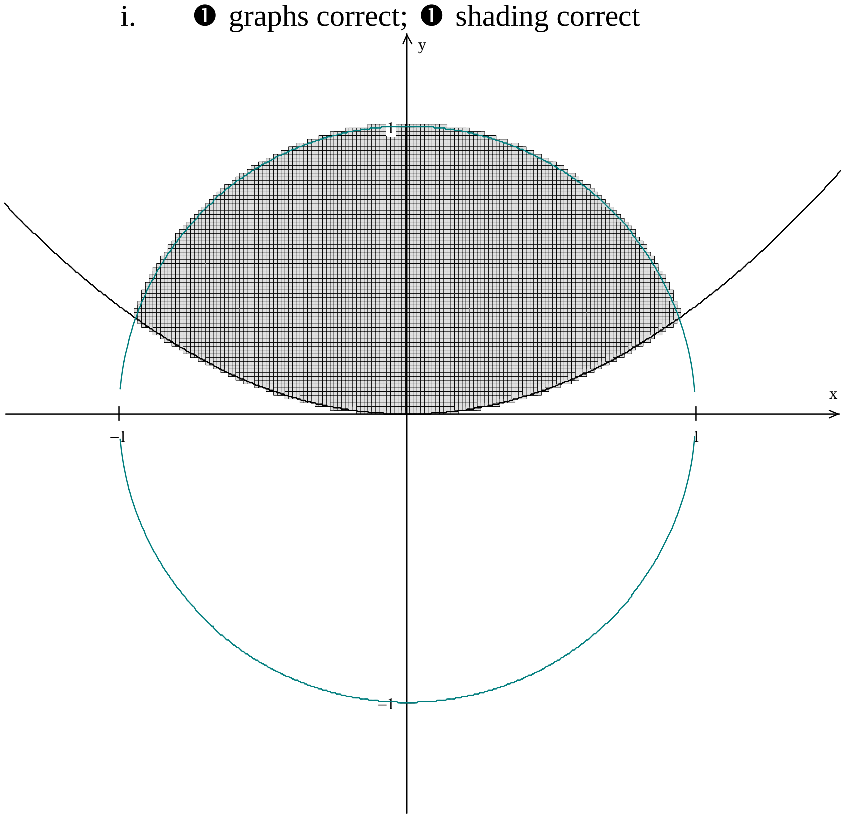
$$= \frac{\sqrt{3}}{18}\left(3456-\frac{3456}{3}\right)$$ ❶

$$= \frac{\sqrt{3}}{18}.2304$$

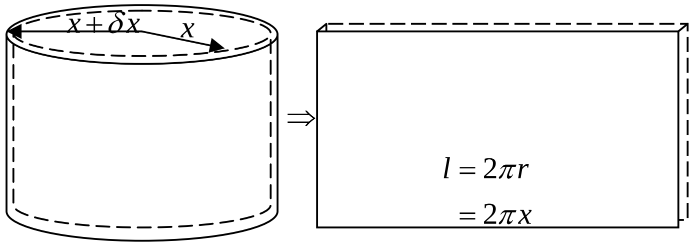
$$= 128\sqrt{3}$$ as reqd.

d)

i. ❶ graphs correct; ❶ shading correct



ii. ❶ for diagrams of shell



$b = y_2 - y_1$

$= \sqrt{1-x^2} - \frac{3}{8}x^2$

Thus $\delta A = lb$

$$= 2\pi x\left(\sqrt{1-x^2} - \frac{3}{8}x^2\right)$$

Then $\delta V = \delta A.\delta x$

$$= 2\pi x\left(\sqrt{1-x^2} - \frac{3}{8}x^2\right)\delta x$$

For bounds: $y = \frac{3}{8}x^2$ into $x^2 + y^2 = 1$:

$$x^2 + \left(\frac{3}{8}x^2\right)^2 = 1$$

$$x^2 + \frac{9x^4}{64} = 1$$

$$64x^2 + 9x^4 = 64$$

$$9x^4 + 64x^2 - 64 = 0$$

Then

$$x^2 = \frac{-64 \pm \sqrt{64^2 - 4.9.(-64)}}{2.9}$$

$$= \frac{-64 \pm \sqrt{6400}}{18}$$

$$= \frac{-64+80}{18}\left(reject\ \frac{-64-80}{18} as\ x\ real\right)$$

$$= \frac{16}{18}$$

$$= \frac{8}{9}$$

$$x = \pm\frac{2\sqrt{2}}{3}$$ ❶

Generally well done to this point.

Hence:

$V \approx \sum \delta V$

$$= \lim_{\delta x \to 0} \sum_{0}^{\frac{2\sqrt{2}}{3}} 2\pi x \left(\sqrt{1-x^2} - \frac{3}{8}x^2 \right) \delta x$$

$$= 2\pi \int_{0}^{\frac{2\sqrt{2}}{3}} x \left(\sqrt{1-x^2} - \frac{3}{8}x^2 \right) dx \text{❶}$$

$$= -\pi \int_{0}^{\frac{2\sqrt{2}}{3}} \frac{3}{4}x^3 - 2x\left(1-x^2\right)^{\frac{1}{2}} dx$$

$$= -\pi \left[\frac{3}{16}x^4 + \frac{2}{3}\left(1-x^2\right)^{\frac{3}{2}} \right]_0^{\frac{2\sqrt{2}}{3}}$$

$$= -\pi \left[\left(\frac{3}{16}.\left(\frac{2\sqrt{2}}{3}\right)^4 + \frac{2}{3}\left(1-\left(\frac{2\sqrt{2}}{3}\right)^2\right)^{\frac{3}{2}} \right) - \left(0+\frac{2}{3}\right) \right]$$

$$= -\pi \left[\frac{3}{16}.\frac{64}{81} + \frac{2}{3}\left(1-\frac{8}{9}\right)^{\frac{3}{2}} - \frac{2}{3} \right]$$

$$= -\pi \left[\frac{3}{16}.\frac{64}{81} + \frac{2}{3}\left(1-\frac{8}{9}\right)^{\frac{3}{2}} - \frac{2}{3} \right]$$

$$= -\pi \left(\frac{-40}{81} \right)$$

$$= \frac{40\pi}{81} cu.units \text{❶}$$

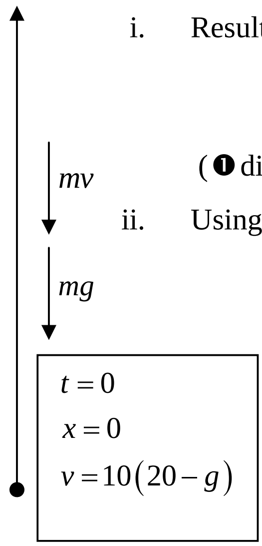
Generally well done to the point of integration, but many students failed to simplify properly.

Q15

Comments

a) Forces Diagram – Upward Motion:

Diagram lacking in most answers.



i. Resultant Force: $m\ddot{x} = -mg - m\dfrac{v}{10}$

$$\ddot{x} = -g - \frac{v}{10}$$

(❶ diagram & forces resolution)

ii. Using $\ddot{x} = \dfrac{dv}{dt}$ gives

$$\frac{dv}{dt} = -g - \frac{v}{10}$$

$$= \frac{-(10g + v)}{10}$$

Then

$$\frac{10\,dv}{10g + v} = -dt$$

Integrating:

$$10\ln(10g + v) = -t + c$$

$t = 0, v = 10(20 - g)$ gives

$$10\ln(10g + 10(20 - g)) = c$$

$$10\ln 200 = c$$

Reasonably well done.

Thus:

$$t = 10\ln 200 - 10\ln(10g + v)$$

$$= 10\ln\left(\frac{200}{10g + v}\right) \text{❶}$$

When $v = 0, t = T$:

$$T = 10\ln\left(\frac{200}{10g}\right) \text{❶}$$

$$= 10\ln\left(\frac{20}{g}\right) \quad \text{as reqd.}$$

iii. Using $\ddot{x} = v\dfrac{dv}{dx}$ gives

$$v\frac{dv}{dx} = -g - \frac{v}{10}$$

$$= \frac{-(10g + v)}{10}$$

$$dx = \frac{-10v\,dv}{10g + v}$$

$$dx = -10.\left(\frac{10g + v - 10g}{10g + v}\right)dv$$

$$= -10\left(1 - \frac{10g}{10g + v}\right)dv$$

Integrating:

$x = -10\left(v - 10g\ln\left(10g + v\right)\right) + c$ ❶

When $x = 0, v = 10(20 - g)$:

$0 = -10\left(10(20-g) - 10g\ln\left(10g + 10(20-g)\right)\right) + c$

$\quad = -10\left(200 - 10g - 10g\ln 200\right) + c$

$c = 100\left(20 - g - g\ln 200\right)$

$x = -10\left(v - 10g\ln\left(10g+v\right)\right) + 100\left(20 - g - g\ln 200\right)$

$\quad = -10v + 100g\ln\left(10g+v\right) + 2000 - 100g - 100g\ln 200$

$\quad = 2000 - 10v - 100g\left[1 + \ln 200 - \ln\left(10g+v\right)\right]$

$\quad = 2000 - 10v - 100g\left[1 + \ln\dfrac{200}{(10g+v)}\right]$ ❶

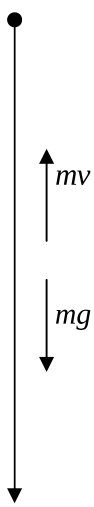
When $v = 0, x = H$:

$H = 2000 - 100g\left[1 + \ln\left(\dfrac{200}{10g}\right)\right]$

$\quad = 2000 - 10g\left[10 + 10\ln\left(\dfrac{20}{g}\right)\right]$ ❶

$\quad = 2000 - 10g\left[10 + T\right]$ as reqd.

iv. Forces Diagram - Downward Motion:



resolving: $m\ddot{x} = mg - m\dfrac{v}{10}$

$\ddot{x} = g - \dfrac{v}{10}$

(❶ diagram & forces resolution)

as $\ddot{x} \to 0$:

$g - \dfrac{v}{10} \to 0$

$\dfrac{v}{10} \to g$

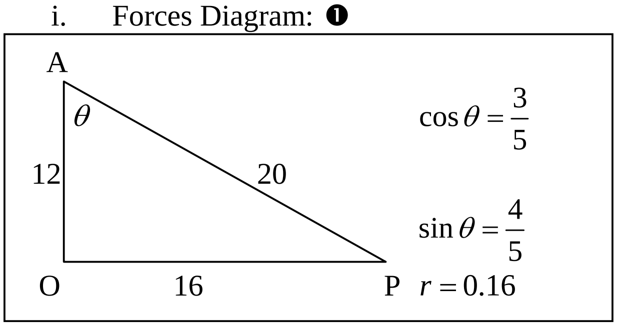
$v \to 10g$

Hence the terminal velocity in this medium is $10g$. ❶

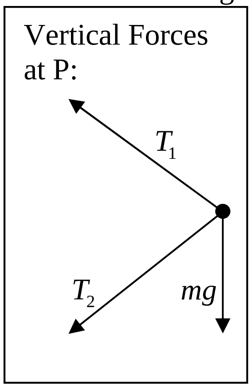
Many forgot that g is now in the positive direction for this case.

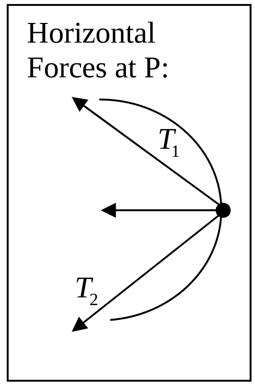
b)

i. Forces Diagram: ❶



ii. Resolving Forces:





Vertically:
No motion

$$mg + T_2 \cos\theta - T_1 \cos\theta = 0 ❶$$

$$\frac{3}{5}(T_1 - T_2) = mg$$

$$T_1 - T_2 = \frac{5mg}{3} \quad ①$$

Horizontally:
circular motion

$$T_1 \sin\theta + T_2 \sin\theta = \frac{mv^2}{r} ❶$$

$$\frac{4}{5}(T_1 + T_2) = \frac{mv^2}{0.16}$$

$$T_1 + T_2 = \frac{500mv^2}{64} \quad ②$$

Many had $r = 16$: need to check units!

Many failed to solve these simultaneous equation by this easy method.

iii. Solving:

①+②:

$$2T_1 = \frac{5mg}{3} + \frac{500mv^2}{64}$$

$$T_1 = \frac{5m}{2}\left(\frac{100v^2}{64} + \frac{g}{3}\right) ❶$$

②-①:

$$2T_2 = \frac{500mv^2}{64} - \frac{5mg}{3}$$

$$T_2 = \frac{5m}{2}\left(\frac{100v^2}{64} - \frac{g}{3}\right) ❶$$

iv. To maintain the system, $T_2 > 0$.
i.e.

$$\frac{5m}{2}\left(\frac{100v^2}{64} - \frac{g}{3}\right) > 0 ❶$$

$$\frac{100v^2}{64} - \frac{g}{3} > 0$$

$$\frac{100v^2}{64} > \frac{g}{3}$$

$$v^2 > \frac{64g}{300}$$

$$v > \frac{8\sqrt{g}}{10\sqrt{3}} ❶$$

$$> \frac{4\sqrt{3g}}{15} \quad \text{as reqd.}$$

Q16

Comments

a)

i. $\frac{d}{dx}\ln\left(\sec x+\tan x\right)$

$=\frac{1}{\sec x+\tan x}.\left(\sec x.\tan x+\sec^2 x\right)$

$=\frac{\sec x\left(\sec x+\tan x\right)}{\left(\sec x+\tan x\right)}$ ❶

$=\sec x$

Mostly well done, but several students could not use the chain rule correctly!

ii.

$\int_0^{\frac{\pi}{4}}\sec x\,dx$

$=\left[\ln\left(\sec x+\tan x\right)\right]_0^{\frac{\pi}{4}}$ ❶

$=\ln\left(\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right)-\ln\left(\sec 0+\tan 0\right)$

$=\ln\left(\sqrt{2}+1\right)-\ln\left(1+0\right)$ ❶

$=\ln\left(\frac{\sqrt{2}+1}{1}\right)$

$=\ln\left(1+\sqrt{2}\right)$ as reqd.

Generally well done.

Those who lost marks generally did not show the links (not enough working lines).

iii.

$I_n=\int_0^{\frac{\pi}{4}}\sec^n x\,dx$

$=\int_0^{\frac{\pi}{4}}\sec^{n-2} x.\sec^2 x\,dx$

$u=\sec^{n-2} x$

$du=\left(n-2\right)\sec^{n-3} x.\sec x.\tan x$ $\quad dv=\sec^2 x$

$=\left(n-2\right)\sec^{n-2} x.\tan x$ $\quad v=\tan x$ ❶

This is a standard “by parts” integration you must know! Many lost easy marks here.

Leads to:

$$I_n = \left[\sec^{n-2} x.\tan x\right]_0^{\frac{\pi}{4}} - (n-2)\int_0^{\frac{\pi}{4}} \sec^{n-2} x.\tan^2 x\,dx$$ ❶

$$= \left[\left(\sec^{n-2}\frac{\pi}{4}.\tan\frac{\pi}{4}\right) - \left(\sec^{n-2} 0.\tan 0\right)\right] - (n-2)\int_0^{\frac{\pi}{4}} \sec^{n-2} x.\left(\sec^2 x - 1\right)dx$$

$$= \left(\sqrt{2}\right)^{n-2} - (n-2)\int_0^{\frac{\pi}{4}} \sec^n x\,dx + (n-2)\int_0^{\frac{\pi}{4}} \sec^{n-2} x\,dx$$

$$= \left(\sqrt{2}\right)^{n-2} - (n-2)I_n + (n-2)I_{n-2}$$ ❶

$$(n-2+1)I_n = \left(\sqrt{2}\right)^{n-2} + (n-2)I_{n-2}$$

$$I_n = \frac{1}{(n-1)}\left[\left(\sqrt{2}\right)^{n-2} + (n-2)I_{n-2}\right]$$

iv. $\therefore I_3 = \frac{1}{(3-1)}\left[\left(\sqrt{2}\right)^{3-2} + (3-2)I_{3-2}\right]$

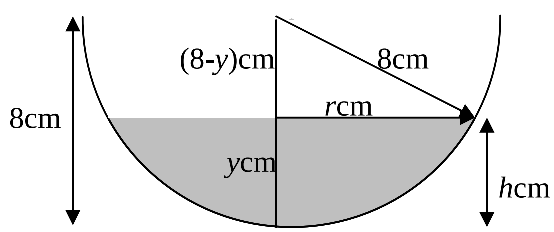
$$= \frac{1}{2}\left[\sqrt{2} + I_1\right]$$

$$= \frac{1}{2}\left[\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right]$$ ❶

Usually well done, but several students could not use the I_1 they found in part (ii) correctly.

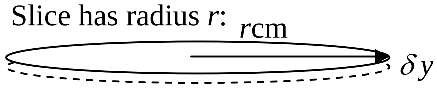
b)

i. Hemisphere to depth of y:



❶

Slice has radius r:



$$r = \sqrt{8^2 - (8-y)^2}$$

$$= \sqrt{8^2 - \left(8^2 - 16y + y^2\right)}$$

$$= \sqrt{8^2 - 8^2 + 16y - y^2}$$

$$= \sqrt{16y - y^2}$$ ❶

Area of cross-section:

$$\delta A = \pi r^2$$

$$= \pi\left(16y - y^2\right)$$

Thin slice of volume:

Diagrams were generally very poor. Most did not explain the relationship to find the radius with reference to the problem as defined. (some did from -8 to $-8+h$ with $x^2 = 8^2 - y^2$, but had poor explanations).

Slice diagram almost never drawn, leading to most of the errors.

Many did not deal with the variables appropriately: h, x and y were used almost interchangeably, resulting in confusion, specially for limits.

Very poor links here in most solutions.

$\delta V = \pi\left(16y - y^2\right)\delta y$

Those that got to this point were generally fine from here.

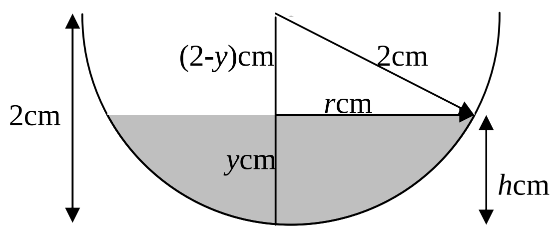
$\therefore$ Volume to depth h is:

$$V \approx \sum \delta V$$
$$= \lim_{\delta x \to 0} \sum_0^h \pi\left(16y - y^2\right)\delta y \text{ ❶}$$
$$= \int_0^h \pi\left(16y - y^2\right)dy$$
$$= \pi\left[8y^2 - \frac{1}{3}y^3\right]_0^h$$
$$= \pi\left(8h^2 - \frac{h^3}{3}\right) \qquad \text{as reqd.}$$

ii. For Case I, we need to subtract the volume of the ice cream to depth h.

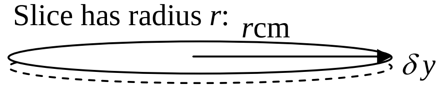
Again, very poor diagrams & links.

Comments from part (i) apply here too.



❶

Slice has radius r:



$$r = \sqrt{2^2 - (2-y)^2}$$
$$= \sqrt{2^2 - \left(2^2 - 4y + y^2\right)}$$
$$= \sqrt{2^2 - 2^2 + 4y - y^2}$$
$$= \sqrt{4y - y^2}$$

Area of cross-section:

$$\delta A = \pi r^2$$
$$= \pi\left(4y - y^2\right)$$

Thin slice of volume:

$$\delta V = \pi\left(4y - y^2\right)\delta y \text{ ❶}$$

$\therefore$ Volume to depth h is:

$$V \approx \sum \delta V$$

$$= \lim_{\delta x \to 0} \sum_{0}^{h} \pi \left(4y - y^2\right) \delta y$$

$$= \int_{0}^{h} \pi \left(4y - y^2\right) dy$$

$$= \pi \left[2y^2 - \frac{1}{3} y^3 \right]_0^h$$

$$= \pi \left(2h^2 - \frac{h^3}{3} \right) \quad ❶$$

Hence volume of milk is

$$V = \pi \left(8h^2 - \frac{h^3}{3} \right) - \pi \left(2h^2 - \frac{h^3}{3} \right) ❶$$

$$= \pi \left(8h^2 - \frac{h^3}{3} - 2h^2 + \frac{h^3}{3} \right)$$

$$= 6\pi h^2 \qquad \text{as reqd.}$$

iii. In Case II, the entire sphere of ice cream is submerged, hence

$$V = \pi \left(8h^2 - \frac{h^3}{3} \right) - \frac{4}{3} \pi . 2^3$$

$$= \pi \left(8h^2 - \frac{h^3}{3} - \frac{32}{3} \right)$$

$$= \frac{\pi}{3} \left(24h^2 - h^3 - 32 \right) ❶ \text{ cubic units.}$$

Most students who attempted this part managed to resolve it correctly, even when unable to complete the first two parts.