Name: ____________________

Teacher: ____________________

Class: ____________________

FORT STREET HIGH SCHOOL

2016

HIGHER SCHOOL CERTIFICATE COURSE

ASSESSMENT TASK 3: TRIAL HSC

Mathematics Extension 2

Time allowed: 3 hours
(plus 5 minutes reading time)

Syllabus Outcomes	Assessment Area Description and Marking Guidelines	Questions
	Chooses and applies appropriate mathematical techniques in order to solve problems effectively	1-10
E2, E3	Applies appropriate strategies to construct arguments and proofs in the areas of complex numbers and polynomials	11, 12
E4, E6	Uses efficient techniques for the algebraic manipulation of conic sections and determining features of a wide variety of graphs	13
E7, E8	Applies further techniques of integration, such as slicing and cylindrical shells, integration by parts and recurrence formulae, to problems	14
E5	Uses ideas and techniques of calculus to solve problems in mechanics involving resolution of forces, resisted motion and circular motion	15
E2-E8	Synthesises mathematical processes to solve harder problems and communicates solutions in an appropriate form	16

Total Marks 100

Section I 10 marks
Multiple Choice, attempt all questions.
Allow about 15 minutes for this section.

Section II 90 Marks
Attempt Questions 11-16.
Allow about 2 hours 45 minutes for this section.

General Instructions:

- Questions 11-16 are to be started in a new booklet.
- The marks allocated for each question are indicated.
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations.
- Marks may be deducted for careless or badly arranged work.
- Board – approved calculators may be used.

Section I	Total 10	Marks
Q1-Q10	/10	
Section II	Total 90	Marks
Q11	/15	
Q12	/15	
Q13	/15	
Q14	/15	
Q15	/15	
Q16	/15	
	Percent	

Section I

1 What is $-\sqrt{3}+i$ expressed in modulus-argument form?

(A) $\sqrt{2}(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})$

(B) $2(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})$

(C) $\sqrt{2}(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6})$

(D) $2(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6})$

2 Consider the hyperbola with the equation $\frac{x^2}{4}-\frac{y^2}{3}=1$.

What are the coordinates of the vertex of the hyperbola?

(A) $(\pm 2, 0)$

(B) $(0, \pm 2)$

(C) $(0, \pm 4)$

(D) $(\pm 4, 0)$

3 The points $P(a\cos\theta,\ b\sin\theta)$ and $Q(a\cos\phi,\ b\sin\phi)$ lie on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the chord PQ subtends a right angle at $(0,0)$. Which of the following is the correct expression?

(A) $\tan\theta\tan\phi=-\frac{b^2}{a^2}$

(B) $\tan\theta\tan\phi=-\frac{a^2}{b^2}$

(C) $\tan\theta\tan\phi=\frac{b^2}{a^2}$

(D) $\tan\theta\tan\phi=\frac{a^2}{b^2}$

4 What is the value of $\int -\sec x dx$? Use the substitution $t=\tan\frac{x}{2}$.

(A) $\ln|(t-1)(t+1)|+c$

(B) $\ln|\frac{1+t}{1-t}|+c$

(C) $\ln|(1-t)(t+1)|+c$

(D) $\ln|\frac{t-1}{t+1}|+c$

5 What is the volume of the solid formed when the region bounded by the curves $y = 2x^3$ and $y = 2\sqrt{x}$ is rotated about the x-axis? Use the method of slicing.

(A) $\frac{5\pi}{14}$ cubic units

(B) $\frac{10\pi}{14}$ cubic units

(C) $\frac{5\pi}{7}$ cubic units

(D) $\frac{10\pi}{7}$ cubic units

6 A particle of mass m is moving in a straight line under the action of a force.

$$F = \frac{m}{x^3}(6 - 10x)$$

Which of the following is an expression for its velocity at any position, if the particle starts from rest at $x = 1$?

(A) $v = \pm\frac{1}{x}\sqrt{(-3 + 10x - 7x^2)}$

(B) $v = \pm x\sqrt{(-3 + 10x - 7x^2)}$

(C) $v = \pm\frac{1}{x}\sqrt{2(-3 + 10x - 7x^2)}$

(D) $v = \pm x\sqrt{2(-3 + 10x - 7x^2)}$

7 The equation $y^3 - xy + x^3 = 7$ implicitly defines y in terms of x.

Which of the following is an expression for $\frac{dy}{dx}$?

(A) $\frac{-3x^2}{3y^2 - 1}$

(B) $\frac{y - 3x^2}{3y^2 - x}$

(C) $\frac{y - 3x^2 + 7}{3y^2 - x}$

(D) $\frac{3y^2 - y + 3x^2}{x}$

8 Let α, β and γ be roots of the equation $x^3+3x^2+4=0$. Which of the following polynomial equations have roots α^2, β^2 and γ^2?

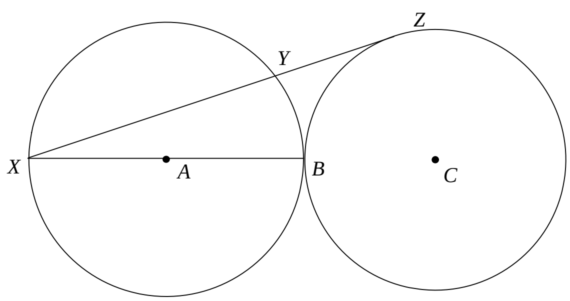
(A) $x^3-9x^2-24x-4=0$

(B) $x^3-9x^2-12x-4=0$

(C) $x^3-9x^2-24x-16=0$

(D) $x^3-9x^2-12x-16=0$

9 Two equal circles touch externally at *B*. *XB* is a diameter of one circle. *XZ* is the tangent from *X* to the other circle and cuts the first circle at *Y*.



Which is the correct expression that relates *XZ* to *XY*?

(A) $3XZ=4XY$

(B) $XZ=2XY$

(C) $2XZ=3XY$

(D) $2XZ=5XY$

10 Which conic has eccentricity $\frac{\sqrt{13}}{3}$?

(A) $\frac{x^2}{3}+\frac{y^2}{2}=1$

(B) $\frac{x^2}{3^2}+\frac{y^2}{2^2}=1$

(C) $\frac{x^2}{3}-\frac{y^2}{2}=1$

(D) $\frac{x^2}{3^2}-\frac{y^2}{2^2}=1$

Section II

Question 11 (15 marks) START A NEW BOOKLET Marks

a) (i) Find real numbers A and B such that

$$\frac{7x+1}{x^2-x-2}=\frac{A}{x+1}+\frac{B}{x-2}$$ 2

(ii) Hence, find $\int \frac{7x+1}{x^2-x-2}\,dx$ 1

b) (i) Prove $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ 2

(ii) Hence evaluate $\int_0^{\frac{\pi}{4}} \frac{1-\tan x}{1+\tan x}\,dx$ 3

c) If $z_1 = 3i$ and $z_2 = 1+i$, find values of

i) $|z_1 - z_2|$ 1

ii) $z_1 + \overline{z_2}$ 1

iii) $\frac{z_1}{z_2}$ 1

d) z is a complex number such that $\arg(z)=\frac{\pi}{3}$ and $|z|\le 2$.

i) Show the locus of the point P representing z in the Argand diagram. 2

ii) Find the possible values of the principal argument of $z-i$ for z on the locus. 2

Question 12 (15 marks) START A NEW BOOKLET Marks

a) Suppose α, β, γ and δ are the four roots of the polynomial equation

$$x^4 + px^3 + qx^2 + rx + s = 0$$

i) find the values of $\alpha + \beta + \gamma + \delta$ and $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$ in terms of p, q, r and s. 2

ii) show that $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = p^2 - 2q$ 2

iii) apply the results in part (ii) to show that $x^4 - 3x^3 + 5x^2 + 7x - 8 = 0$ cannot have four real roots. 1

b) Let $P(x) = x^5 - 10x^2 + 15x - 6$.

i) show that $x = 1$ is a root of $P(x)$ of multiplicity three. 2

ii) Hence, or otherwise, find the two complex roots of $P(x)$. 2

c) If α, β, γ are the roots of $x^3 - 3x^2 + 2x - 1 = 0$ find the equation whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$. 2

d) Evaluate $\int_0^2 t\, e^{-t}\, dt$. 4

Question 13 (15 marks) START A NEW BOOKLET Marks

a) Function $f(x)$ is defined by $f(x)=\log_e\left(\dfrac{2}{x-1}\right)$.

i) State the domain of $f(x)$. 1

ii) Write the equation of any asymptote to the graph of $y=f(x)$. 1

iii) Find the x-intercept(s) of $y=f(x)$. 1

iv) Is the function increasing or decreasing? Justify your answer. 2

v) use your answers to the above questions to sketch graphs of the following.

α) $y=f(x)$ 1

β) $y=\left|f(x)\right|$ 1

γ) $y=\ln\left(f(x)\right)$ 2

b) $P\left(2p,\dfrac{2}{p}\right)$ is a variable point on the hyperbola $xy=4$. The normal to the hyperbola at P meets the hyperbola again at $Q\left(2q,\dfrac{2}{q}\right)$. M is the midpoint of PQ.

i) Show that $q=\dfrac{-1}{p^3}$ 2

ii) Show that M has co-ordinates $\left[\dfrac{1}{p}\left(p^2-\dfrac{1}{p^2}\right),p\left(\dfrac{1}{p^2}-p^2\right)\right]$. 1

iii) Show that the locus of M is given by the equation $-x^3y^3=\left(y^2-x^2\right)^2$. 3

Question 14 (15 marks) START A NEW BOOKLET Marks

a) Let $I_n = \int_0^{\frac{\pi}{2}} (\sin x)^n \, dx$, where n is an integer, $n \geq 0$.

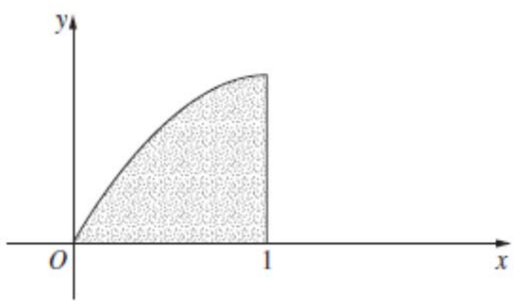
i) Using integration by parts, show that, for $n \geq 2$ **4**

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2}$$

ii) Hence show that **4**

$$\int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx = \frac{\pi (2n)!}{2^{2n+1} (n!)^2}$$

b) The diagram shows the graph of $f(x) = \dfrac{x}{1+x^2}$ for $0 \leq x \leq 1$. **4**



The area bounded by $y = f(x)$, the line $x = 1$ and the x -axis is rotated about the line $x = 1$ to form a solid.

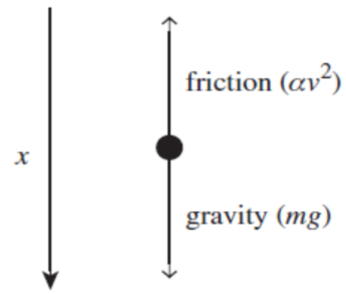
Use the method of cylindrical shells to find the volume of the solid.

c) Evaluate $\displaystyle\int \frac{\ln x}{(1+\ln x)^2} \, dx$ **3**

Question 15 (15 marks) START A NEW BOOKLET Marks

a) In the case of turbulent flow, the frictional force on an object travelling through a fluid is modelled as being proportional to the square of the travelling velocity.

In the case where a particle is falling vertically through the atmosphere, gravity is providing downward acceleration and the resistive frictional force is effectively acting upward, since friction always opposes motion. The situation is represented diagrammatically below.



Expressing the object's net force in terms of its velocity and a positive coefficient α yields

$$m\ddot{x} = mg - \alpha v^2$$

According to this representation, the net acceleration of the object is therefore given by

$$\ddot{x} = g - \beta v^2 \text{ where } \beta = \frac{\alpha}{m}$$

The coefficient β depends on the surface profile and mass of the object.

i) Show that setting the initial height of the object as $x = 0$ and assuming that the initial velocity is zero yields the following equation of motion. 3

$$\ln\left(g - \beta v^2\right) = -2\beta x + \ln g$$

ii) Show that this allows the velocity to be written as a function of position in the following way. 2

$$v(x) = \sqrt{\frac{g}{\beta}\left(1 - e^{-2\beta x}\right)}$$

iii) By using the result from (ii), state the object's terminal velocity. 1

iv) Calculate the terminal velocity of a table tennis ball in free fall through the atmosphere. Take $g = 9.8\,ms^{-2}$ and $\beta = 0.2\,m^{-1}$ and give your answer in kilometres per hour. 1

v) How far in metres will the table tennis ball fall before it reaches 80% of its terminal velocity? 2

b) A particle of mass *m*, attached by a light rod to a pivot point *C*, moves with constant speed in a horizontal circle whose centre *O* is distant *h* metres below *C*.

i. Show that the time *T* seconds taken for one revolution of the particle is given by $T = 2\pi\sqrt{\frac{h}{g}}$, where *g* metres per second per second is the acceleration due to gravity. 3

ii. discuss the effect, if any, on the motion of the particle if its speed is doubled. 3

Question 16 (15 marks) START A NEW BOOKLET Marks

a) A water sprinkler is positioned at a point O on level ground at a distance of x metres from a vertical wall. Water is projected from the sprinkler at v m/s at an angle θ to the ground. Neglecting air resistance and taking g as the acceleration due to gravity. At time t, the horizontal and vertical displacement of the water from the point of projection are $x = vt\cos\theta$, and $y = vt\sin\theta - \frac{1}{2}gt^2$ respectively (DO NOT PROVE THESE RESULTS).

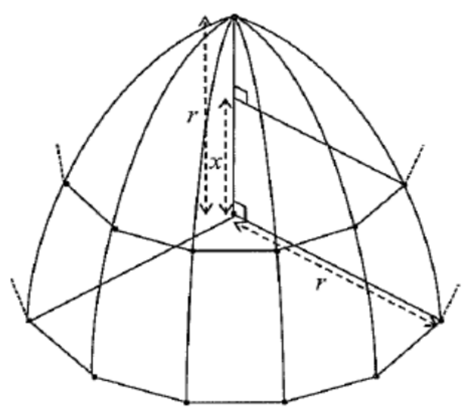
i) Show that $gx^2\tan^2\theta - 2xv^2\tan\theta + 2yv^2 + gx^2 = 0$ 2

ii) If there are two possible angles of projection, prove that $\left(v^2 - gy\right)^2 > g^2\left(x^2 + y^2\right)$. 2

iii) show that the water from sprinkler hits the base of the wall only if $v = \sqrt{gx}$ 1

iv) Given $v = 2\sqrt{gx}$, prove that the maximum height that can be reached by the water projected by the sprinkler up the wall is $\frac{15x}{8}$ metres. 3

b) The diagram below shows part of a polygonal dome. Each cross-section is a regular *n*-sided polygon.



The vertex of the dome is r units directly above the centre of the polygonal base, which is r units from each vertex. A circular arc joins the top of the dome to each vertex of the base.

i) Show that the area of the horizontal cross-section x units from the base is given by $\frac{n}{2}\left(r^2 - x^2\right)\sin\left(\frac{2\pi}{n}\right)$ 2

ii) Hence show that the volume of the dome is given by $\frac{nr^3}{3}\sin\left(\frac{2\pi}{n}\right)$. 3

iii) Show that as $n \to \infty$, the volume of the dome approaches that of a hemisphere. 2

End of Paper

Questions 1-10: Multiple Choice

1. D	2. A	3. B	4. D	5. D	6. C	7. B	8. C	9. C	10. D

Question 11:	Marking	Comments
a)		
i) $\frac{7x+1}{x^2-x-2}=\frac{A}{x+1}+\frac{B}{x-2}$, hence equating numerators		
$7x+1=A(x-2)+B(x+1)$		
$x=2:\quad 15=A(2-2)+B(2+1)\Rightarrow B=5$	❶ for correct B value	
$x=-1:\quad -6=A(-1-2)+B(-1+1)\Rightarrow A=2$	❶ for correct A value	*Well done*
$\therefore\ \frac{7x+1}{x^2-x-2}=\frac{2}{x+1}+\frac{5}{x-2}$		
ii) $\int\frac{7x+1}{x^2-x-2}\,dx$		
$=\int\frac{2}{x+1}+\frac{5}{x-2}\,dx$		
$=2\ln(x+1)+5\ln(x-2)+c$	❶ for correct integration	
b)		
i) Let $x=a-u\quad x=0\quad a=u$		
$dx=-du\quad x=a\quad u=0$	❶ for set-up	
Then		

$\int_0^a f(x)\ dx$		
$= \int_a^0 f(a-u)\ -du$	❶ for substitution	*Well done*
$= \int_0^a f(a-u)\ du$		
$= \int_0^a f(a-x)\ dx$	❶ for conclusion	
(as a definite integral is independent of the variable used.)		
ii) $\int_0^{\frac{\pi}{4}} \frac{1-\tan x}{1+\tan x}\ dx$		
$= \int_0^{\frac{\pi}{4}} \frac{1-\tan\left(\frac{\pi}{4}-x\right)}{1+\tan\left(\frac{\pi}{4}-x\right)}\ dx$		*Well done*
$= \int_0^{\frac{\pi}{4}} \frac{1-\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}}{1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}}\ dx$ (using (i) above)	❶ for using pt (i) result	

$$=\int_0^{\frac{\pi}{4}} \frac{1+\tan\frac{\pi}{4}\tan x-\left(\tan\frac{\pi}{4}-\tan x\right)}{1+\tan\frac{\pi}{4}\tan x}\cdot\frac{1+\tan\frac{\pi}{4}\tan x}{1+\tan\frac{\pi}{4}\tan x+\tan\frac{\pi}{4}-\tan x}\,dx$$

$$=\int_0^{\frac{\pi}{4}} \frac{1+\tan x-1+\tan x}{1+\tan x+1-\tan x}\,dx$$

$$=\int_0^{\frac{\pi}{4}} \frac{2\tan x}{2}\,dx$$

❶ for resolving to simple integral

$$=\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x}\,dx$$

$$=\left[-\ln(\cos x)\right]_0^{\frac{\pi}{4}}$$

$$=-\ln\frac{1}{\sqrt{2}}$$

$$=\frac{1}{2}\ln 2$$

❶ for value

c)

i) $|z_1-z_2|$

$$=\sqrt{(-1)^2+2^2}$$

$$=\sqrt{5}$$

❶ for value — *Well done*

ii) $z_1+\overline{z_2}$

$$=3i+1-i$$

$$=1+2i$$

❶ for value

iii) $\frac{z_1}{z_2}$

$= \frac{3i}{1+i} \cdot \frac{1-i}{1-i}$

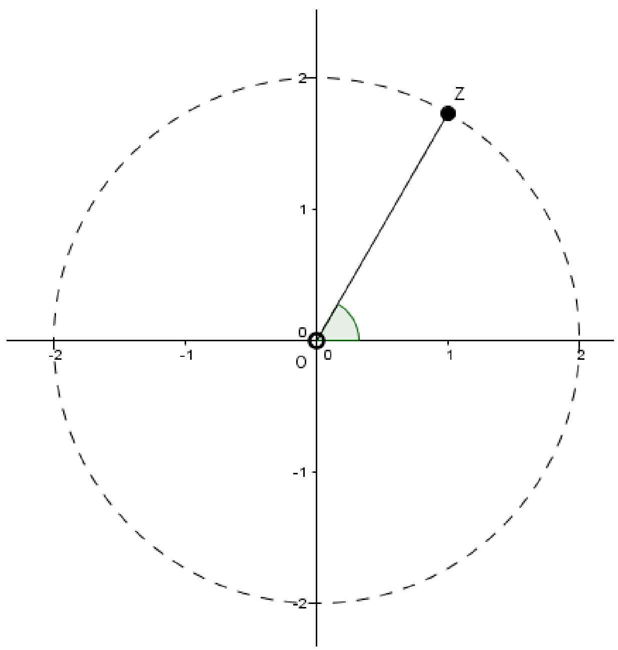
$= \frac{3i - 3i^2}{1-(-1)}$

$= \frac{3+3i}{2}$

❶ for value

d)

i)



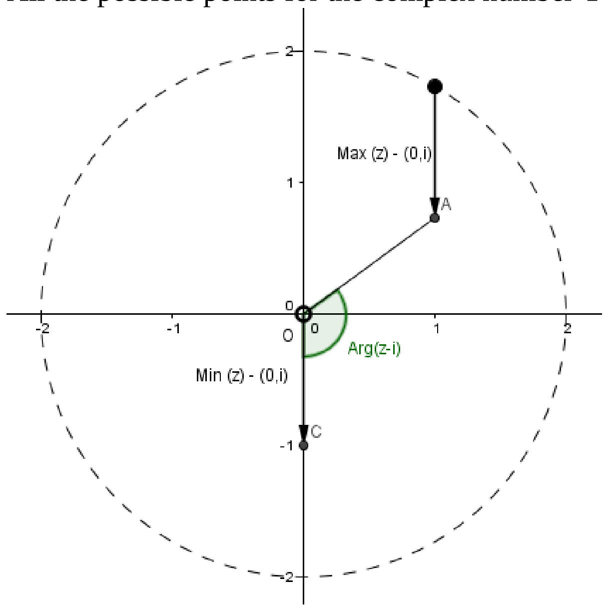
The locus of P is the interval OZ, with the point at O excluded.

❶ for diagram

❶ for explanation

Well done

ii) All the possible points for the complex number $z-i$ lie on the interval *AC*.



When $|z|=2$, $z-i$ is at A. As $|z|\to 0$, $z-i\to C$. Hence:

$\arg(C)<\arg(z-i)\le \arg\left(A\left(1,\sqrt{3}-1\right)\right)$

$\frac{-\pi}{2}<\arg(z-i)\le \tan^{-1}\left(\sqrt{3}-1\right)$

❶ for value $\frac{-\pi}{2}$

❶ for value $\tan^{-1}\left(\sqrt{3}-1\right)$

Poorly done. Most students did not get even the minimum value for $\arg(-i)$

<u>Question 12:</u>

Solution	Marking	Comments
a)		
i) $\alpha+\beta+\gamma+\delta=\frac{-b}{a}$ $=-p$	❶ for value	
$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=\frac{c}{a}$ $=q$	❶ for value	
ii) $\alpha^2+\beta^2+\gamma^2+\delta^2$ $=(\alpha+\beta+\gamma+\delta)^2-2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$	❶ for expansion/ conncetion	
$=(-p)^2-2(q)$		*Well done*
$=p^2-2q$	❶ for value	
iii) $\alpha^2+\beta^2+\gamma^2+\delta^2$ $=p^2-2q$ $=(-3)^2-2\times5$ $=-1$		
<0 Hence some roots must be complex, so not all roots are real.	❶ for conclusion with justification	

b)

i) $P(x) = x^5 - 10x^2 + 15x - 6$

$P'(x) = 5x^4 - 20x + 15$

$P''(x) = 20x^3 - 20$

$P(1) = 1^5 - 10.1^2 + 15\times 1 - 6$

$= 0$

❶ for working

$P'(x) = 5.1^4 - 20\times 1 + 15$

$= 0$

Well done

$P''(x) = 20.1^3 - 20$

$= 0$

❶ for justification

Hence $x = 1$ is a root of multiplicity 3.

ii)

$$
\begin{array}{r}
x^2 + 3x + 6 \\
x^3 - 3x^2 + 3x - 1 \enclose{longdiv}{x^5 + 0x^4 + 0x^3 - 10x^2 + 15x - 6} \\
x^5 - 3x^4 + 3x^3 - x^2 \\
3x^4 - 3x^3 - 9x^2 + 15x \\
3x^4 - 9x^3 + 9x^2 - 3x \\
6x^2 - 18x^2 + 18x - 6 \\
6x^2 - 18x^2 + 18x - 6 \\
. \quad . \quad . \quad .
\end{array}
$$

❶ for $Q(x) = x^2 + 3x + 6$

Thus $P(x) = x^5 - 10x^2 + 15x - 6$

$= (x-1)^3 Q(x)$

$= (x-1)^3 (x^2 + 3x + 6)$

So $x^2 + 3x + 6 = 0$ gives

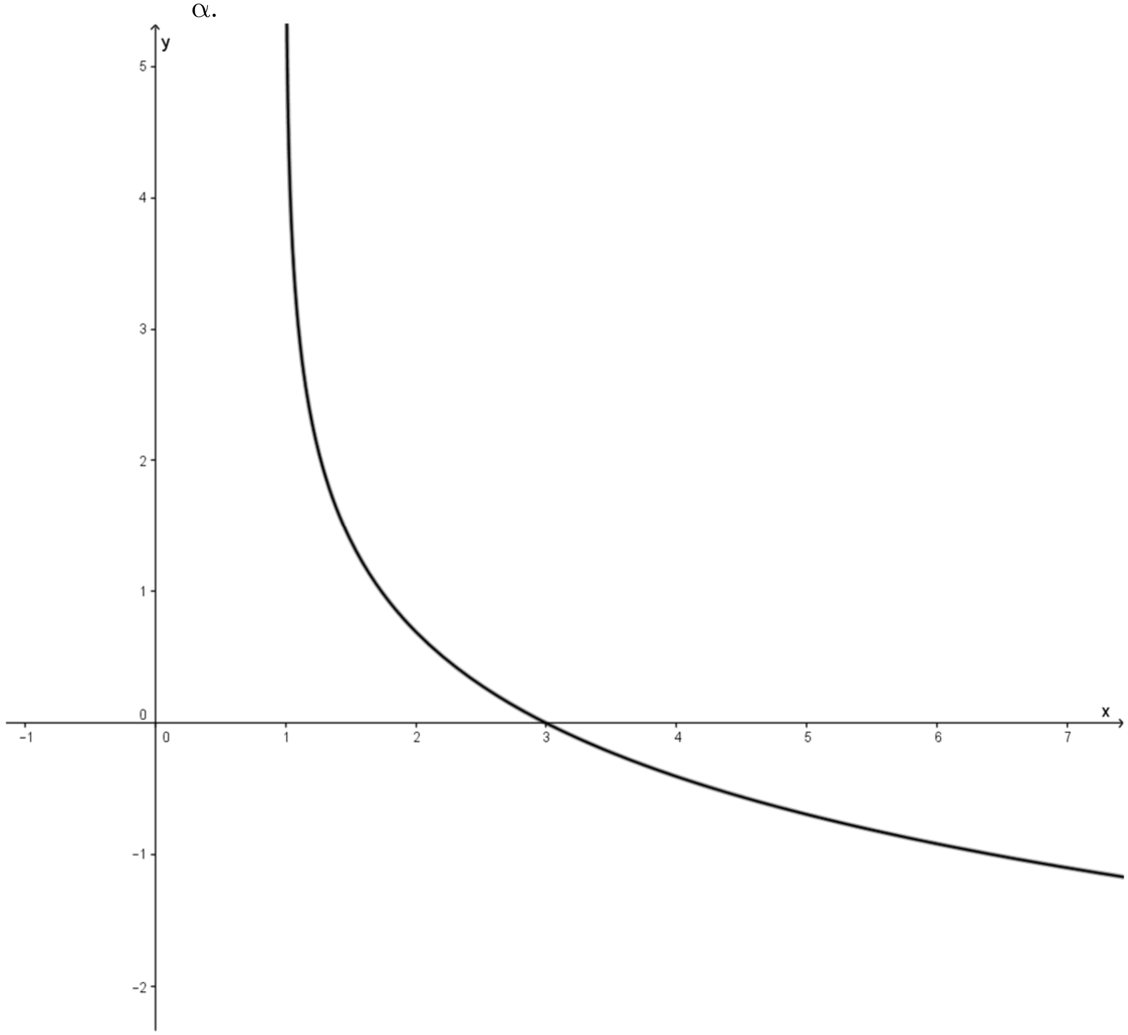
$x = \dfrac{-3 \pm \sqrt{3^2 - 4.1.6}}{2}$ $= \dfrac{-3 \pm \sqrt{-15}}{2}$ $= \dfrac{-3}{2} \pm i\dfrac{\sqrt{15}}{2}$ as the two complex roots.	❶ for values	
c) $x = \dfrac{1}{\alpha}$ $\alpha = \dfrac{1}{x}$ As $\alpha^3 - 3\alpha^2 + 2\alpha - 1 = 0$ $\left(\dfrac{1}{x}\right)^3 - 3\left(\dfrac{1}{x}\right)^2 + 2\left(\dfrac{1}{x}\right) - 1 = 0$ $1 - 3x + 2x^2 - x^3 = 0$ $x^3 - 2x^2 + 3x - 1 = 0$ is the required equation.	❶ for correct method ❶ for correct equation	*Well done*

d) Let $u = t \quad \frac{dv}{dt} = e^{-t}$ $\frac{du}{dt} = 1 \quad v = -e^{-t}$ Then $\int_0^2 t\,e^{-t}\,dt$ $= \left[-te^{-t}\right]_0^2 - \int_0^2 -e^{-t}\,dt$ $= \left(-2e^{-2} - 0\right) - \left[-e^{-t}\right]_0^2$ $= -2e^{-2} - \left(-e^{-2} - 1\right)$ $= 1 - 3e^{-2}$ $\left(= \frac{e^2 - 3}{e^2}\right)$	❶ for correct "by parts" set-up ❶ for resolving ❶ for correct answer	*Well done*

Question 13:	Marking	Comments
a) For $f(x)=\log_e\left(\frac{2}{x-1}\right)$:		
i) Domain needs $\frac{2}{x-1}>0$ $x-1>0$ $x>1$	❶ for value	
ii) $x=1$	❶ for value	*Well done for (i) – (iii)*
iii) $\ln\left(\frac{2}{x-1}\right)=0$ $\frac{2}{x-1}=1$ $2=x-1$ $x=3$ Hence the x-intercept is $(3,0)$.	❶ for point	
iv) $f(x)=\ln\left(\frac{2}{x-1}\right)$ $=\ln 2-\ln(x-1)$ $f'(x)=0-\frac{1}{x-1}$ <0 as $x>1$ from (i). Hence the function is always decreasing.	❶ for working ❶ for justification	*Some had errors in* $\frac{dy}{dx}$ *or didn't simplify sufficiently to support argument that* $\frac{dy}{dx}<0$. *Some incorrect limits used instead of sign of* $\frac{dy}{dx}$.

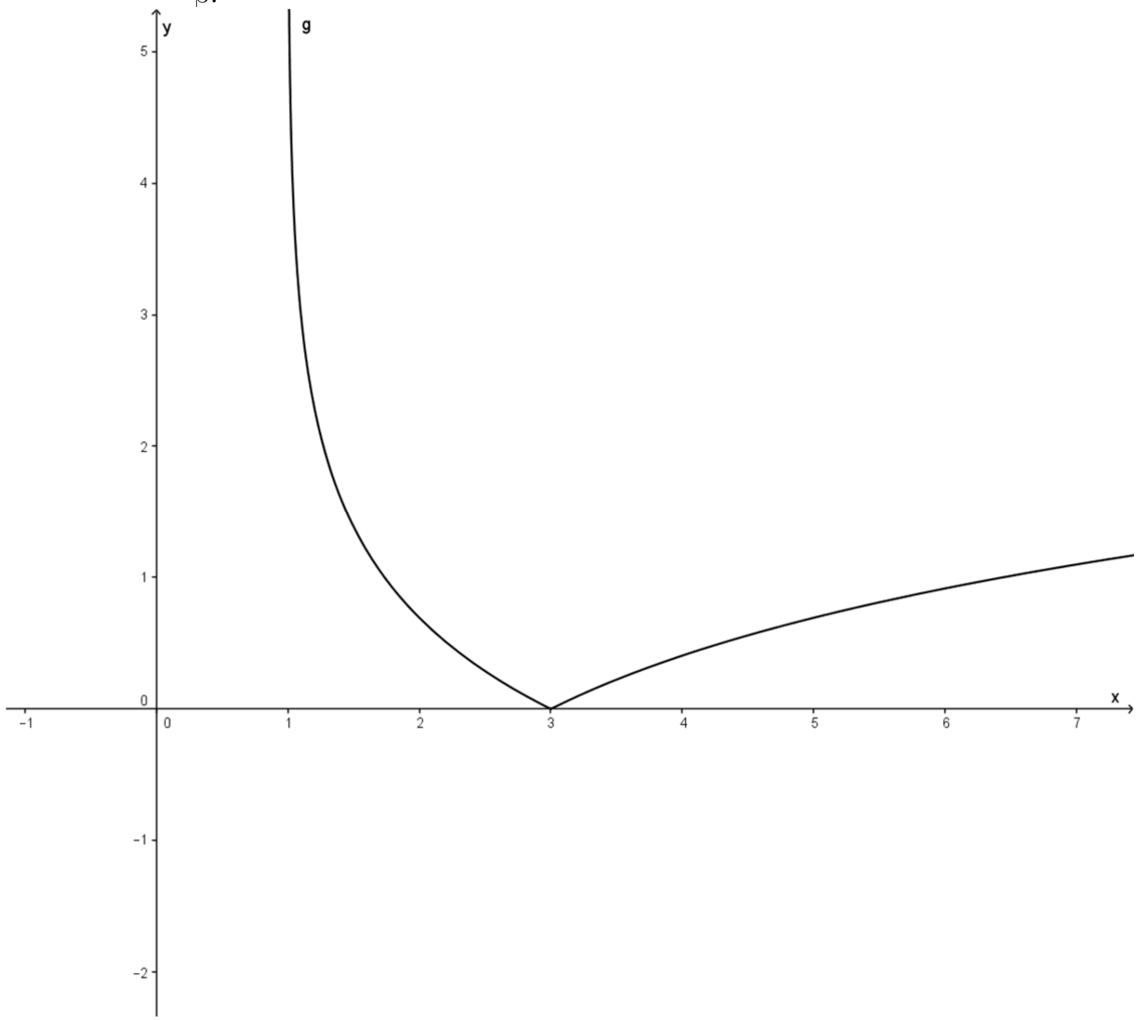
v)

α.

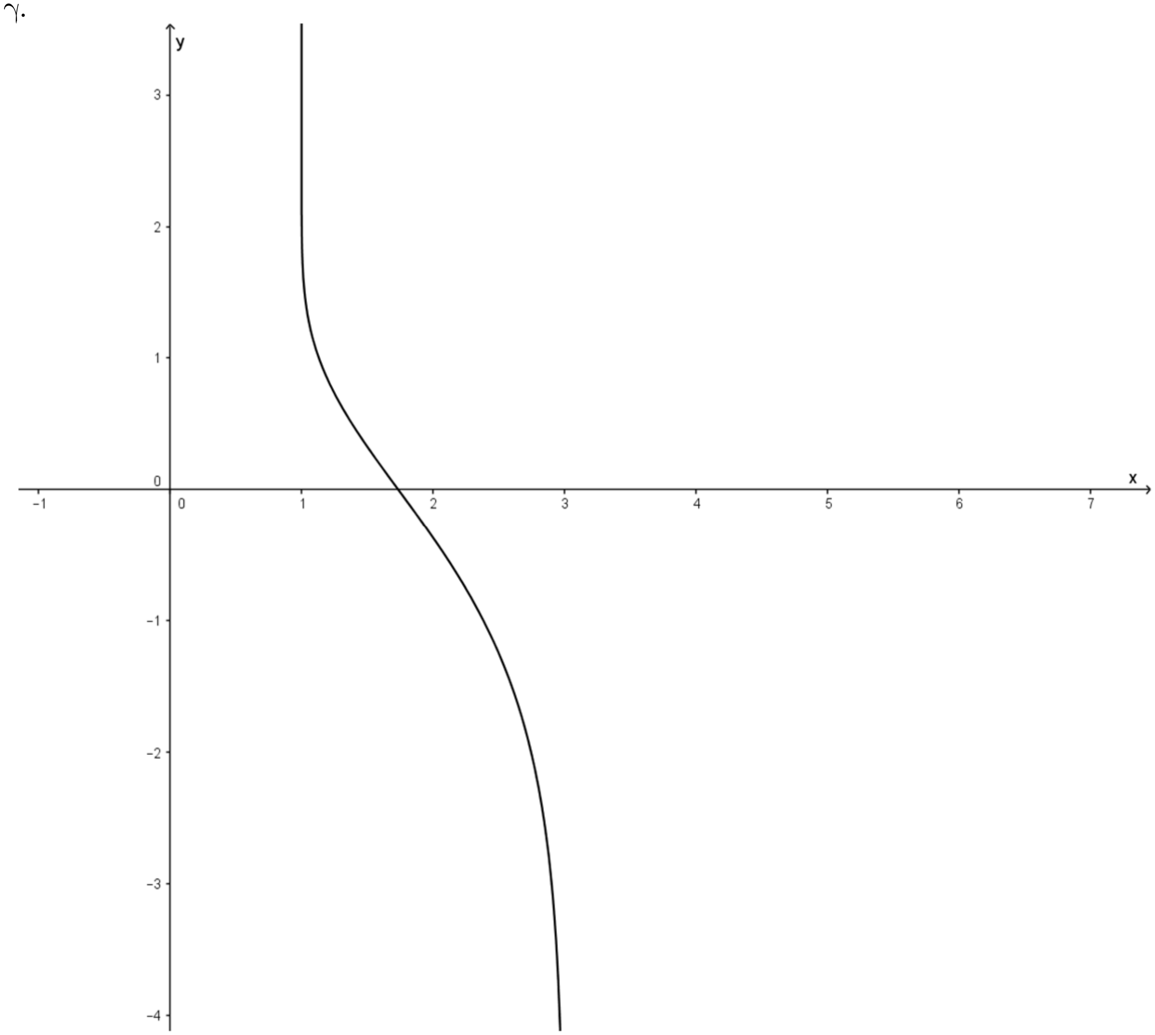


❶ for graph

Well done

β. 	❶ for graph	*Well done*

γ.



❶ for graph

❶ for intercept/ asymptotes

A few did not realize restriction on domain of $1 < x < 3$.

b)

i) $y = \frac{4}{x}$

$\frac{dy}{dx} = \frac{-4}{x^2}$

At $P\left(2p, \frac{2}{p}\right)$:

$m = \frac{-1}{p^2}$, so normal has gradient p^2 . Hence

Gradient of PQ:

$$m_{PQ} = \frac{\frac{2}{q} - \frac{2}{p}}{2q - 2p}$$

$$= \frac{2\left(\frac{p-q}{pq}\right)}{2(q-p)}$$

$$= \frac{-1}{pq}$$

As these are the same (equal):

$$p^2 = \frac{-1}{pq}$$

$$q = \frac{-1}{p^3}$$

	(b) (i) and (ii) Generally well done
❶ for gradient	
❶ for equating	

ii) $x_m = \dfrac{2q+2p}{2}$ and $y_m = \dfrac{\frac{2}{q}+\frac{2}{p}}{2}$

$$= p+q \qquad = \frac{1}{p}+\frac{1}{q}$$

$$= p - \frac{1}{p^3} \qquad = \frac{1}{p} - p^3$$

$$= \frac{1}{p}\left(p^2 - \frac{1}{p^2}\right)$$

❶ for both values

iii)

$$\frac{y}{x} = \frac{\frac{1}{p}\left(p^2 - \frac{1}{p^2}\right)}{p\left(\frac{1}{p^2} - p^2\right)}$$

$$= p^2 \frac{\left(p^2 - \frac{1}{p^2}\right)}{-\left(p^2 - \frac{1}{p^2}\right)}$$

$$= -p^2$$

$$y = -p^2 x$$

$$p^2 = \frac{-y}{x} \qquad ①$$

Then from

❶ for y to x relationship

(iii) many students tried to use the incorrect method of showing LHS=RHS. This does not show the equation is the locus of M. It just shows that the identity is true (without relating it to the locus) which is not what the question required.

$$y = p\left(\frac{1}{p^2} - p^2\right)$$

$$y^2 = p^2\left(\frac{1}{p^2} - p^2\right)^2$$

$$y^2 = \frac{-y}{x}\left(\frac{-x}{y} - \frac{-y}{x}\right)^2$$ using ① above.

❶ for substitution

$$y^2 = \frac{-y}{x}\left(\frac{y^2 - x^2}{xy}\right)^2$$

$$= \frac{-1}{x^3 y}\left(y^2 - x^2\right)^2$$

❶ for algebra to result

$$-x^3y^3 = \left(y^2 - x^2\right)^2$$ as reqd.

Question 14:	Marking	Comments
a) i) $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\, dx$		
$= \int_0^{\frac{\pi}{2}} \sin^{n-1} x . \sin x\, dx$		
Let $u = \sin^{n-1} x$ $\quad dv = \sin x$	❶ for by parts set-up	
$du = (n-1)\sin^{n-2} x . \cos x$ $\quad v = -\cos x$		
Then by parts:		*Generally well done*
$I_n = [0-0] + (n-1)\int_0^{\frac{\pi}{2}} (1-\sin^2 x).\sin^{n-2} x\, dx$	❶ resolves by parts…	
$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x - \sin^n x\, dx$		
$= (n-1)(I_{n-2} - I_n)$	❶ resolves to I_n, I_{n-2} relationship	
Hence $I_n = (n-1) I_{n-2} - (n-1) I_n$		
$(n-1) I_n + I_n = (n-1) I_{n-2}$	❶ resolves algebra to result	
$nI_n = (n-1) I_{n-2}$		
$I_n = \frac{(n-1)}{n} I_{n-2}$ as reqd.		

ii) To show $\int_0^{\frac{\pi}{2}} \sin^{2n} x\,dx = \dfrac{\pi(2n)!}{2^{2n+1}(n!)^2}$

i.e.

$$\int_0^{\frac{\pi}{2}} \sin^{2n} x\,dx = I_{2n}$$

$$= \left(\frac{2n-1}{2n}\right) I_{n-2}$$

$$= \left(\frac{2n-1}{2n}\right)\left(\frac{2n-3}{2n-2}\right) I_{n-4}$$

$$= \left(\frac{2n-1}{2n}\right)\left(\frac{2n-3}{2n-2}\right) \ldots I_2$$

$$= \left(\frac{2n-1}{2n}\right)\left(\frac{2n-3}{2n-2}\right) \ldots \frac{1}{2}.I_0$$

❶ establishes pattern

Now $I_0 = \int_0^{\frac{\pi}{2}} \sin^0 x\,dx$

$$= \int_0^{\frac{\pi}{2}} 1\,dx$$

$$= [x]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}$$

❶ finds I_0

Many had problems establishing a pattern, and only a few found the value for I_0.

From the pattern $\left(\dfrac{2n-1}{2n}\right)\left(\dfrac{2n-3}{2n-2}\right)\ldots$ we note the missing terms to be in sequence in the numerator are $2n, 2n-2, 2n-4, \ldots$ hence inserting these:

❶ identifies missing terms to complete pattern

$$I_{2n}=\frac{2n}{2n}\left(\frac{2n-1}{2n}\right)\frac{2n-2}{2n-2}\left(\frac{2n-3}{2n-2}\right)\frac{2n-4}{2n-4}\cdots\frac{1}{2}\cdot\frac{\pi}{2}$$

$$=\frac{2n.2n-1.2n-2.2n-3...2.1.\pi}{(2n)^2(2n-2)^2...4^2.2^2.2}$$

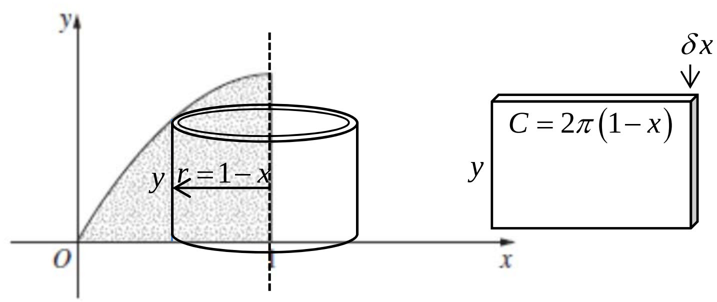
$$=\frac{\pi(2n)!}{2^2n^2.2^2(n-1)^2.2^2(n-2)^2..2^22^2.2^21^2.2}$$

$$=\frac{\pi(2n)!}{\left(2^2\right)^n.2.\left(n(n-1)(n-2)...2.1\right)^2}$$

$$=\frac{\pi(2n)!}{2^{2n+1}(n!)^2}$$ as reqd.

❶ resolves factorials and powers correctly

b)



Hence the shell is rectangular with area

$\delta A=lb$

$=2\pi(1-x)y$ where $y=\dfrac{x}{1+x^2}$

Thus the volume of the shell is

$\delta V=2\pi(1-x)y.\delta x$

❶ set-up: diagram with derivation of δV in appropriate detail

Set-up was generally poorly done. Radius was often incorrectly identified as x.

$$V=\lim_{\delta x\to 0}\sum_0^1 \delta V$$

$$=\lim_{\delta x\to 0}\sum_0^1 2\pi\left(1-x\right)y.\delta x$$

$$=2\pi\int_0^1 \left(1-x\right)y.dx$$

$$=2\pi\int_0^1 \left(1-x\right)\frac{x}{1+x^2}.dx$$

$$=2\pi\int_0^1 \frac{x-x^2}{1+x^2}.dx$$

$$=2\pi\int_0^1 \frac{-1-x^2+1+x}{1+x^2}.dx$$

$$=2\pi\int_0^1 \frac{-\left(1+x^2\right)+1+x}{1+x^2}.dx$$

$$=2\pi\int_0^1 -1+\frac{1}{1+x^2}+\frac{x}{1+x^2}.dx$$

$$=2\pi\left[-x+\tan^{-1}x+\frac{1}{2}\ln\left(1+x^2\right)\right]_0^1$$

$$=2\pi\left[\left(-1+\tan^{-1}1+\frac{1}{2}\ln\left(1+1\right)\right)-\left(-0+\tan^{-1}0+\frac{1}{2}\ln\left(1+0\right)\right)\right]$$

$$=\pi\left(\ln 2+\frac{\pi}{2}-2\right)$$

❶ resolves sum to integral

❶ resolves into partial fractions (any method)

❶ answer

c) $\displaystyle\int \frac{\ln x}{(1+\ln x)^2}\, dx$

Let $u = 1+\ln x$ *so* $\ln x = u-1$, then

$$du = \frac{1}{x}dx \qquad x = e^{u-1}$$

❶ uses appropriate substitution

$$dx = e^{u-1}\, du$$

$$\int \frac{\ln x}{(1+\ln x)^2}\, dx$$

$$= \int \frac{u-1}{u^2}.e^{u-1}\, du$$

$$= \int \left(\frac{1}{u} - \frac{1}{u^2}\right).e^{u-1}\, du$$

Generally well done

$$= \int \frac{e^{u-1}}{u} du - \int \frac{e^{u-1}}{u^2} du$$

$$= \int \frac{e^{u-1}}{u} du - \left[-\left(\frac{e^{u-1}}{u}\right) - \int \frac{-1}{u} e^{u-1} du \right]$$

$$= \int \frac{e^{u-1}}{u} du + \left(\frac{e^{u-1}}{u}\right) - \int \frac{e^{u-1}}{u} du$$

❶ resolves to simple integration

$$= \left(\frac{e^{u-1}}{u}\right) + c$$

$$= \frac{x}{1+\ln x} + c$$

❶ answer back into original variable

Question 15:

a)

i)

$$\ddot{x} = g - \beta v^2$$

$$v\frac{dv}{dx} = g - \beta v^2$$

$$\frac{vdv}{g - \beta v^2} = dx$$

$$\frac{-2\beta vdv}{g - \beta v^2} = -2\beta dx$$

$$\ln\left(g - \beta v^2\right) = -2\beta x + c$$

When $x = 0, v = 0$:

$$-2\beta \times 0 + c = \ln\left(g - \beta \times 0^2\right)$$

$$c = \ln g$$

$$\therefore \ln\left(g - \beta v^2\right) = -2\beta x + \ln g$$

Marking:
❶ resolves to integration
❶ resolves integral correctly
❶ correct c value

Comments: *Generally well done. Several students successfully used* $\int_0^x \sim dx = \int_0^v \sim dv$ *method.*

ii)

$$-2\beta x = \ln\left(g - \beta v^2\right) - \ln g$$

$$-2\beta x = \ln\left(\frac{g - \beta v^2}{g}\right)$$

$$e^{-2\beta x} = \frac{g - \beta v^2}{g}$$

$$ge^{-2\beta x} = g - \beta v^2$$

$$\beta v^2 = g - ge^{-2\beta x}$$

$$v^2 = \frac{g}{\beta}\left(1 - e^{-2\beta x}\right)$$

$$v = \sqrt{\frac{g}{\beta}\left(1 - e^{-2\beta x}\right)} \text{ (as } v > 0\text{)}$$

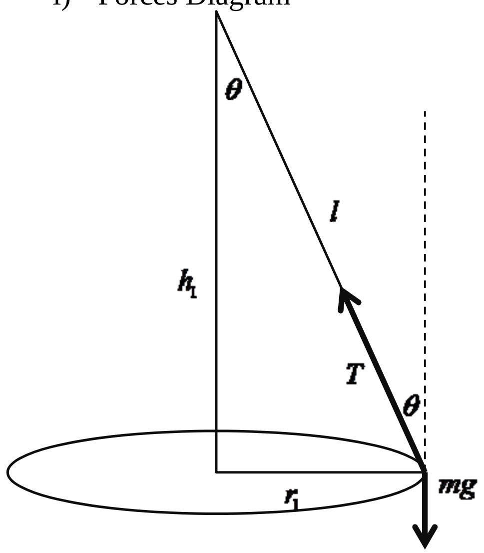
Marking:
❶ resolves to exponentials
❶ correct algebra to result

Comments: *Well done.*

Solution	Marking	Comments
iii) Terminal velocity when $x \to \infty, \Rightarrow e^{-2\beta x} \to 0$, hence $v_T = \sqrt{\frac{g}{\beta}(1-0)}$ $= \sqrt{\frac{g}{\beta}}$ as reqd.	❶ correct reasoning	*Some used the acceleration definition to get to the same result. Don't substitute ∞ into equations – it's not a number!*
iv) $v_T = \sqrt{\frac{g}{\beta}}$ $= \sqrt{\frac{9.8}{0.2}}$ $= 7\,m/s$ $= \frac{7 \times 60 \times 60}{1000}\,km/h$ $\approx 25.2\,km/h$	❶ correct result	*Some did not read requirements and left the answer in m/s.*
v) From (i): $-2\beta x = \ln g - \ln\left(g - \beta v^2\right)$ $x = \frac{-1}{2\beta}\ln\left(\frac{g}{g-\beta v^2}\right)$ $= \frac{-1}{2\times 0.2}\ln\left(\frac{9.8}{9.8 - 0.2\times(0.8\times 7)^2}\right)$ $\approx 2.55m$	❶ substitutions correct ❶ correct answer	*Generally well done.*

b)

i) Forces Diagram



Resolving Forces:

Vertically: no motion

$$\sum F_i = 0$$

$$T\cos\theta - mg = 0$$

$$T\cos\theta = mg \quad ①$$

Horizontally: circular motion

$$\sum F_i = mr\omega^2$$

$$T\sin\theta = mr\omega^2 \quad ②$$

$$② \div ①:\ \tan\theta = \frac{mr\omega^2}{mg}$$

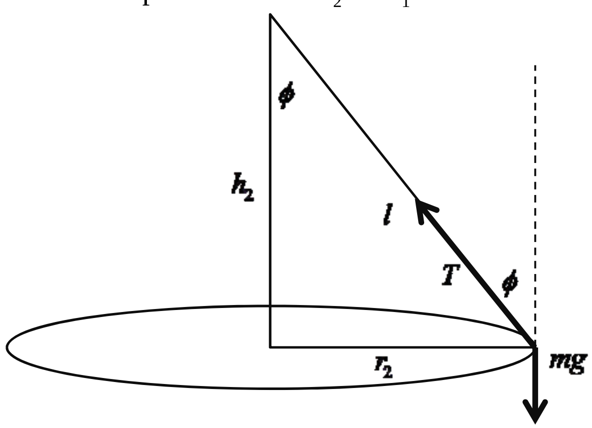
$$= \frac{r\omega^2}{g}$$

But from diagram: $\tan\theta = \dfrac{r}{h}$, hence

❶ forces diagram and resolution of forces correct

Set-up and forces diagram, with resolution of forces, usually poorly done.

$\frac{r}{h} = \frac{r\omega^2}{g}$ $\omega^2 = \frac{g}{h}$ $\omega = \sqrt{\frac{g}{h}}$ For Period: $T = \frac{2\pi}{\omega}$ $= \frac{2\pi}{\sqrt{\frac{g}{h}}}$ $= 2\pi\sqrt{\frac{h}{g}}$ as reqd. ii) When the speed doubles: $v_2 = 2v_1$	❶ linking the two expressions for $\tan\theta$. ❶ resolved for ω with correct resolution to required result.



We know from $h=\dfrac{g}{\omega^2}$ that h decreases as ω increases and so clearly $\phi>\theta$.

So from part (i):

$v_1=r_1\omega_1$

$h_1=\dfrac{g}{\omega_1^2}$

$T_1=\dfrac{2\pi}{\omega_1}$

compared to double speed:

$v_2=r_2\omega_2$ ①

$h_2=\dfrac{g}{\omega_2^2}$ ②

$T_2=\dfrac{2\pi}{\omega_2}$ ③

With $v_2=2v_1$, this gives the following relationships:

①:

$$\omega_2=\frac{v_2}{r_2}=\frac{2v_1}{r_2}=\left(\frac{2r_1}{r_2}\right).\omega_1$$

②:

$$h_2=\frac{g}{\omega_2^2}=\frac{g}{\left(\frac{2r_1}{r_2}\right).\omega_1^2}=\left(\frac{r_2}{2r_1}\right)\frac{g}{\omega_1^2}=\left(\frac{r_2}{2r_1}\right)h_1$$

❶ appropriate working to support conclusions below

Generally poorly done.
Many assumed a doubling of ω rather than v, thus not reading the question carefully, and erroneously finding that angle or radius doubled.

As $h_2 < h_1$, this leads to $\left(\frac{r_2}{2r_1}\right)h_1 < h_1$ $r_2 < 2r_1$ ③: $T_2 = \frac{2\pi}{\omega_2}$ $= \frac{2\pi}{\left(\frac{2r_1}{r_2}\right).\omega_1}$ $= \left(\frac{r_2}{2r_1}\right)T_1$ Thus height and period decrease, radius and angle increase (but NOT by doubling). Note also that $\tan\theta = \frac{r_1}{h_1}$ $\tan\phi = \frac{r_2}{h_2}$ $= \frac{r_2}{\left(\frac{r_2}{2r_1}\right)h_1}$ $= 2\frac{r_1}{h_1}$ $= 2\tan\theta$ $\phi = \tan^{-1}\left(2\tan\theta\right)$ (i.e. the angle is not doubled)	➊ height, radius and angle (at least two mentioned) ➊ effect on period	*Period is specifically mentioned in the previous part, so reference should be made to this! Ensure you understand that the parts are linked!*

Question 16:	Marking	Comments
a)		
i) $x = vt\cos\theta$ gives $t = \dfrac{x}{v\cos\theta}$		
Substituting this into $y = vt\sin\theta - \dfrac{1}{2}gt^2$ gives	❶ resolves for *t* and substitutes	
$y = v\sin\theta.\dfrac{x}{v\cos\theta} - \dfrac{1}{2}g\left(\dfrac{x}{v\cos\theta}\right)^2$		
$= x\tan\theta - \dfrac{gx^2}{2v^2\cos^2\theta}$	❶ links $\sec\theta$ to $\tan\theta$	*Generally well done.*
$2v^2y = 2v^2x\tan\theta - gx^2\sec^2\theta$		
$= 2v^2x\tan\theta - gx^2\left(1+\tan^2\theta\right)$		
$= 2v^2x\tan\theta - gx^2 - gx^2\tan^2\theta$	❶ resolves to required eqn.	
$0 = gx^2\tan^2\theta - 2v^2x\tan\theta + gx^2 + 2v^2y$ as reqd.		
ii) Considering the equation as a quadratic in $\tan\theta$, for two possible angles $\Delta > 0$. Hence		
$b^2 - 4ac > 0$		
$\left(2v^2x\right)^2 - 4.gx^2.\left(2v^2y + gx^2\right) > 0$	❶ link to discriminant and set-up of inequality	*Some students did not realize to use the discriminant.*
$4v^4x^2 - 8gx^2v^2y - 4g^2x^4 > 0$		
$v^4 - 2gv^2y - g^2x^2 > 0$		
$v^4 - 2gv^2y > g^2x^2$	❶ correct algebra to result	
$v^4 - 2gv^2y + g^2y^2 > g^2x^2 + g^2y^2$		
$\left(v^2 - gy\right)^2 > g^2\left(x^2 + y^2\right)$		

iii) To hit the base of the wall substitute $y=0$ in (ii): $\left(v^2-0\right)^2 > g^2\left(x^2+0^2\right)$ $v^4 > g^2\left(x^2\right)$ $v > \sqrt{gx}$, noting g, x, v are all >0. Hence $v=\sqrt{gx}$ to hit the base of the wall.	❶ correct reasoning	*Many students did not use the inequality derived in (ii).*
iv) Substituting $v=2\sqrt{gx}$ into (ii): $\left(\left(2\sqrt{gx}\right)^2-gy\right)^2 > g^2\left(x^2+y^2\right)$ $\left(4gx-gy\right)^2 > g^2\left(x^2+y^2\right)$ $g^2\left(4x-y\right)^2 > g^2\left(x^2+y^2\right)$ $16x^2-8xy+y^2 > x^2+y^2$ $16x^2-8xy > x^2$ $15x^2 > 8xy$ $y < \frac{15x}{8}$ (as $x>0$)	❶ correct inequality ❶ correct algebra to eliminate y^2 ❶ correct reasoning	*Having missed the link earlier, many did not use this method, but used the alternate method below.*
ALTERNATE METHOD (used by many) Substituting $v=2\sqrt{gx}$ into (i): $0 = gx^2\tan^2\theta-8gx^2\tan\theta+8gxy+gx^2$ $= x\tan^2\theta-8x\tan\theta+8y+x$ $y = x\tan\theta-\frac{x}{8}\tan^2\theta-\frac{x}{8}$ ① $\therefore \frac{dy}{d\theta} = x\sec^2\theta-\frac{x}{4}\tan\theta\sec^2\theta$ $\frac{dy}{d\theta}=0$ gives	ALTERNATE MARKING ❶ finding $\tan\theta=4$	

$0 = x\sec^2\theta - \frac{x}{4}\tan\theta\sec^2\theta$

$= \frac{x}{4}\sec^2\theta\left(4-\tan\theta\right)$

Hence $\tan\theta = 4$ as $\sec^2\theta \neq 0$.

$\tan\theta = 4$ into ①:

$y = 4x - \frac{x}{8}.16 - \frac{x}{8}$

$= \frac{15x}{8}$ as reqd.

Also:

θ	$\tan^{-1}(3)$	$\tan^{-1}(4)$	$\tan^{-1}(5)$
y'	$+$	0	$-$

Hence the value is a maximum.

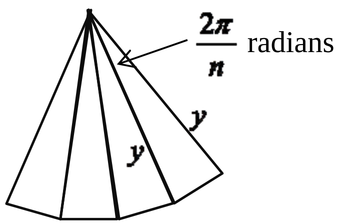
❶ proving the result.

❶ showing a max value.

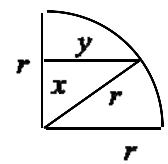
Many students did not confirm that the required value was a maximum.

b)

i) Top view:



Side view



From the side view: $y = \sqrt{r^2 - x^2}$

Many students did not demonstrate the relationship between x, y and r in a right triangle. Most students recognised the need to use $A = \frac{1}{2}ab\sin C$.

From the top view, one triangle has

$$A = \frac{1}{2}ab\sin C$$
$$= \frac{1}{2}y^2\sin\left(\frac{2\pi}{n}\right)$$

❶ diagrams and set-up for one cross-section

So the total cross-sectional area is

$$A_T = \frac{n}{2}y^2\sin\left(\frac{2\pi}{n}\right)$$
$$= \frac{n}{2}\left(r^2 - x^2\right)\sin\left(\frac{2\pi}{n}\right)$$

❶ resolves to given result

ii) $\delta V = A_T\,\delta x$

$$= \frac{n}{2}\left(r^2 - x^2\right)\sin\left(\frac{2\pi}{n}\right)\delta x$$

Hence

$$V = \sum \delta V$$
$$= \lim_{\delta x\to 0}\sum_{0}^{r}\frac{n}{2}\left(r^2 - x^2\right)\sin\left(\frac{2\pi}{n}\right)\delta x$$
$$= \int_0^r \frac{n}{2}\left(r^2 - x^2\right)\sin\left(\frac{2\pi}{n}\right)dx$$

Well done.

❶ derivation of δV leading to integral

$$= \frac{n}{2}\sin\left(\frac{2\pi}{n}\right)\left[r^2x - \frac{1}{3}x^3\right]_0^r$$
$$= \frac{n}{2}\sin\left(\frac{2\pi}{n}\right)\left[\left(r^3 - \frac{1}{3}r^3\right) - (0)\right]$$

❶ resolves for r values

$$= \frac{n}{2}.\frac{2r^3}{3}.\sin\left(\frac{2\pi}{n}\right)$$
$$= \frac{nr^3}{3}\sin\left(\frac{2\pi}{n}\right) \qquad \text{as reqd.}$$

❶ resolves to result

iii) Noting that $n \to \infty$ implies $\frac{2\pi}{n} \to 0$, then $\lim_{n\to\infty} \frac{nr^3}{3}\sin\left(\frac{2\pi}{n}\right)$ $= \lim_{\frac{2\pi}{n}\to 0} \frac{nr^3}{3}\sin\left(\frac{2\pi}{n}\right)$	❶ resolves limit correctly	*Hardly any students realized that as* $n \to \infty$, $\frac{2\pi}{n} \to 0$ *and hence were unable to correctly resolve* $\lim_{x\to 0}\frac{\sin x}{x} = 1$.
$= \lim_{\frac{2\pi}{n}\to 0} \frac{nr^3}{3}.\frac{2\pi}{n}.\frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}}$ $= \frac{2\pi r^3}{3}\lim_{\frac{2\pi}{n}\to 0}\frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}}$ $= \frac{2\pi r^3}{3}.1$ $= \frac{2\pi r^3}{3}$ $= \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$ which is half the volume of a sphere (a hemisphere) as reqd.	❶ demonstrates conclusion with supporting working	