FORT STREET HIGH SCHOOL

Name: ____________________

Teacher: ____________________

Class: ____________________

2018

HIGHER SCHOOL CERTIFICATE COURSE

ASSESSMENT TASK 3: TRIAL HSC

Mathematics Extension 2

Time allowed: 3 hours
(plus 5 minutes reading time)

Syllabus Outcomes	Assessment Area Description and Marking Guidelines	Questions
E1	Chooses and applies appropriate mathematical techniques in order to solve a broad range of problems effectively	1-10
E3	Uses the relationship between algebraic and geometric representations of complex numbers	11
E6	Combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions	12
E4	Uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials	13
E7, E8	Applies further techniques of integration, such as slicing and cylindrical shells, integration by parts and recurrence formulae, to problems	14,15
E2-E8	Synthesises mathematical processes to solve harder problems and communicates solutions in an appropriate form	16

Total Marks 100

Section I 10 marks
Multiple Choice, attempt all questions.
Allow about 15 minutes for this section.

Section II 90 Marks
Attempt Questions 11-16.
Allow about 2 hours 45 minutes for this section.

General Instructions:

- Questions 11-16 are to be started in a new booklet.
- The marks allocated for each question are indicated.
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations.
- Marks may be deducted for careless or badly arranged work.
- Board – approved calculators may be used.

Section I	Total 10	Marks
Q1-Q10	/10	
Section II	Total 90	Marks
Q11	/15	
Q12	/15	
Q13	/15	
Q14	/15	
Q15	/15	
Q16	/15	
	Percent	

Section I (10 marks)

Attempt questions 1–10. Allow about 15 minutes for this section.

Use the multiple-choice answer sheet, on the last page of this question booklet, for Questions 1–10.

Question 1.

Let $z = 2 - 3i$ and $w = 3 + 2i$. What is the value of $z - \overline{w}$?

(A) $5 - i$ (B) $-1 - 5i$ (C) $-1 - i$ (D) 0

Question 2.

For the ellipse $\frac{x^2}{9} + \frac{y^2}{6} = 1$ what is the value of $a^2 e^2$

(A) 15

(B) 107

(C) 3

(D) 45

Question 3.

The equation $0 = 4x^3 + 2x^2 - 5x + 3$ has roots α, β and γ. Which polynomial has roots $2\alpha, 2\beta$ and 2γ.

(A) $0 = x^3 + x^2 - 5x + 6$

(B) $0 = 8x^3 + 4x^2 - 10x + 6$

(C) $0 = 2x^3 + x^2 - \frac{5}{2}x + \frac{3}{2}$

(D) $0 = 32x^3 + 8x^2 - 10x + 3$

Question 4.

If $y\cos x + x\sin y = \frac{\pi}{6}$, what is the value of $\frac{dy}{dx}$ at $\left(0, \frac{\pi}{6}\right)$.

(A) 2

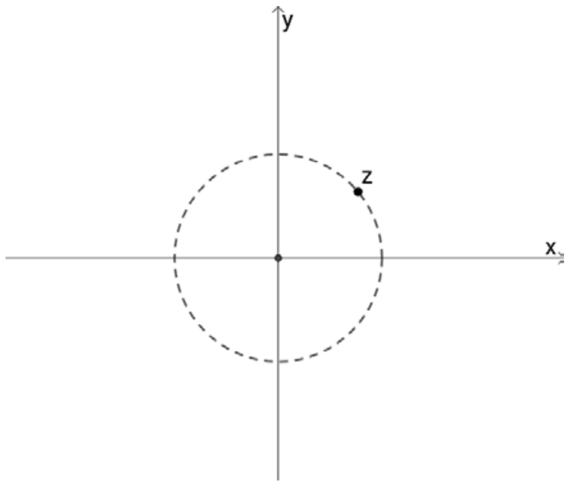
(B) $\frac{1}{2}$

(C) -2

(D) $-\frac{1}{2}$

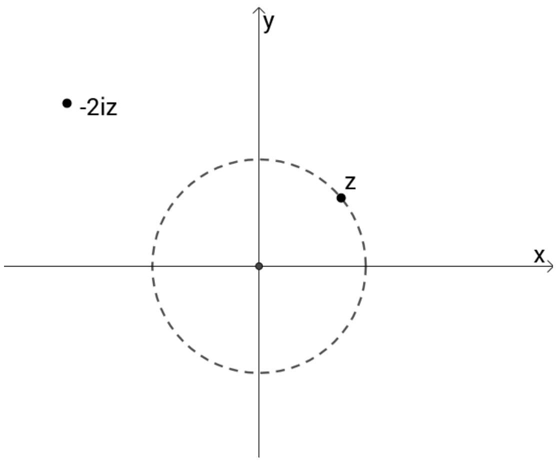
Question 5.

The complex number z is shown on the Argand diagram below.

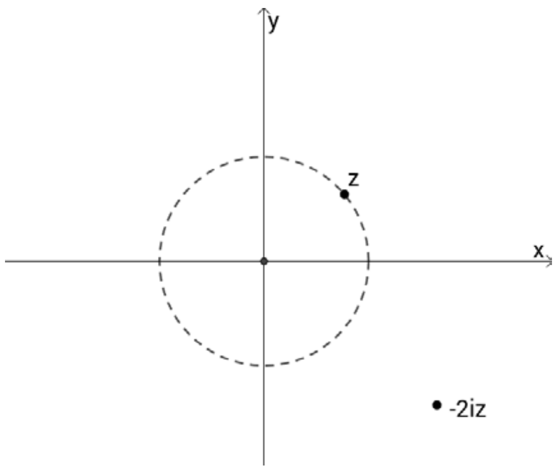


Which of the following best represents $-2iz$?

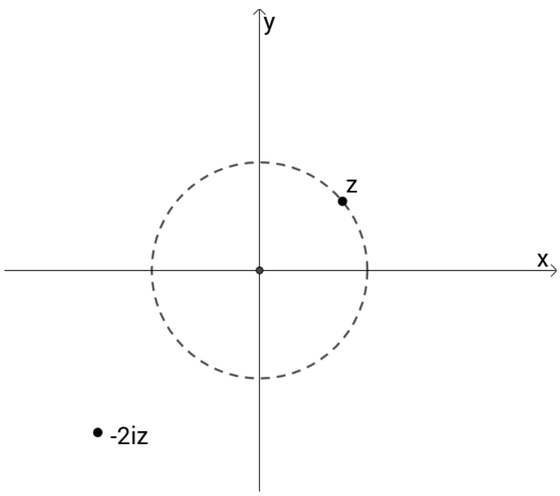
(A)



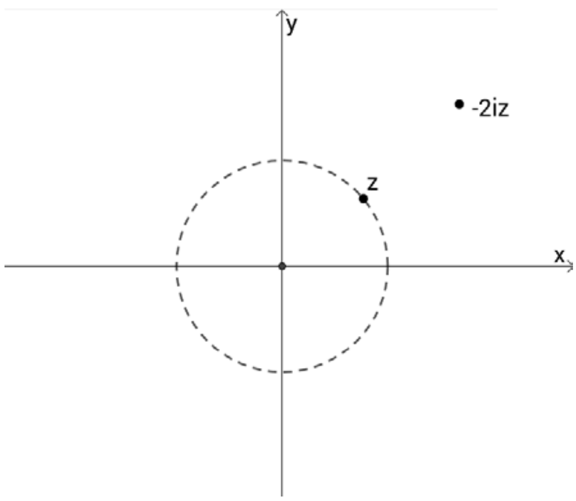
(B)



(C)

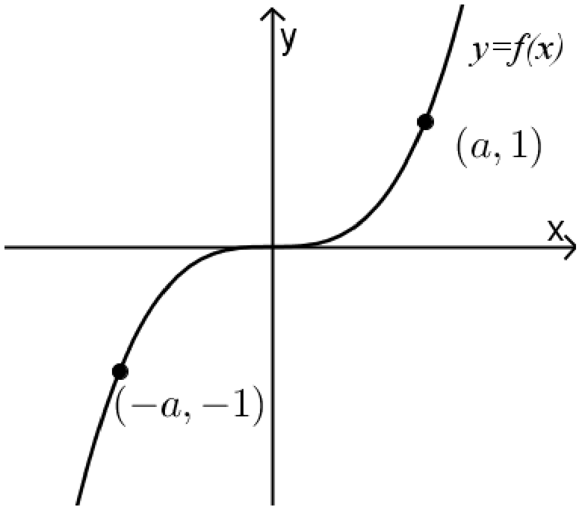


(D)



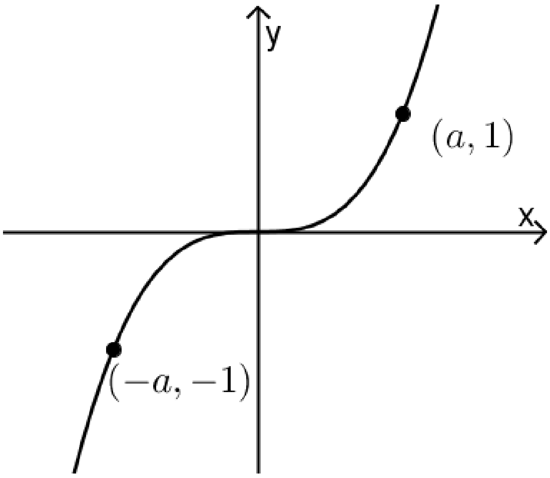
Question 6.

The graph of $y = f\left(x\right)$ is shown below.

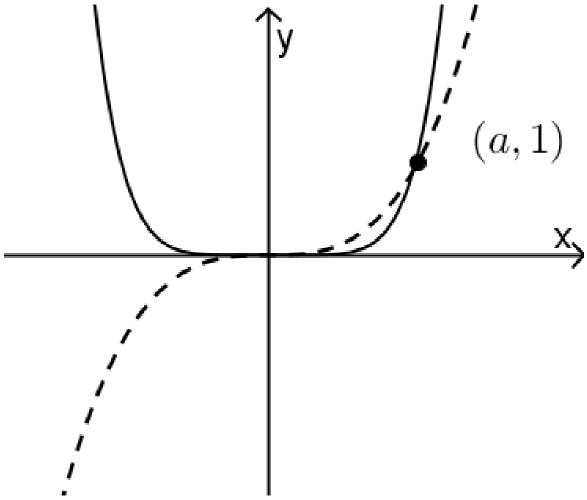


Which of the following graphs best represents $y = \sqrt{\left[f\left(x\right)\right]^2}$? Note that, where possible, the graph of $y = f\left(x\right)$ is shown as the dashed curve.

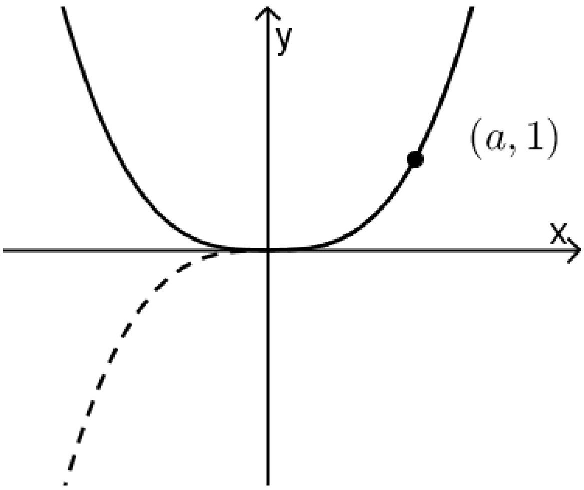
(A)



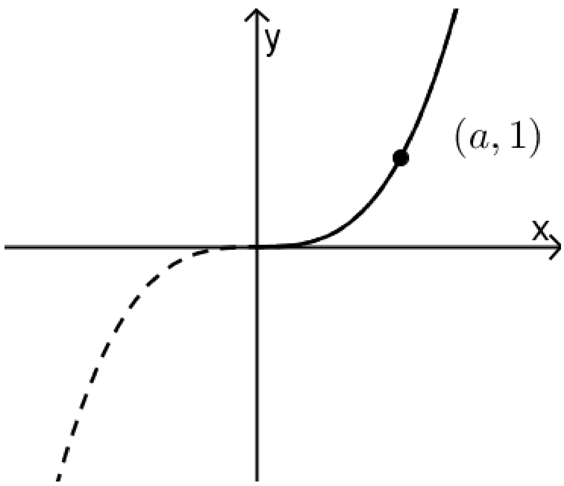
(B)



(C)



(D)



Question 7.

A partical is dropped vertically in a medium where the resistance is $\frac{k}{v^3}$. If down is taken as the positive direction, the terminal velocity is

(A) $\sqrt[3]{\frac{mg}{k}}$

(B) $\sqrt[3]{\frac{mk}{g}}$

(C) $\sqrt[3]{\frac{k}{mg}}$

(D) $\sqrt[3]{\frac{gk}{m}}$

Question 8.

$\frac{6}{2x^2-5x+2}$ expressed as a sum of partial fractions is

(A) $\frac{2}{2x-1}-\frac{4}{x-2}$

(B) $\frac{2}{x-2}-\frac{4}{2x-1}$

(C) $\frac{2}{2x-1}-\frac{4}{x+2}$

(D) $\frac{4}{2x-1}-\frac{2}{x-2}$

Question 9.

$$\int \sin^3\theta\cos^2\theta \; d\theta =$$

(A) $\dfrac{\sin^4\theta\cos^3\theta}{12} + C$

(B) $\dfrac{1}{5}\cos^5\theta - \dfrac{1}{3}\cos^3\theta + C$

(C) $\dfrac{1}{3}\cos^3\theta - \dfrac{1}{5}\cos^5\theta + C$

(D) $-\dfrac{\sin^4\theta\cos^3\theta}{12} + C$

Question 10.

In the Argand diagram, *ABC* is an equilateral triangle and the vertices *A* and *B* correspond to the complex numbers *w* and *z*.

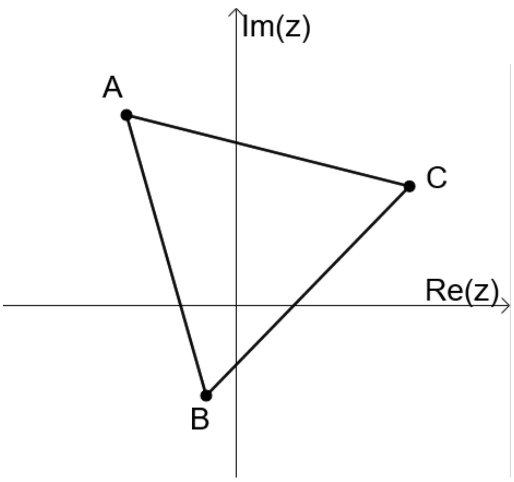
What complex number corresponds to the vector $\overrightarrow{CA}$?

(A) $(z-w)\left(\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i\right)$

(B) $(z-w)\left(\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}i\right)$

(C) $(w-z)\left(\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i\right)$

(D) $(w-z)\left(\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}i\right)$



Section II (90 marks)

Attempt Questions 11–16. Allow about 2 hours and 45 minutes for this section.

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/ or calculations.

Question 11 (15 marks) Use a *separate* writing booklet

(a) (i) Write $1-\sqrt{3}i$ in Mod-Arg form **2**

(ii) Hence, write $z=\left(1-\sqrt{3}i\right)^{10}$ in $a+ib$ form. **2**

(b) (i) Calculate $\sqrt{-8+6i}$ **2**

(ii) Hence, solve $2z^2-(3+i)z+2=0$ **2**

(c) Sketch on an Argand diagram, the locus of z where $\text{Im}\left[(4-8i)z\right]\geq -8$ **2**

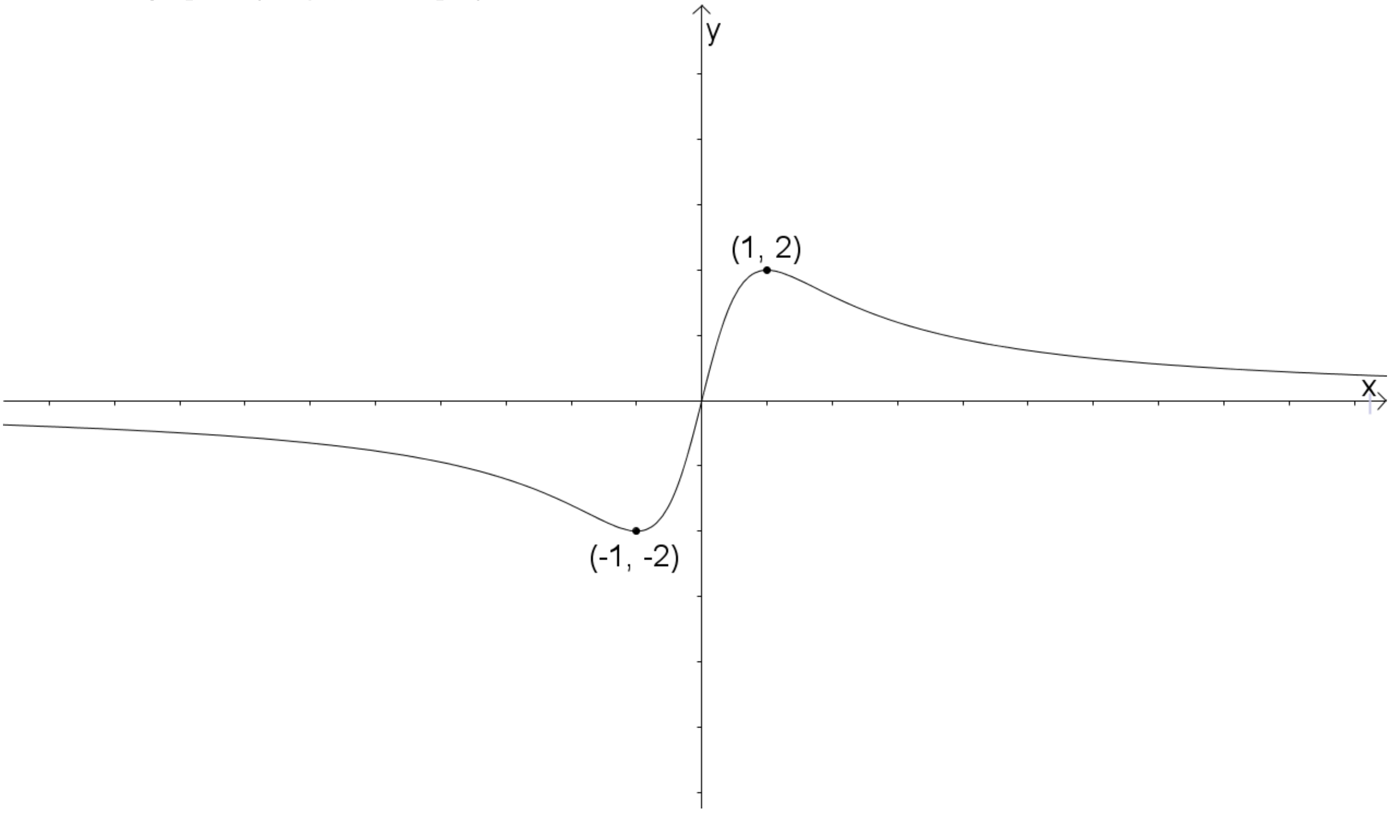
(d) (i) Use De Moivre's theorem and a binomial expansion to show that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ **2**

(ii) Find 3 distinct solutions of the equation $8x^3-6x+1=0$ **2**

(iii) Hence show that $\sin\dfrac{\pi}{18}+\sin\dfrac{13\pi}{18}=\sin\dfrac{11\pi}{18}$ **1**

Question 12 (15 marks) Use a *separate* writing booklet

(a) The graph of $y = f(x)$ is displayed below.



Without using calculus, sketch each of the graphs below.

(i) $y = \left[f(x)\right]^2$ **3**

(ii) $y = \dfrac{1}{f(x)}$ **3**

(b) (i) Show that $\int_{-a}^{a} f(x)\,dx = \int_{0}^{a}\left[f(x) + f(-x)\right]dx$ **2**

(ii) Hence find $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \dfrac{1}{1+\sin x}\,dx$ **3**

(c) The region enclosed by the curve $y = 4x^2 - x^4$ and the x-axis between $x = 0$ and $x = 2$ is rotated around the y-axis. The curve has a maximum turning point at $\left(\sqrt{2}, 4\right)$. Use annular discs to find the volume of the solid. **4**

Question 13 (15 marks) Use a *separate* writing booklet

(a) (i) State the equation of the hyperbola on which the point P with co-ordinates $\left(3\sec\theta, \sqrt{7}\tan\theta\right)$ will always lie. **1**

(ii) Find the eccentricity of the hyperbola **1**

(iii) Show that the normal to the hyperbola at P has the equation $\dfrac{3x}{\sec\theta}+\dfrac{\sqrt{7}y}{\tan\theta}=16$. **3**

(iv) Given $0<\theta<\dfrac{\pi}{2}$, show that N, the point of intersection of the normal with the y-axis, has coordinates $N\left(0,\dfrac{16\tan\theta}{\sqrt{7}}\right)$. **1**

(b) Solve $z^4+z=0$ over the field of complex numbers. **3**

(c) The polynomial $x^3-4x-1=0$ has roots α,β and γ .

Find an equation with roots $\dfrac{\alpha}{\beta\gamma},\dfrac{\beta}{\gamma\alpha}$ and $\dfrac{\gamma}{\alpha\beta}$. **3**

(d) Write $x^3-5x^2+4x+10$ as the product of a linear and quadratic factor over the real numbers given that $3+i$ is a zero. **3**

Question 14 (15 marks) Use a *separate* writing booklet

(a) i) Express $\dfrac{7x^2-5x+4}{(x-1)(x^2+1)}$ as a sum of partial fractions. **2**

ii) Hence, find $\displaystyle\int \dfrac{7x^2-5x+4}{(x-1)(x^2+1)}\,dx$ **2**

(b) Evaluate $\displaystyle\int_0^{\frac{1}{4}} \frac{x^2 dx}{\sqrt{1-4x^2}}$ **3**

(c) The region enclosed by the *y*-axis and the curve $y=e^x$ from $y=1$ to $y=2$ is rotated about the *x*-axis. **3**

Using the method of cylindrical shells, find the volume of the solid formed.

(d) Let *O* be the origin and $P(x_1, y_1)$ a variable point on the hyperbola $x^2-y^2=a^2$. The tangent at the point *P* cuts the *x*-axis at *A* and the *y*-axis at *B*.

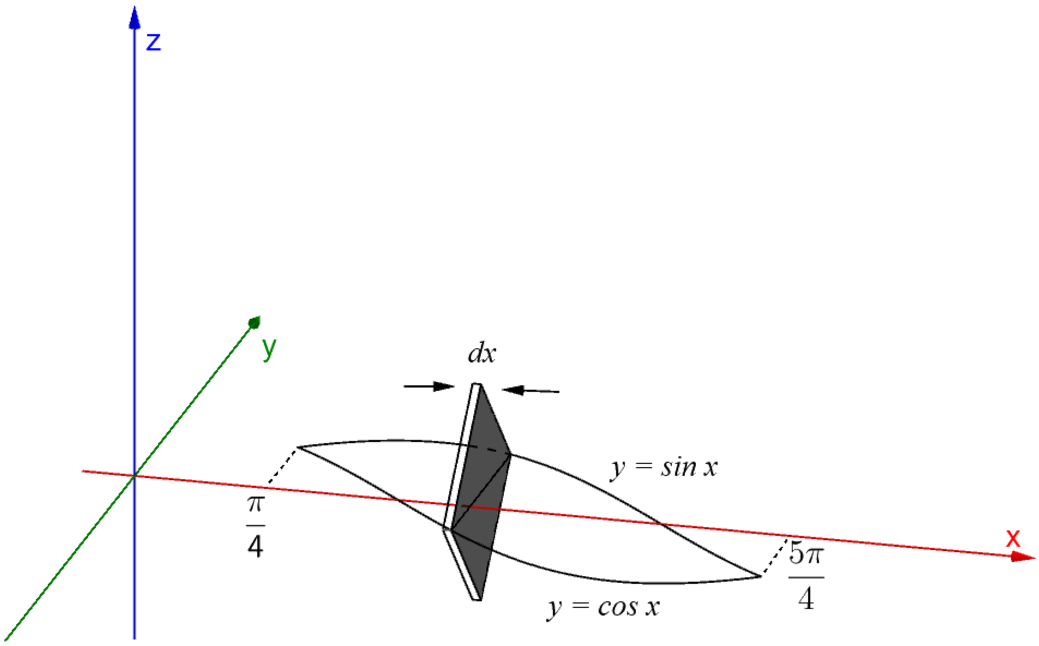
i) Show that the tangent at *P* has equation $xx_1 - yy_1 = a^2$ **2**

ii) *R* (*x*, *y*) is a point such that $-a < x < a$, $x \neq 0$ and *OARB* is a rectangle. Show that the locus of *R* is

$\dfrac{a^2}{x^2}-\dfrac{a^2}{y^2}=1$ **3**

Question 15 (15 marks) Use a *separate* writing booklet

(a)



A solid is formed from an infinite number of square slices *dx* thick. Each square slice is parallel to the *y*-*z* plane and has a diagonal extending from $y = \sin x$ to $y = \cos x$. The curves $y = \sin x$ and $y = \cos x$ are restricted to the domain $\frac{\pi}{4} \le x \le \frac{5\pi}{4}$.

Find the volume of the solid formed. **4**

(b) As a 10kg particle moves through the air it experiences a resistive force, in newtons, equal to one-tenth of the square of its velocity, $\frac{v^2}{10}$. The particle is projected vertically upwards from the ground with initial velocity $u\,ms^{-1}$. After reaching its highest point it then falls back to the ground.

Assuming that gravity is $10\,ms^{-2}$,

(i) find the time T it takes for the particle to reach its highest point. (Give your answer in exact form). **3**

(ii) show that the greatest height H is $50\log_e\left(1+\frac{u^2}{1000}\right) metres$. **4**

(iii) show that the particles velocity, $w\ \ ms^{-1}$, when it reaches the ground is given by

$$w^2 = \frac{1000u^2}{1000+u^2}$$ **4**

Question 16 (15 marks) Use a *separate* writing booklet

(a) Let $I_n = \int_0^2 \left(4 - x^2\right)^n dx$ where n is an integer and $n \geq 0$.

(i) Show that $I_n = \dfrac{8n}{2n+1} I_{n-1}$. **4**

(ii) Hence find I_3 . **2**

(b) Any smooth, continuous and repeatedly differentiable function $f\left(x\right)$, may be approximated near $x = 0$ by a polynomial function $A\left(x\right)$ defined as

$$A\left(x\right) = \sum_{r=0}^{n} \frac{f^{(r)}\left(0\right) \times x^r}{r!} .$$

Where $f^{(r)}\left(0\right)$ is the *r*th derivative of $f\left(x\right)$ evaluated at $x = 0$. That is,

$$f^{(0)}\left(0\right) = f\left(0\right)$$
$$f^{(1)}\left(0\right) = f'\left(0\right)$$
$$f^{(2)}\left(0\right) = f''\left(0\right)$$
$$\ldots$$

Using $n = 5$, write $A\left(x\right)$ in expanded form for each of the following functions:

(i) $f\left(x\right) = e^x$ **2**

(ii) $f\left(x\right) = \sin x$ **2**

(iii) $f\left(x\right) = \cos x$ **2**

(c) (i) You may assume the results in (b) can be extended to complex numbers. By considering the expansions from (b) as $n \to \infty$, deduce an identity for $e^{i\theta}$ in terms of $\sin\theta$ and $\cos\theta$. Show steps of reasoning. **2**

(ii) Hence, evaluate $e^{\left(\frac{i\pi}{3}\right)}$ in $a + ib$ form. **1**

End of examination.

2018 Mathematics Extension 2 Trial Solutions

1	C
2	C
3	A
4	D
5	B
6	C
7	C
8	B
9	B
10	C

Question11

Marker's Comments

(a) i)

$$\left|1-\sqrt{3}i\right| = \sqrt{1^2+\left(\sqrt{3}\right)^2} = 2 \text{ ❶}$$

$$Arg\left(1-\sqrt{3}i\right) = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right) = -\frac{\pi}{3} \text{ ❶}$$

$$2\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)$$

Well done. Take care to calculate the correct argument.

ii)

$$\left[2cis\left(\frac{-\pi}{3}\right)\right]^{10}$$

$$=1024cis\left(\frac{-10\pi}{3}\right) \text{ ❶}$$

$$=1024cis\left(\frac{2\pi}{3}\right)$$

$$=1024\left(\frac{-1}{2}+\frac{\sqrt{3}}{2}i\right)$$

$$=-512+512\sqrt{3}i \text{ ❶}$$

Well done. Take care to answer the question i.e. write in $a+ib$ form.

(b) i)

Well done.

$$Let\ x+iy=\sqrt{-8+6i}$$

$$(x+iy)^2=-8+6i$$

$$x^2-y^2+2xyi\quad=-8+6i$$

$$\therefore x^2-y^2=-8 \text{ and } 2xy=6$$ ❶

$$xy=3$$

$$x^4-x^2y^2=-8x^2$$

$$x^4+8x^2-9=0$$

$$x^2=\left(x^2+9\right)\left(x^2-1\right)$$

$$x^2=1$$

$$x=\pm1$$

$$y=\pm3$$

$$\therefore \sqrt{-8+6i}=\pm(1+3i)$$ ❶

ii)

Well done. Some errors carried from part i)

$$2z^2-(3+i)z+2=0$$

$$z=\frac{3+i\pm\sqrt{(3+i)^2-16}}{4}$$

$$z=\frac{3+i\pm\sqrt{9+6i-1-16}}{4}$$

$$z=\frac{3+i\pm\sqrt{-8+6i}}{4}$$ ❶

$$z=\frac{3+i\pm(1+3i)}{4}$$

$$z=1+i\ or\ \frac{1}{2}-\frac{1}{2}i$$ ❶

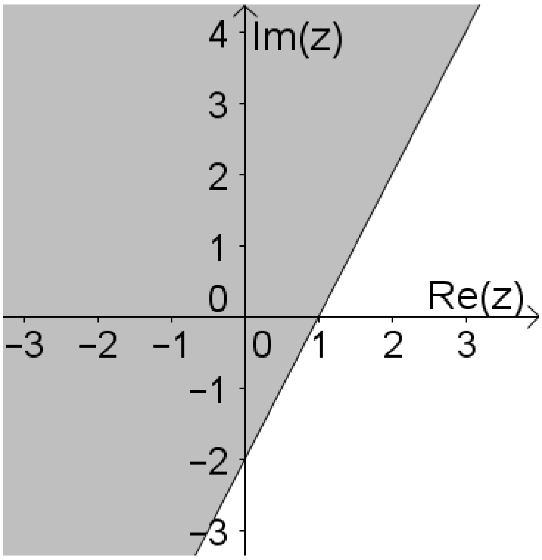
(c)

$Let\ z = x + iy$

$\text{Im}\left[(4-8i)(x+iy)\right] \geq -8$

$4y - 8x \geq -8$

$y \geq 2x - 2$ ❶



Some students wrote $\text{Im}(4y-8x) \geq -8$ which is incorrect. The question is to find the locus. Some did not shade the region at all or the correct region. Some graphed the incorrect line.

(d) i)

By DeMoivres

$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$ ...①

Using Pascals triangle

$(\cos\theta + i\sin\theta)^3$

$= \cos^3\theta + 3\cos^2\theta \times i\sin\theta + 3\cos\theta \times i^2\sin^2\theta + i^3\sin^3\theta$

$= \cos^3\theta - 3\cos\theta\sin^2\theta + i\left(3\cos^2\theta\sin\theta - \sin^3\theta\right)$ ...② ❶

Equating imaginary parts of ① and ②

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$ ❶

$\sin 3\theta = 3\left(1-\sin^2\theta\right)\sin\theta - \sin^3\theta$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

Mostly well done. If a question specifies the method required to be used then you must use this.

(d) ii)

$$8x^3-6x+1=0$$

$$4x^3-3x+\frac{1}{2}=0$$

$$4x^3-3x=-\frac{1}{2}$$

$$3x-4x^3=\frac{1}{2}$$

Let $x=\sin\theta$,

$$3\sin\theta-4\sin^3\theta=\frac{1}{2}$$

$$\sin 3\theta=\frac{1}{2} \quad ❶$$

$$3\theta=\frac{\pi}{6}+2k\pi$$

$$3\theta=\frac{\pi+12k\pi}{6}$$

$$\theta=\frac{\pi+12k\pi}{18}$$

for $k=-1,0,1$

$$\theta=\frac{-11\pi}{18},\frac{\pi}{18},\frac{13\pi}{18}$$

$$\therefore x=\sin\left(\frac{-11\pi}{18}\right),\sin\frac{\pi}{18},\sin\frac{13\pi}{18} \quad ❶$$

Not very well done, many students wrote $\sin\frac{\pi}{18},\sin\frac{13\pi}{18},\sin\frac{5\pi}{18}$ as the three distinct solutions but $\sin\frac{5\pi}{18}=\sin\frac{13\pi}{18}$

(d) iii)

By Sum of roots of $8x^3-6x+1=0$,

$$\sin\left(-\frac{11\pi}{18}\right)+\sin\frac{\pi}{18}+\sin\frac{13\pi}{18}=-\frac{b}{a}$$

$$-\sin\frac{11\pi}{18}+\sin\frac{\pi}{18}+\sin\frac{13\pi}{18}=0 \quad ❶$$

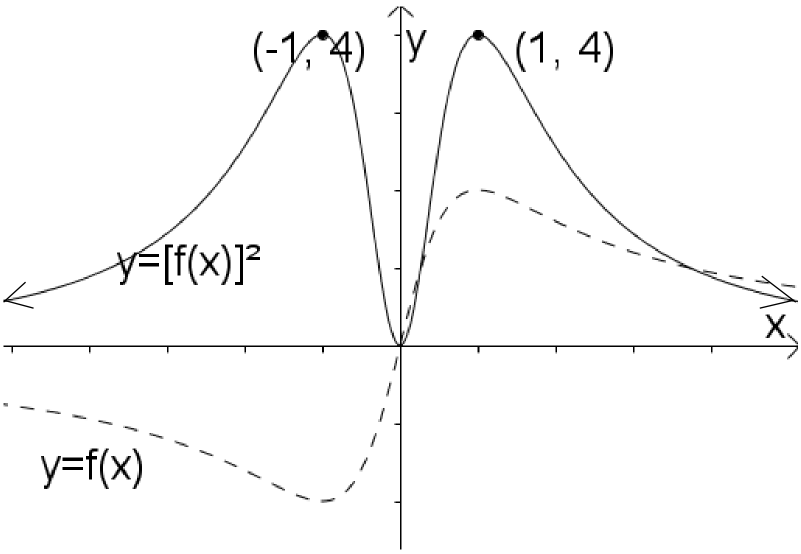
$$\sin\frac{\pi}{18}+\sin\frac{13\pi}{18}=\sin\frac{11\pi}{18}$$

Not very well done

Question12

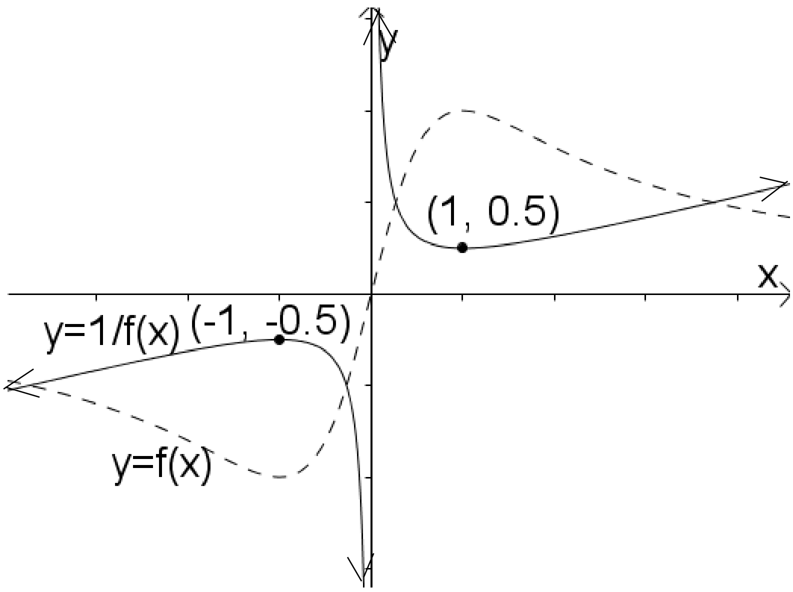
Marker's Comments

(a) i)



Generally well done but a few students had incorrectly drawn the turning point at the origin as a sharp corner rather than as a smooth curve.

ii)



Well done

Question 12

(b) i)

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{-a}^{0} f(x)\,dx$$

Changing the variable of the second integral;

Let $x = -u$

$dx = -du$

When $x = -a,\ u = a$

When $x = 0,\ u = 0$ ❶

$$\begin{aligned}\int_{-a}^{a} f(x)\,dx &= \int_{0}^{a} f(x)\,dx + \int_{a}^{0} f(-u)\times -1\,du \\ &= \int_{0}^{a} f(x)\,dx - \int_{a}^{0} f(-u)\,du \\ &= \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(-u)\,du \quad ❶ \\ &= \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(-x)\,dx \\ &= \int_{0}^{a} f(x) + f(-x)\,dx\end{aligned}$$

Well done but a few students struggled to complete the solution due to a poor choice of substitution.

ii)

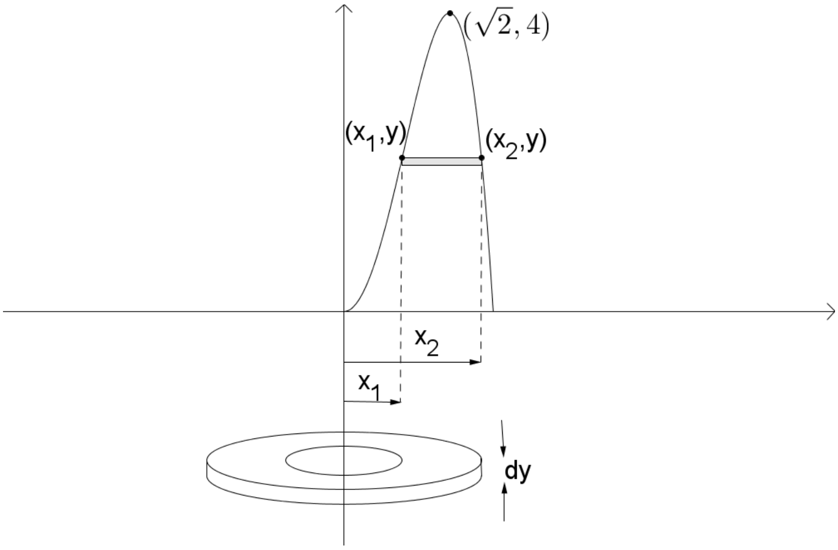
$$\begin{aligned}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1+\sin x}\,dx &= \int_{0}^{\frac{\pi}{4}} \frac{1}{1+\sin x} + \frac{1}{1+\sin(-x)}\,dx \quad ❶ \\ &= \int_{0}^{\frac{\pi}{4}} \frac{1}{1+\sin x} + \frac{1}{1-\sin x}\,dx \\ &= \int_{0}^{\frac{\pi}{4}} \frac{1-\sin x + 1 + \sin x}{1-\sin^2 x}\,dx \\ &= 2\int_{0}^{\frac{\pi}{4}} \frac{1}{\cos^2 x}\,dx \\ &= 2\int_{0}^{\frac{\pi}{4}} \sec^2 x\,dx \quad ❶ \\ &= 2\left[\tan x\right]_{0}^{\frac{\pi}{4}} \\ &= 2\left[1-0\right] \\ &= 2 \quad ❶\end{aligned}$$

Well done. Most students successfully got to

$$2\int_{0}^{\frac{\pi}{4}} \frac{1}{\cos^2 x}\,dx$$

but then some chose to use the t-results to proceed from there. This resulted in a much longer solution with many students making errors. Students are reminded that 2-unit results are still relevant in the Extension 2 exam.

(c)



$$dV = \pi\left(x_2{}^2 - x_1{}^2\right)dy$$

$$y = 4x^2 - x^4$$
$$x^4 - 4x^2 + y = 0$$
$$x^2 = \frac{4 \pm \sqrt{16-4y}}{2}$$
$$x^2 = 2 \pm \sqrt{4-y} \quad ❶$$
$$as\, x_2 \geq x_1$$
$$x_2{}^2 = 2+\sqrt{4-y} \; and \; x_1{}^2 = 2-\sqrt{4-y}$$
$$x_2{}^2 - x_1{}^2 = 2\sqrt{4-y}$$

$$\therefore dV = 2\pi\sqrt{4-y}\ dy$$
$$V = \lim_{dy\to 0}\sum_{y=0}^{4} 2\pi\sqrt{4-y}\ dy$$
$$V = 2\pi\int_0^4 (4-y)^{\frac{1}{2}}\, dy \quad ❶$$
$$V = 2\pi\left[\frac{(4-y)^{\frac{3}{2}}}{\frac{3}{2}\times -1}\right]_0^4 \quad ❶$$
$$V = \frac{-4}{3}\pi\left[(4-y)^{\frac{3}{2}}\right]_0^4$$
$$V = \frac{-4}{3}\pi\left[0 - 4^{\frac{3}{2}}\right]$$
$$V = \frac{32}{3}\pi\, units^3 \quad ❶$$

Lots of students made errors when incorrectly stating that the roots of the quadratic in x^2 ,

$$x^4 - 4x^2 + y = 0$$
$$\left(x^2\right)^2 - 4x^2 + y = 0$$

has roots x_1 and x_2 rather than x_1^2 and x_2^2.

Other students did not follow the instructions in the question and calculated the volume by cylindrical shells rather than annular discs.

Question 13

(a)

i) $\dfrac{x^2}{9}-\dfrac{y^2}{7}=1$ ❶

Generally well done.

ii)

$b^2=a^2\left(e^2-1\right)$

$7=9\left(e^2-1\right)$

$\dfrac{16}{9}=e^2$

$For\ a\ hyperbola\ e>1$

$\therefore e=\dfrac{4}{3}$ ❶

Generally well done.

Errors tended to be with students who could not recall the correct formula.

iii)

$x=3\sec\theta=3\left(\cos\theta\right)^{-1}$

$\dfrac{dx}{d\theta}=-3\left(\cos\theta\right)^{-2}\times-\sin\theta$

$\dfrac{dx}{d\theta}=3\sec\theta\tan\theta$

$y=\sqrt{7}\tan\theta$

$\dfrac{dy}{d\theta}=\sqrt{7}\sec^2\theta$

By the chain rule

$\dfrac{dy}{dx}=\dfrac{dy}{d\theta}\times\dfrac{d\theta}{dx}$

$=\dfrac{\sqrt{7}\sec^2\theta}{3\sec\theta\tan\theta}$

$=\dfrac{\sqrt{7}\sec\theta}{3\tan\theta}$ ❶

$=\dfrac{\sqrt{7}}{3\sin\theta}$

$\therefore$ gradient of normal will be $m_N=\dfrac{-3\sin\theta}{\sqrt{7}}$

Generally well done.

Errors tended to be careless arithmetic mistakes.

The equation of the normal will be

$$y - y_1 = m\left(x - x_1\right)$$

$$y - \sqrt{7}\tan\theta = \frac{-3\sin\theta}{\sqrt{7}}\left(x - 3\sec\theta\right) \text{ ❶}$$

$$\sqrt{7}y - 7\tan\theta = -3\sin\theta x + 9\tan\theta$$

$$3\sin\theta x + \sqrt{7}y = 16\tan\theta$$

$$\frac{3x}{\sec\theta} + \frac{\sqrt{7}y}{\tan\theta} = 16 \text{ ❶}$$

iv)

$When\ x = 0,$

Wow. Easy mark.

$$0 + \frac{\sqrt{7}y}{\tan\theta} = 16$$

$$y = \frac{16\tan\theta}{\sqrt{7}}$$

$$\therefore N = \left(0, \frac{16\tan\theta}{\sqrt{7}}\right)$$

(b)

$$0 = z^4 + z$$

$$0 = z\left(z^3 + 1\right)$$

$$0 = z\left(z+1\right)\left(z^2 - z + 1\right) \text{ ❶}$$

$$0 = z\left(z+1\right)\left(z^2 - z + \frac{1}{4} + \frac{3}{4}\right)$$

$$0 = z\left(z+1\right)\left(\left(z - \frac{1}{2}\right)^2 - \left(\frac{i\sqrt{3}}{2}\right)^2\right) \text{ ❶}$$

$$0 = z\left(z+1\right)\left(z - \frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\left(z - \frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$$

$$z = 0, -1, \frac{1}{2} + \frac{i\sqrt{3}}{2}, \frac{1}{2} - \frac{i\sqrt{3}}{2} \text{ ❶}$$

Students attempted a variety of ways.

Errors included:

- Too many solutions (eg 5 for a deg 4 poly)
- Arithmetic mistakes.

(c)

$$\frac{\alpha}{\beta\gamma},\frac{\beta}{\gamma\alpha},\frac{\gamma}{\alpha\beta}$$

$$=\frac{\alpha^2}{\alpha\beta\gamma},\frac{\beta^2}{\alpha\beta\gamma},\frac{\gamma^2}{\alpha\beta\gamma}$$

$$=\alpha^2,\beta^2,\beta^2$$ ❶

$$\left(as\ \alpha\beta\gamma=1\right)$$

So the required polynomial is

$$\left(\sqrt{x}\right)^3-4\sqrt{x}-1=0$$ ❶

$$x\sqrt{x}-4\sqrt{x}=1$$

$$\sqrt{x}\left(x-4\right)=1$$

$$x\left(x-4\right)^2=1$$

$$x\left(x^2-8x+16\right)=1$$

$$x^3-8x^2+16x-1=0$$ ❶

Many students did not realise you could rearrange the roots to $\alpha^2,\beta^2,\gamma^2$.

Many students tried to find the relationship between the sum of the roots and the coefficients of the required poly. Typically these students got 'bogged' down with working and were unable to find the equation.

(d)

Because the co-efficients of the polynomial are real, as $3+i$ is a zero then so too is the conjugate $3-i$ ❶

$$\therefore\left(x-3-i\right)\left(x-3+i\right)$$

$$=\left(x^2-2\operatorname{Re}\left(3-i\right)x+\left|3-i\right|^2\right)$$

$$=\left(x^2-6x+10\right)$$ ❶

$$\therefore\left(x^2-6x+10\right)\left(x+a\right)\equiv x^3-5x^2+4x+10$$

By inspection of the constant terms and leading co-efficients, or by using polynomial division,
$a=1.$

$$\therefore\left(x^2-6x+10\right)\left(x+1\right)\equiv x^3-5x^2+4x+10$$ ❶

Mostly well done.

Errors tended to be arithmetic.

Question 14

(a)i

$$\frac{7x^2-5x+4}{(x-1)(x^2+1)} \equiv \frac{A}{x-1}+\frac{Bx+C}{x^2+1}$$

$$\therefore 7x^2-5x+4 = A(x^2+1)+(Bx+C)(x-1)$$

When $x=i$, *When* $x=1$,

$-3-5i=(Bi+C)(i-1)$ $6=2A$

$-3-5i=-B-C+(-B+C)i$ $A=3$ ❶

Equating real and imaginary parts

$B+C=3$...①

$B-C=5$...②

①+② $2B=8$

$B=4$

①−② $2C=-2$

$C=-1$

$$\therefore \frac{7x^2-5x+4}{(x-1)(x^2+1)} \equiv \frac{3}{x-1}+\frac{4x-1}{x^2+1}$$ ❶

This question was very well answered

(a)ii

$$\int \frac{7x^2-5x+4}{(x-1)(x^2+1)}\,dx$$

$$=\int \frac{3}{x-1}+\frac{4x-1}{x^2+1}\,dx$$

$$=3\int \frac{1}{x-1}\,dx+2\int \frac{2x}{x^2+1}\,dx-\int \frac{1}{x^2+1}\,dx$$ ❶

$$=3\ln|x-1|+2\ln|x^2+1|-\tan^{-1}x+c$$ ❶

Well done.
Take care to avoid arithmetic errors.

(b)

$$I=\int_0^{\frac{1}{4}}\frac{x^2dx}{\sqrt{1-4x^2}}$$

$$Let\, 2x=\sin\theta$$

$$x=\frac{1}{2}\sin\theta$$

$$dx=\frac{1}{2}\cos\theta\ d\theta$$

$$When\, x=\frac{1}{4},\ \theta=\frac{\pi}{6}$$

$$When\, x=0,\ \theta=0.$$ ❶

$$I=\int_0^{\frac{\pi}{6}}\frac{\frac{1}{4}\sin^2\theta}{\sqrt{1-\sin^2\theta}}\times\frac{1}{2}\cos\theta\ d\theta$$

$$=\frac{1}{8}\int_0^{\frac{\pi}{6}}\sin^2\theta\ d\theta$$

$$=\frac{1}{16}\int_0^{\frac{\pi}{6}}1-\cos 2\theta\ d\theta$$

$$=\frac{1}{16}\left[\theta-\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{6}}$$ ❶

$$=\frac{1}{16}\left[\frac{\pi}{6}-\frac{1}{2}\sin\frac{\pi}{3}-\left(0-\frac{1}{2}\sin 0\right)\right]$$

$$=\frac{1}{16}\left(\frac{\pi}{6}-\frac{\sqrt{3}}{4}\right)$$

$$=\frac{1}{32}\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right)$$ ❶

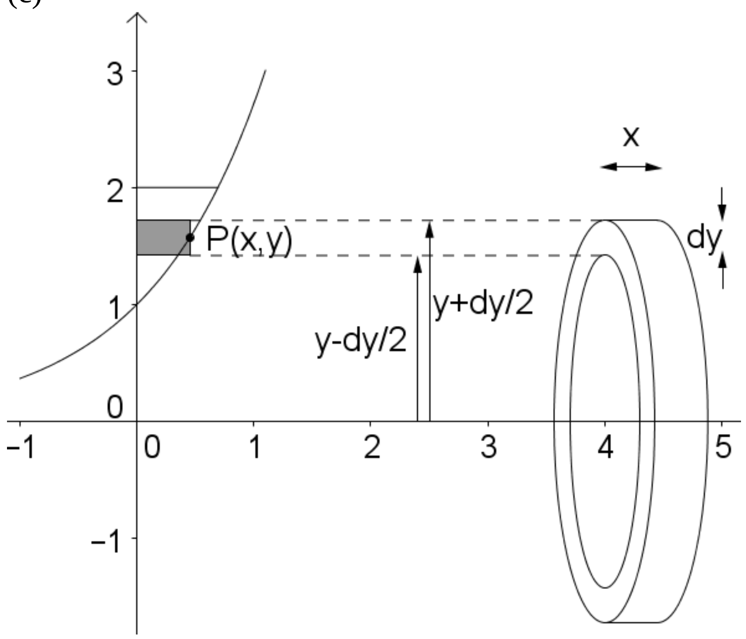
This question was mostly well done.

It was answered using a range of methods.

Some student's used inefficient approaches that involved making two substitutions.

Experience through greater practise will develop your understanding of the best substitution to make.

(c)



$$dV = \pi\left[\left(y+\frac{dy}{2}\right)^2-\left(y-\frac{dy}{2}\right)^2\right]x$$
$$= \pi\left[y^2+ydy+\frac{(dy)^2}{4}-\left(y^2-ydy+\frac{(dy)^2}{4}\right)\right]x$$
$$= \pi\times 2y\,dy\times x$$
$$= 2\pi xy\,dy$$
$$= 2\pi\ln(y)\times y\,dy \text{ ❶}$$

$$V = \lim_{dy\to 0}\sum_{y=1}^{y=2} 2\pi y\ln y\,dy$$

$$V = 2\pi\int_1^2 y\ln y\,dy$$

$$V = 2\pi\left\{\left[\frac{y^2}{2}\ln y\right]_1^2-\int_1^2\frac{y^2}{2}\times\frac{1}{y}dy\right\} \text{ ❶}$$

$$V = 2\pi\left\{2\ln 2-0-\int_1^2\frac{y}{2}dy\right\}$$

$$V = 2\pi\left\{2\ln 2-\left[\frac{y^2}{4}\right]_1^2\right\}$$

$$V = 2\pi\left[2\ln 2-\left(1-\frac{1}{4}\right)\right]$$

$$V = 2\pi\left(2\ln 2-\frac{3}{4}\right)units^3 \text{ ❶}$$

Some students incorrectly stated the radius of each cylindrical shell is $2-y$.

Few incorrectly used x values in the bounds.

It is a good idea to evaluate your exact volume answer to check it is reasonable i.e. a positive not negative value.

(d) i)

$$x^2 - y^2 = a^2$$

$$2x - 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{x}{y}$$

Gradient of tangent at (x_1, y_1) is $\frac{x_1}{y_1}$ ❶

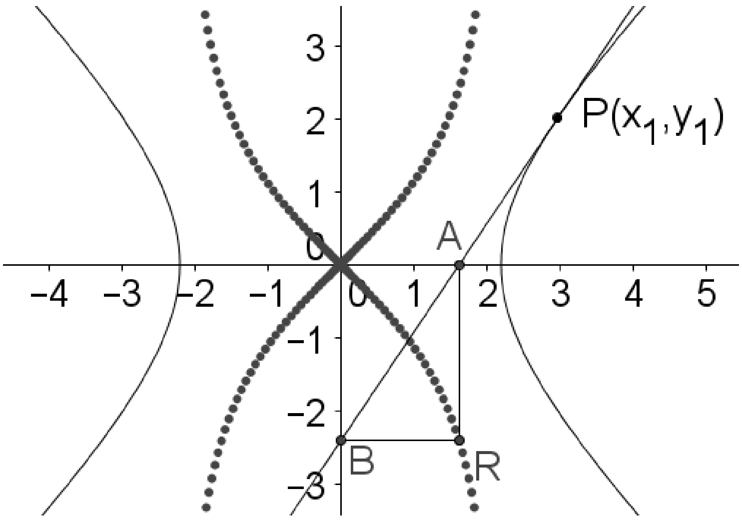
Equation of tangent is

$$(y - y_1) = \frac{x_1}{y_1}(x - x_1)$$

$$y_1 y - {y_1}^2 = x_1 x - {x_1}^2$$

$${x_1}^2 - {y_1}^2 = x_1 x - y_1 y \text{ ❶}$$

$$a^2 = x_1 x - y_1 y$$



(d) ii)

When $y = 0$,	*When* $x = 0$,
$x_1 x = a^2$	$-y_1 y = a^2$
$x = \frac{a^2}{x_1}$ ...①	$y = \frac{-a^2}{y_1}$ ...②
$\therefore A\left(\frac{a^2}{x_1}, 0\right)$	$\therefore B\left(0, -\frac{a^2}{y_1}\right)$ ❶

As $OARB$ is a rectangle, R is $\left(\frac{a^2}{x_1}, -\frac{a^2}{y_1}\right)$.

To find the equation of the locus, eliminate x_1 and y_1 from equations ① and ②.

This question was very well answered

Some students recognised that midpoint_{OR} and midpoint_{AB} are equal

i.e. $M_{OR} = \left(\frac{x}{2}, \frac{y}{2}\right)$

$$M_{AB} = \left(\frac{a^2}{2x_1}, -\frac{a^2}{2y_1}\right)$$

and then proceeded to work out the locus.

$x = \frac{a^2}{x_1}$ ...① and $y = \frac{-a^2}{y_1}$ ...②

$x_1 = \frac{a^2}{x}$ $y_1 = \frac{-a^2}{y}$ ❶

But (x_1, y_1) is a point on the hyperbola and so it must satisfy the equation, $a^2 = x^2 - y^2$

$\therefore a^2 = x_1{}^2 - y_1{}^2$

$$a^2 = \left(\frac{a^2}{x}\right)^2 - \left(\frac{-a^2}{y}\right)^2$$ ❶

$$a^2 = \frac{a^4}{x^2} - \frac{a^4}{y^2}$$

$$1 = \frac{a^2}{x^2} - \frac{a^2}{y^2}$$

Many students successfully determined the coordinates of R and then made x_1 and y_1 the subject but did not know how to proceed from this point.

Question 15

(a)

Area of square = Area of rhombus

$$= \frac{1}{2} \, product\ of\ diagonals$$

$$dV = \frac{1}{2}\left(\sin x - \cos x\right)^2 dx$$

$$dV = \frac{1}{2}\left(\sin^2 x - 2\sin x\cos x + \cos^2 x\right)dx$$

$$dV = \frac{1}{2}\left(1 - \sin 2x\right)dx \text{ ❶}$$

$$V = \lim_{dx \to 0} \sum_{x=\frac{\pi}{4}}^{x=\frac{5\pi}{4}} \frac{1}{2}\left(1 - \sin 2x\right)dx \text{ ❶}$$

$$V = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} 1 - \sin 2x \, dx$$

$$V = \frac{1}{2}\left[x + \frac{1}{2}\cos 2x\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \text{ ❶}$$

$$V = \frac{1}{2}\left\{\frac{5\pi}{4} + \frac{1}{2}\cos\frac{5\pi}{2} - \left(\frac{\pi}{4} + \frac{1}{2}\cos\frac{\pi}{2}\right)\right\}$$

$$V = \frac{1}{2}\left\{\frac{5\pi}{4} + 0 - \left(\frac{\pi}{4} + 0\right)\right\}$$

$$V = \frac{\pi}{2}\, units^3 \text{ ❶}$$

Many students experienced difficulty with this question.

Typically students had trouble finding the area of a square.

Some students over complicated it by firstly finding a side length.

Many students overlooked that a square has the properties of a rhombus.

(b) i)

Resultant force $= ma$

$$ma = -mg - \frac{v^2}{10}$$

$$10a = -100 - \frac{v^2}{10}$$

$$a = -\frac{1000 + v^2}{100} \quad ❶$$

$$\frac{dv}{dt} = -\frac{1000 + v^2}{100}$$

$$\frac{dt}{dv} = -\frac{100}{1000 + v^2}$$

$$t = -100\int \frac{1}{\left(10\sqrt{10}\right)^2 + v^2}\,dv$$

$$t = \frac{-100}{10\sqrt{10}}\tan^{-1}\frac{v}{10\sqrt{10}} + c$$

$$t = -\sqrt{10}\tan^{-1}\frac{v}{10\sqrt{10}} + c \quad ❶$$

$$\text{when}\, t = 0, v = u$$

$$0 = -\sqrt{10}\tan^{-1}\frac{u}{10\sqrt{10}} + c$$

$$c = \sqrt{10}\tan^{-1}\frac{u}{10\sqrt{10}}$$

$$\therefore t = \sqrt{10}\tan^{-1}\frac{u}{10\sqrt{10}} - \sqrt{10}\tan^{-1}\frac{v}{10\sqrt{10}}$$

$$\text{when}\, t = T, v = 0$$

$$\therefore T = \sqrt{10}\tan^{-1}\frac{u}{10\sqrt{10}} \quad ❶$$

Many students experienced difficulty.

Typically students would take ‘short cuts’ and not develop an equation for acceleration. This created errors which were carried.

Some students chose a definite integral method but did not use it correctly. Care should be taken to be familiar with your chosen method.

(b) ii)

$$a = -\frac{1000+v^2}{100}$$

$$v\frac{dv}{dx} = -\frac{1000+v^2}{100}$$ ❶

$$\frac{dv}{dx} = -\frac{1000+v^2}{100v}$$

$$\frac{dx}{dv} = -\frac{100v}{1000+v^2}$$

$$x = -50\int \frac{2v}{1000+v^2}\,dv$$

$$x = -50\ln\left|1000+v^2\right| + c$$ ❶

$$x = -50\ln\left(1000+v^2\right) + c$$

$when\, x = 0, v = u$

$$0 = -50\ln\left(1000+u^2\right) + c$$

$$c = 50\ln\left(1000+u^2\right)$$ ❶

$$\therefore v = 50\ln\left(1000+u^2\right) - 50\ln\left(1000+v^2\right)$$

$$x = 50\ln\left(\frac{1000+u^2}{1000+v^2}\right)$$

$when\, x = H, v = 0$ ❶

$$\therefore H = 50\ln\left(\frac{1000+u^2}{1000}\right)$$

Many students experienced difficulty.

Typically students would take 'short cuts' and not develop an equation for acceleration. This created errors which were carried.

Some students chose a definite integral method but did not use it correctly. Care should be taken to be familiar with your chosen method.

Many students realised they did not have the correct answer and tried to 'fudge' their result. Not getting the error should alert you to your mistake. Find that error and fix it!

(b) iii)

$$ma = mg - \frac{v^2}{10}$$

$$10a = 100 - \frac{v^2}{10}$$

$$a = 10 - \frac{v^2}{100} \ ❶$$

$$v\frac{dv}{dx} = \frac{1000 - v^2}{100}$$

$$\frac{dv}{dx} = \frac{1000 - v^2}{100v}$$

$$\frac{dx}{dv} = \frac{100v}{1000 - v^2}$$

$$dx = \frac{100v}{1000 - v^2}\,dv \ ❶$$

$$x = -50\int \frac{-2v}{1000 - v^2}\,dv$$

$$x = -50\ln\left|1000 - v^2\right| + c$$

$$when\ x = 0, v = 0$$

$$0 = -50\ln(1000) + c$$

$$c = 50\ln(1000)$$

$$\therefore x = 50\ln(1000) - 50\ln\left|1000 - v^2\right|$$

$$x = 50\ln\left|\frac{1000}{1000 - v^2}\right| \ ❶$$

$$when\ x = H, v = w$$

$$\therefore H = 50\ln\left|\frac{1000}{1000 - w^2}\right|$$

Using the result of bii,

$$50\ln\left(\frac{1000 + u^2}{1000}\right) = 50\ln\left|\frac{1000}{1000 - w^2}\right| (****)$$

$$\frac{1000 + u^2}{1000} = \frac{1000}{1000 - w^2}$$

$$1000 - w^2 = \frac{1000000}{1000 + u^2}$$

$$-w^2 = \frac{1000000}{1000 + u^2} - 1000$$

$$-w^2 = \frac{1000000 - 1000000 - 1000u^2}{1000 + u^2}$$

$$-w^2 = \frac{-1000u^2}{1000 + u^2}$$

$$w^2 = \frac{1000u^2}{1000 + u^2} \ ❶$$

Note: At (****), it is not necessary to consider the case where $-\frac{1000 + u^2}{1000} = \frac{1000}{1000 - w^2}$ because a calculation of the terminal velocity shows that $\frac{1000}{1000 - w^2} > 0$

Errors were made as described above.

Additionally some students had trouble comparing results found in parts (ii) and (iii) because of poor setting out, often confusing what to substitute and where.

Clear communication is required.

Question 16

(a) i)

$$I_n = \int_0^2 \left(4-x^2\right)^n dx$$

$$Let\, u = \left(4-x^2\right)^n \qquad dv = 1$$

$$du = n\left(4-x^2\right)^{n-1} \times -2x \qquad v = x$$

$$du = -2nx\left(4-x^2\right)^{n-1} \quad ❶$$

$$I_n = \left[x\left(4-x^2\right)\right]_0^2 + 2n\int_0^2 x^2\left(4-x^2\right)^{n-1} dx \quad ❶$$

$$I_n = 0 - 2n\int_0^2 -x^2\left(4-x^2\right)^{n-1} dx$$

$$I_n = 0 - 2n\int_0^2 \left(4-x^2-4\right)\left(4-x^2\right)^{n-1} dx$$

$$I_n = -2n\int_0^2 \left(4-x^2\right)\left(4-x^2\right)^{n-1} - 4\left(4-x^2\right)^{n-1} dx \quad ❶$$

$$I_n = -2n\int_0^2 \left(4-x^2\right)^n dx + 8n\int_0^2 \left(4-x^2\right)^{n-1} dx$$

$$(2n+1)I_n = 8nI_{n-1} \quad ❶$$

$$I_n = \frac{8n}{2n+1}I_{n-1}$$

Some students completed a successful solution but many failed to make good choices for u and dv in the integration by parts.

Other students got confused at the stage

$$I_n = 0 - 2n\int_0^2 -x^2\left(4-x^2\right)^{n-1} dx$$

$$I_n = 0 - 2n\int_0^2 \left(4-x^2-4\right)\left(4-x^2\right)^{n-1} dx$$

(a) ii)

$$I_n = \frac{8n}{2n+1}I_{n-1}$$

$$I_3 = \frac{24}{7}I_2$$

$$= \frac{24}{7}\times\frac{16}{5}I_1$$

$$= \frac{24}{7}\times\frac{16}{5}\times\frac{8}{3}I_0 \quad ❶$$

$$= \frac{1024}{35}\int_0^2 \left(4-x^2\right)^0 dx$$

$$= \frac{1024}{35}\int_0^2 1\,dx$$

$$= \frac{1024}{35}[x]_0^2$$

$$= \frac{2048}{35} \quad ❶$$

Well done but a few students miscalculated I_0

(b) i)

$f(x)=e^x$

$f'(x)=f''(x)=f'''(x)=f^{(3)}(x)=f^{(4)}(x)=f^{(5)}(x)=e^x$

$f(0)=f'(0)=f''(0)=f'''(0)=f^{(4)}(0)=f^{(5)}(0)=1$ ❶

$\therefore$ *For* $f(x)=e^x$,

$$A(x)=\sum_{r=0}^{5}\frac{f^{(r)}(0)\times x^r}{r!}$$

$$=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}$$ ❶

ii)

$f(x)=\sin x,\ \ f(0)=0$

$f'(x)=\cos x,\ f'(0)=1$

$f''(x)=-\sin x,\ f''(0)=0$

$f'''(x)=-\cos x,\ f'''(0)=-1$

$f^{(4)}(x)=\sin x,\ f^{(4)}(0)=0$

$f^{(5)}(x)=\cos x,\ f^{(5)}(0)=1$ ❶

$\therefore$ *For* $f(x)=\sin x$,

$$A(x)=\sum_{r=0}^{5}\frac{f^{(r)}(0)\times x^r}{r!}$$

$$=x-\frac{x^3}{3!}+\frac{x^5}{5!}$$ ❶

For bi, ii and iii

A few students completed a correct solution. Many students, as they were expanding the series, failed to calculate the nth derivative at zero but instead used $f^{(n)}(x)$.

iii)

$$f(x)=\cos x,\quad f(0)=1$$

$$f'(x)=-\sin x,\ f'(0)=0$$

$$f''(x)=-\cos x,\ f''(0)=-1$$

$$f'''(x)=\sin x,\ f'''(0)=0$$

$$f^{(4)}(x)=\cos x,\ f^{(4)}(0)=1$$

$$f^{(5)}(x)=-\sin x,\ f^{(5)}(0)=0 \text{ ❶}$$

$$\therefore\ For\ f(x)=\cos x,$$

$$A(x)=\sum_{r=0}^{5}\frac{f^{(r)}(0)\times x^r}{r!}$$

$$=1-\frac{x^2}{2!}+\frac{x^4}{4!} \text{ ❶}$$

iv)

$$as\, n\to\infty,$$

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\frac{x^6}{6!}+\frac{x^7}{7!}+\frac{x^8}{8!}\ldots$$

$$e^{i\theta}=1+i\theta+\frac{i^2\theta^2}{2!}+\frac{i^3\theta^3}{3!}+\frac{i^4\theta^4}{4!}+\frac{i^5\theta^5}{5!}+\frac{i^6\theta^6}{6!}+\frac{i^7\theta^7}{7!}+\frac{i^8\theta^8}{8!}\ldots$$

$$=1+i\theta-\frac{\theta^2}{2!}-\frac{i\theta^3}{3!}+\frac{\theta^4}{4!}+\frac{i\theta^5}{5!}-\frac{\theta^6}{6!}-\frac{i\theta^7}{7!}+\frac{\theta^8}{8!}\ldots \text{❶}$$

$$=\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\frac{\theta^6}{6!}+\frac{\theta^8}{8!}\ldots\right)+\left(i\theta-\frac{i\theta^3}{3!}+\frac{i\theta^5}{5!}-\frac{i\theta^7}{7!}+\ldots\right)$$

$$=\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\frac{\theta^6}{6!}+\frac{\theta^8}{8!}\ldots\right)+i\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\frac{\theta^7}{7!}+\ldots\right) \text{ ❶}$$

$$\therefore e^{i\theta}=\cos\theta+i\sin\theta$$

Hardly any students attempted this part of the question.

v)

$$e^{i\frac{\pi}{3}}=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$$

$$=\frac{1}{2}+i\frac{\sqrt{3}}{2} \text{ ❶}$$

Those who completed iv) generally got v) right.