Name: ______________________

Teacher: ______________________

Class: ______________________

FORT STREET HIGH SCHOOL

2019

HIGHER SCHOOL CERTIFICATE COURSE

ASSESSMENT TASK 4: TRIAL HSC

Mathematics Extension 2

Time allowed: 3 hours
(plus 5 minutes reading time)

Syllabus Outcomes	Assessment Area Description and Marking Guidelines
E3	Uses the relationship between algebraic and geometric representations of complex numbers and conic sections
E4	Uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving polynomials and conic sections
E5	Uses ideas and techniques of calculus to solve problems in mechanics involving resolution of forces and resisted motion
E6	Combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions
E7	Uses the techniques of slicing and cylindrical shells to determine volume
E8	Applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems
E9	Communicates abstract ideas and relationships using appropriate notation and logical argument

Total Marks 100

Section I 10 marks
Multiple Choice, attempt all questions.
Allow about 15 minutes for this section.

Section II 90 Marks
Attempt Questions 11-16.
Allow about 2 hours 45 minutes for this section.

Section I	Total 10	Marks
Q1-Q10	/10	
Section II	Total 90	Marks
Q11	/15	
Q12	/15	
Q13	/15	
Q14	/15	
Q15	/15	
Q16	/15	
	Percent	

General Instructions:

- Questions 1-10 are to be answered on the multiple choice answer sheet that is located on the last page of this booklet.
- Questions 11-16 are to be started in a new answer booklet.
- The marks allocated for each question are indicated.
- In Questions 11-16, show relevant mathematical reasoning and/or calculations.
- Marks may not be awarded for careless or poorly arranged work.
- NESA-approved calculators may be used.
- A reference sheet is provided.

Section I (10 marks)

Attempt questions 1–10. Allow about 15 minutes for this section.

Use the multiple-choice answer sheet, for Questions 1–10.

Question 1.

If $z = 4 - i$ and $w = 3 + 5i$, what is the value of $z - \overline{w}$?

(A) $-1-4i$ (B) $-1+4i$ (C) $1-4i$ (D) $1+4i$

Question 2.

What is the eccentricity for the hyperbola $\frac{y^2}{4} - \frac{x^2}{3} = 1$?

(A) $\frac{\sqrt{7}}{3}$ (B) $\frac{5}{2}$ (C) $\frac{5}{\sqrt{3}}$ (D) $\frac{\sqrt{7}}{2}$

Question 3.

$\frac{3x-4}{x^2-x-6}$ expressed as a sum of partial fractions is

(A) $\frac{3}{x-2} - \frac{13}{x+3}$ (B) $\frac{3}{x+3} + \frac{2}{x-2}$ (C) $\frac{2}{x+2} - \frac{1}{x-3}$ (D) $\frac{2}{x+2} + \frac{1}{x-3}$

Question 4.

$\int \frac{3x^2}{\left(1+x^3\right)^5} \, dx =$

(A) $\frac{-1}{4\left(1+x^3\right)^4} + C$ (B) $\frac{4}{\left(1+x^3\right)^4} + C$ (C) $\frac{-6}{\left(1+x^3\right)^6} + C$ (D) $\frac{1}{6\left(1+x^3\right)^6} + C$

Question 5.

If $\int_1^2 f(x)\,dx = 5$, what is the value of $\int_2^1 f(3-x)\,dx$?

(A) 3 (B) 5 (C) -5 (D) -3

Question 6.

At how many points do the graphs of $y = |x|$ and $y = |\sin x|$ intersect?

(A) 1 (B) 2 (C) 3 (D) 0

Question 7.

If ω is a complex cubed root of unity, then which of the following is **NOT** correct ?

(A) $\omega^2 = -\omega - 1$

(B) ω^2 is the other complex root.

(C) $\omega^6 = \omega^2 + \omega$

(D) $\overline{\omega}$ is also a root

Question 8.

What is the number of asymptotes on the graph of $y = \dfrac{2x^4}{x^3 + 1}$?

(A) 1 (B) 2 (C) 3 (D) 0

Question 9.

A partical of mass m is dropped vertically in a medium where gravity is g and the terminal velocity is $\sqrt{\dfrac{mg}{k}}$.

If down is taken as the positive direction, then the equation of motion is

(A) $a = mg - kv^2$

(B) $a = kv^2 - mg$

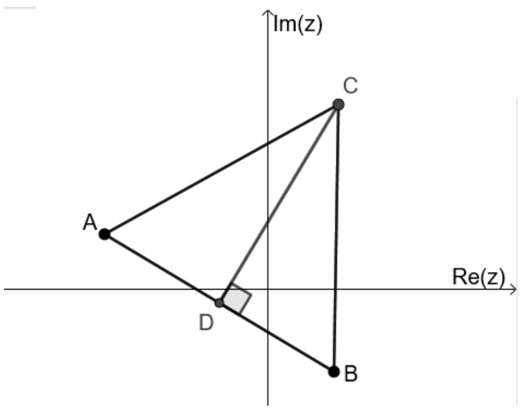
(C) $a = mg - kv^3$

(D) $a = mg - kv$

Question 10.

In the Argand diagram, ABC is an equilateral triangle and the vertices A and C correspond to the complex numbers w and z respectively.

What complex number below corresponds to the vector $\overrightarrow{DB}$?



(A) $(z-w)\left(\dfrac{1}{4}+\dfrac{\sqrt{3}}{4}i\right)$

(B) $(z-w)\left(\dfrac{1}{4}-\dfrac{\sqrt{3}}{4}i\right)$

(C) $(w-z)\left(\dfrac{1}{4}+\dfrac{\sqrt{3}}{4}i\right)$

(D) $(w-z)\left(\dfrac{1}{4}-\dfrac{\sqrt{3}}{4}i\right)$

End of Section 1

Section II (90 marks)

Attempt Questions 11–16. Allow about 2 hours and 45 minutes for this section.

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/ or calculations.

Question 11 (15 marks) Use a *separate* writing booklet

(a) For the complex number $w = 1 - i\sqrt{3}$,

(i) Express w in modulus argument form. **2**

(ii) Hence, find w^{-4} in $a+ib$ form. **2**

(b) (i) Show that $(1-2i)^2 = -3-4i$ **1**

(ii) Hence, solve $z^2 - 5z + (7+i) = 0$ **2**

(c) On separate Argand diagrams, sketch regions defined by the following, where z is a complex number.

(i) $|z| < 3$ **1**

(ii) $0 \le \arg(z+2-i) \le \frac{\pi}{2}$ **2**

(d) (i) Given $z = \cos\theta + i\sin\theta$, show that $z^n + z^{-n} = 2\cos n\theta$ **1**

(ii) By expanding $\left(z+z^{-1}\right)^3$ and using the result of (i), express $\cos^3\theta$ in terms of $\cos 3\theta$ and $\cos\theta$. **2**

(iii) Using your answer to part (ii), evaluate $\int_0^{\frac{\pi}{6}} \cos^3\theta \, d\theta$ **2**

End of Question 11

Question 12 (15 marks) Use a *separate* writing booklet.

(a) The polynomial $P(x) = 2x^3 - 13x^2 + 32x - 13$, has a zero at $x = 3 - 2i$. Determine the other zeroes of $P(x)$. **2**

(b) (i) Show that 2 is a root of the polynomial $P(x) = x^3 + 3x - 14$. **1**

(ii) If the other two roots of $P(x)$ are α and β, find the polynomial whose roots are

5, $\alpha + 3$ and $\beta + 3$. **2**

(c) (i) If α is a multiple root of the polynomial equation $P(x) = 0$, prove that $P'(\alpha) = 0$. **2**

(ii) Find all roots of the equation $18x^3 + 3x^2 - 28x + 12 = 0$ given that two of the roots are equal. **3**

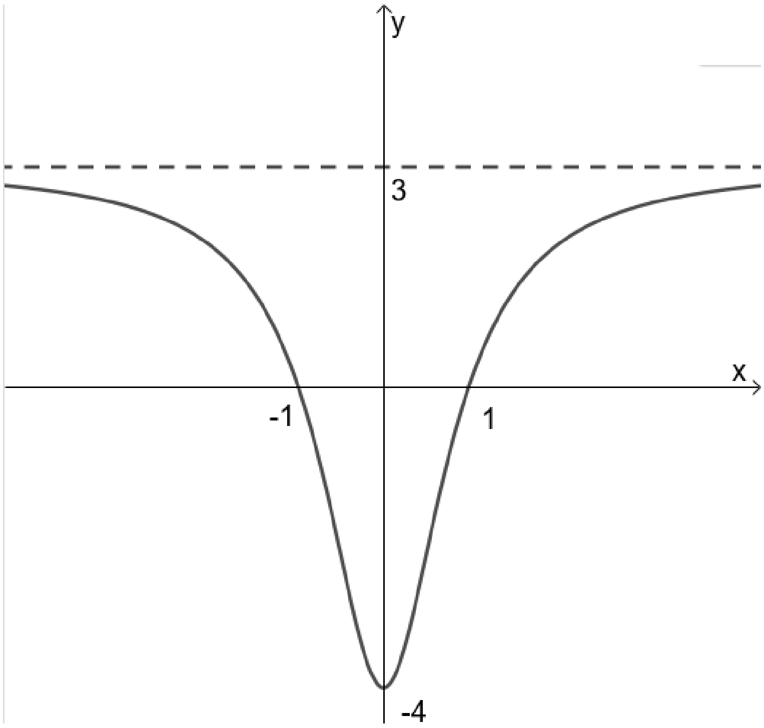
(d) (i) Find the six sixth roots of -1, expressing each in the form $x + iy$ where x and y are real. **3**

(ii) Hence or otherwise, find the four roots of the equation $z^4 - z^2 + 1 = 0$. **2**

End of Question 12

Question 13 (15 marks) Use a *separate* writing booklet.

(a) The graph of $y = f(x)$ is shown below.



On separate diagrams, draw the following graphs, clearly indicating important features.

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = \left[f(x)\right]^2$ **2**

(iii) $y = \log_e\left(f(x)\right)$ **2**

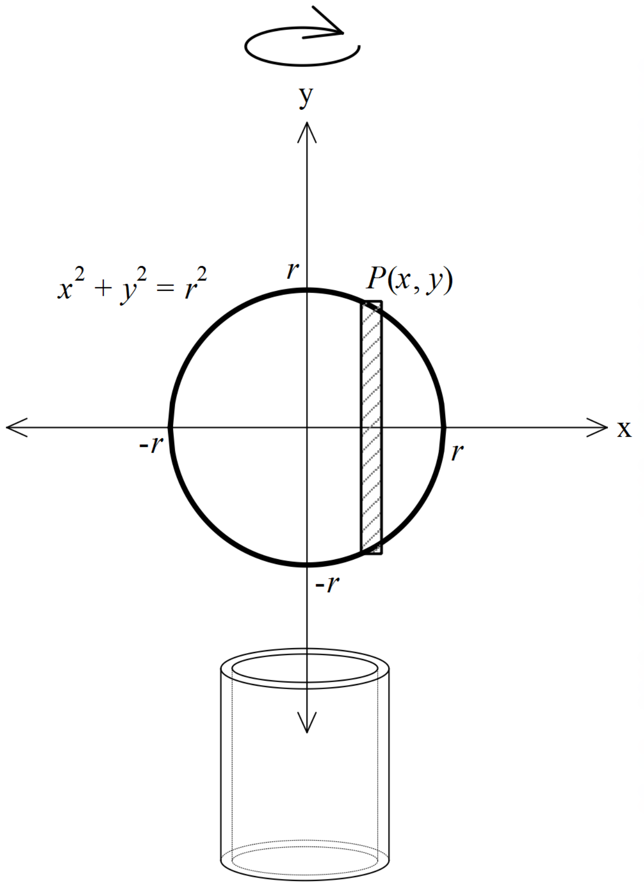
(b) Find the gradient of the tangent to the curve $x^2 - 3xy + 2y^2 = 3$ at the point $(5, 2)$. **2**

(c) If pqr represents a three digit number (i.e. $pqr \equiv 100p + 10q + r$) and $p + q + r = 3A$ where A is a positive integer, show that the number pqr is divisible by 3. **2**

Question 13 continues on the next page

Question 13 - continued

(d) A circle centred at the origin with equation $x^2 + y^2 = r^2$ and a variable point $P(x, y)$ on the circle are shown below.



(i) Draw an appropriate diagram for the cylindrical shell, labelling its dimensions. **1**

(ii) Prove, using the method of cylindrical shells, that the volume of a sphere is $\frac{4}{3}\pi r^3$. **4**

End of Question 13

Question 14 (15 marks) Use a *separate* writing booklet

(a) Use the technique of integration by parts to find $\int x^2 e^x \, dx$ **2**

(b) (i) Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x+\sin x}$ using the substitution $t = \tan\frac{x}{2}$. **3**

(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \frac{x\,dx}{1+\cos x+\sin x}$ using the substitution $u = \frac{\pi}{2} - x$ **3**

(c) Find $\int \sqrt{\frac{1-x}{1+x}}\,dx$ **3**

(d) The normal to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $P\left(a\cos\phi, b\sin\phi\right)$ cuts the x-axis at the point M and the y-axis at N.

(i) Prove that the equation of the normal at P is $\frac{ax}{\cos\phi}-\frac{by}{\sin\phi}=a^2-b^2$ **2**

(ii) Hence prove that $\frac{PM}{PN}=\frac{b^2}{a^2}$ **2**

End of Question 14

Question 15 (15 marks) Use a *separate* writing booklet

(a) Two points, $P\left(cp, \dfrac{c}{p}\right)$ and $Q\left(cq, \dfrac{c}{q}\right)$, lie on the rectangular hyperbola $xy = c^2$

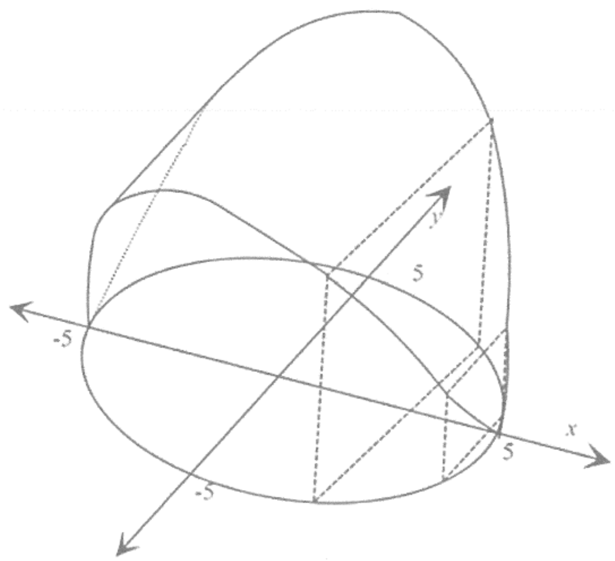
(i) Show that the equation of the tangent at P is $x + p^2 y = 2cp$ **2**

(ii) The tangents through P and Q meet at T. Find the co-ordinates of T in terms of c, p and q. **2**

(iii) The point T lies on another hyperbola $xy = k^2$, for all positions of P and Q. **1**

Show that $\dfrac{pq}{(p+q)^2} = \dfrac{k^2}{4c^2}$

(b) Let S be a solid where the base is the region bounded by the circle $x^2 + y^2 = 25$ and each cross-section taken perpendicular to the x-axis is a square.



Find the volume of the solid S **3**

(c) Let $I_n = \int_0^1 x\left(1 - x^5\right)^n dx$ where n is an integer and $n \geq 0$

(i) Show that $I_n = \dfrac{5n}{5n+2} I_{n-1}$, for $n \geq 1$. **4**

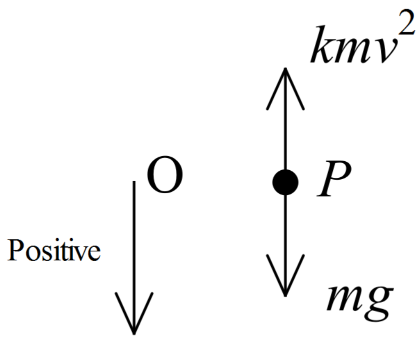
(ii) Show that $I_n = \dfrac{5^n \times n!}{2 \times 7 \times 12 \times \ldots \times (5n+2)}$, for $n \geq 1$. **2**

(iii) Hence evaluate I_4 **1**

End of Question 15

Question 16 (15 marks) Use a *separate* writing booklet

(a) In the case of turbulent flow, the resistive frictional force on an object travelling through a fluid is often modelled as being proportional to the square of the travelling velocity. In the case where a particle is falling vertically through the atmosphere, gravity is providing downward acceleration and the resistive frictional force is effectively acting upward, since friction always opposes motion. The situation is represented diagrammatically below.



Applying Newton's second law to the forces acting on the particle gives

$$ma = mg - \mu v^2$$

According to this representation, the equation of motion of the object is therefore given by

$$a = g - kv^2 \text{ , where } km = \mu \text{ .}$$

The co-efficient k depends on the surface profile and the mass of the object.

(i) Find the terminal velocity V of the particle in terms of g and k. **1**

(ii) Show that setting the initial height of the object as $x = 0$ and assuming that the initial velocity is also zero yields the following relationship between x and v. **3**

$$x = -\frac{1}{2k}\ln\left(1 - \frac{v^2}{V^2}\right)$$

(iii) Show that the relationship between v and t is **3**

$$t = \frac{1}{2\sqrt{gk}}\ln\left(\frac{V+v}{V-v}\right)$$

(iv) Taking $g = 9.8\,ms^{-2}$ and $k = 0.2\,ms^{-1}$, calculate how far the particle would fall before it reaches 80% of its terminal velocity and the time at which this occurs. **2**

Question 16 continues on next page

Question 16 continued

(b) $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\gamma, b\sin\gamma)$ lie on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

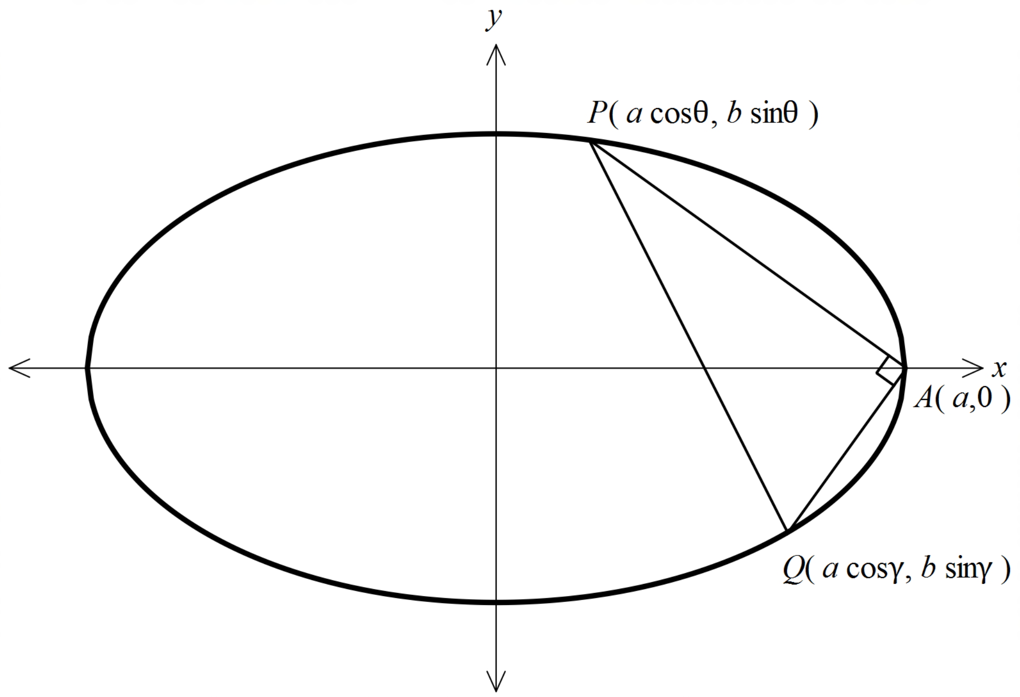
You are given that the chord PQ has equation

$$\frac{x}{a}\cos\left(\frac{\theta+\gamma}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta+\gamma}{2}\right)=\cos\left(\frac{\theta-\gamma}{2}\right)$$

DO NOT PROVE THIS

(i) If chord PQ subtends a right angle at $A(a,0)$ show that **1**

$$\frac{b\sin\theta}{a(\cos\theta-1)}\times\frac{b\sin\gamma}{a(\cos\gamma-1)}=-1$$



(ii) Hence, show that **2**

$$\tan\frac{\theta}{2}\times\tan\frac{\gamma}{2}=\frac{-b^2}{a^2}$$

(iii) Hence, show that the x-intercept of the chord PQ is a constant which can be expressed in terms of a and the eccentricity e. **3**

End of examination.

Student Number________________________

Teacher ________________________

Mathematics: Multiple Choice Answer Sheet

You may remove this page from the question booklet. Completely fill the response oval representing the most correct answer.

1. A ◯ B ◯ C ◯ D ◯

2. A ◯ B ◯ C ◯ D ◯

3. A ◯ B ◯ C ◯ D ◯

4. A ◯ B ◯ C ◯ D ◯

5. A ◯ B ◯ C ◯ D ◯

6. A ◯ B ◯ C ◯ D ◯

7. A ◯ B ◯ C ◯ D ◯

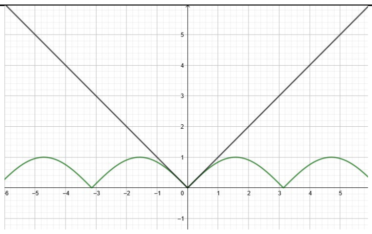
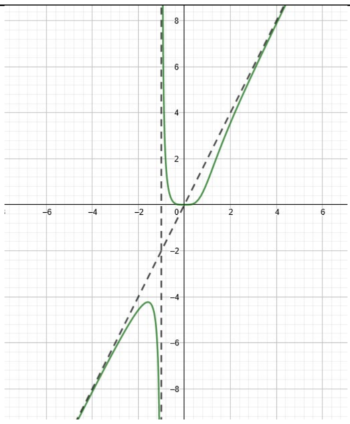
8. A ◯ B ◯ C ◯ D ◯

9. A ◯ B ◯ C ◯ D ◯

10. A ◯ B ◯ C ◯ D ◯

2019 Mathematics Extension 2 Trial Solutions

1	D	$4-i-(3-5i)$ $=1+4i$
2	D	$a^2=b^2(e^2-1)$ $3=4(e^2-1)$ $e^2-1=\dfrac{3}{4}$ $e^2=\dfrac{7}{4}$ $e=\dfrac{\sqrt{7}}{2}$
3	D	$2(x-3)+1(x+2)=3x-4$ and $(x+2)(x-3)=x^2-x-6$ The partial fractions of (A) and (B) have the wrong denominators. By cross multiplying the denominators of (C) and (D) with the other partial fractions numerators, (D) is the option that gives 3x-4.
4	A	$\int(1+x^3)^{-5}d(1+x^3)$ $=\dfrac{(1+x^3)^{-4}}{-4}+c$ $=\dfrac{-1}{4(1+x^3)^4}+c$
5	C	$I=\int_2^1 f(3-x)\,dx$ Let $u=3-x$ $du=-dx$ when $x=1,\ u=2$ when $x=2,\ u=1$ $\therefore I=-\int_1^2 f(u)\,du$ $=-\int_1^2 f(x)\,dx$ $=-5$

6	A	 Note that $y = x$ is a tangent to $y = \sin x$ at the origin.
7	C	$LHS = \omega^6 = \left(\omega^3\right)^2 = 1^2 = 1$ $RHS = \omega^2 + \omega = \omega^2 + \omega + 1 - 1 = 0 - 1 = -1$ $\therefore LHS \neq RHS$
8	B	 $\frac{2x^4}{x^3+1} = 2x - \frac{2x}{x^3+1} = 2x - \frac{2x}{(x+1)\left(x^2+x+1\right)}$ Note for $x^2 + x + 1, \Delta < 0$ So only vertical asymptote is $x = -1$ and oblique asymptote is $y = 2x$.
9	A	Let V_t = terminal velocity and v = velocity $V_t = \sqrt{\frac{mg}{k}}$ $V_t^2 = \frac{mg}{k}$ $kV_t^2 = mg$ $kV_t^2 - mg = 0 \quad or \quad 0 = mg - kV_t^2$ But V_t occurs when $a = 0$ $\therefore kv^2 - mg = a \quad or \quad a = mg - kv^2$ But down is positive, gravity acts downward and resistance always acts in opposing direction of movement. $\therefore a = mg - kv^2$

10	B	$\overrightarrow{DB}=\frac{1}{2}\overrightarrow{AB}$ $=\frac{1}{2}\overrightarrow{AC}cis\left(\frac{-\pi}{3}\right)$ $=\frac{1}{2}\left(\overrightarrow{OC}-\overrightarrow{OA}\right)cis\left(\frac{-\pi}{3}\right)$ $=\frac{1}{2}(z-w)\left[cos\left(\frac{-\pi}{3}\right)+i\sin\left(\frac{-\pi}{3}\right)\right]$ $=\frac{1}{2}(z-w)\left[cos\left(\frac{\pi}{3}\right)-i\sin\left(\frac{\pi}{3}\right)\right]$ $=\frac{1}{2}(z-w)\left[\frac{1}{2}-\frac{i\sqrt{3}}{2}\right]$ $=(z-w)\left[\frac{1}{4}-\frac{i\sqrt{3}}{4}\right]$

Question11

Marker's Comments

(a) i)

$$\left|1-\sqrt{3}i\right| = \sqrt{1^2+\left(\sqrt{3}\right)^2} = 2$$

$$Arg\left(1-\sqrt{3}i\right) = \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right) = -\frac{\pi}{3}$$

$$2\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)$$

Well done. Take care to calculate the correct argument.

❷ Correct solution

❶ either modulus or argument correct.

ii)

$$\left[2cis\left(\frac{-\pi}{3}\right)\right]^{-4}$$

$$=\frac{1}{16}cis\left(\frac{4\pi}{3}\right)$$

$$=\frac{1}{16}\left(\frac{-1}{2}-\frac{\sqrt{3}}{2}i\right)$$

$$=-\left(\frac{1}{32}+\frac{\sqrt{3}}{32}i\right)$$

Well done. Take care to answer the question i.e. write in $a+ib$ form.

❷ correct solution

❶ correct application of DeMoivres theorem

(b) i) Well done.

$$\left(1-2i\right)^2 = 1^2 - 2\times1\times2i + \left(2i\right)^2$$
$$= -3-4i$$

❶ correct proof

ii) Well done.

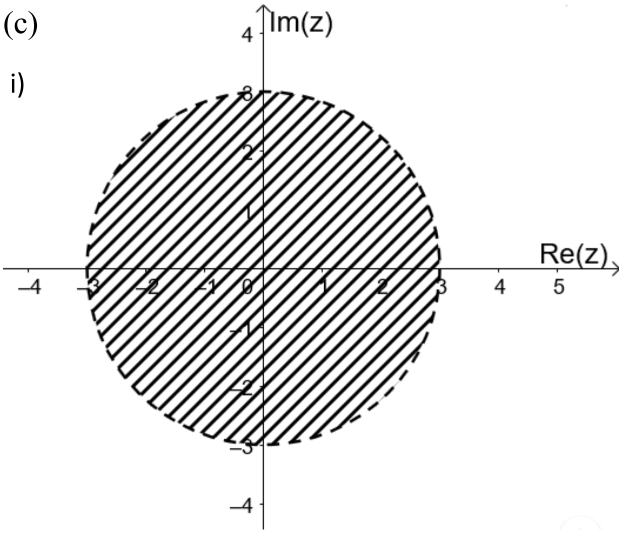
$$z^2 - 5z + \left(7+i\right) = 0$$
$$z = \frac{5 \pm \sqrt{5^2 - 4\times\left(7+i\right)}}{2}$$
$$z = \frac{5 \pm \sqrt{25-28-4i}}{2}$$
$$z = \frac{5 \pm \sqrt{-3-4i}}{2}$$
$$z = \frac{5 \pm \left(1-2i\right)}{2}$$
$$z = \frac{6-2i}{2} \ or \ \frac{4+2i}{2}$$
$$z = 3-i \ or \ 2+i$$

❷ for correct solution.

❶ for correctly calculating discriminant.

(c)

i)

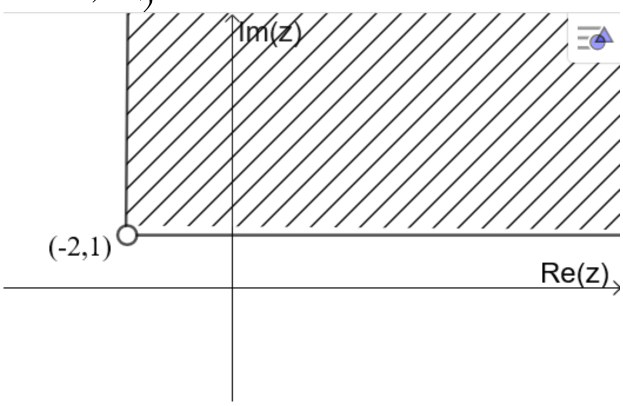


❶ Correct locus.

Markers comment

Well done

ii) $0 \le \arg(z+2-i) \le \dfrac{\pi}{2}$



❷ for correct locus

❶ for omitting the circle at (-2,1) or for partially correct locus

Markers comment

A few students forgot to circle point at (-2,1).

(d)

i)

$$
\begin{aligned}
z^n &= (\cos\theta + i\sin\theta)^n \\
&= \cos n\theta + i\sin n\theta \\
z^{-n} &= (\cos\theta + i\sin\theta)^{-n} \\
&= \cos n\theta - i\sin n\theta \\
&\therefore z^n + z^{-n} = 2\cos n\theta
\end{aligned}
$$

❶ Correct proof.

Markers comment

Well done

ii)

$$
\begin{aligned}
&\left(z+z^{-1}\right)^{3}\\
&=z^{3}+3z^{2}z^{-1}+3zz^{-2}+z^{-3}\\
&=z^{3}+3z+3z^{-1}+z^{-3}\\
&=z^{3}+z^{-3}+3\left(z+z^{-1}\right)\\
&=2\cos 3\theta+3\left(2\cos\theta\right)\\
&=2\cos 3\theta+6\cos\theta\\
&\textit{Also}\\
&\left(z+z^{-1}\right)^{3}\\
&=\left(2\cos\theta\right)^{3}\\
&=8\cos^{3}\theta
\end{aligned}
$$

$$8\cos^{3}\theta=2\cos 3\theta+6\cos\theta$$

$$\cos^{3}\theta=\frac{1}{4}\cos 3\theta+\frac{3}{4}\cos\theta$$

❷ for correct proof.

❶ for correctly expanding $\left(z+z^{-1}\right)^{3}$.

Markers comment

A few students omitted

$$
\begin{aligned}
&\left(z+z^{-1}\right)^{3}\\
&=\left(2\cos\theta\right)^{3}\\
&=8\cos^{3}\theta
\end{aligned}
$$

(d) iii)

$$\int_{0}^{\frac{\pi}{6}}\cos^{3}\theta\,d\theta$$

$$=\frac{1}{4}\int_{0}^{\frac{\pi}{6}}\cos 3\theta+3\cos\theta\,d\theta$$

$$=\frac{1}{4}\left[\frac{\sin 3\theta}{3}+3\sin\theta\right]_{0}^{\frac{\pi}{6}}$$

$$=\frac{1}{4}\left(\frac{1}{3}+\frac{3}{2}-\left(0+0\right)\right)$$

$$=\frac{11}{36}$$

❷ for correct solution.

❶ for correct primitive.

Markers comment

Well done. A few carried errors from ii)

Question12

(a) i) As the co-efficients are real, then $3+2i$ is also a root.

$$\left[x-(3-2i)\right]\left[x-(3+2i)\right](ax+b) \equiv 2x^3-13x^2+32x-13$$
$$(x^2-6x+13)(ax+b) \equiv$$

By equating co-efficients of x^3 and the constant terms

$a=2$ and $b=-1$. So the other zeroes are $3+2i$ and $\frac{1}{2}$

❷ for correct solution.

❶ for identifying 3+2i as a root.

Markers comment

Generally well done. Students used a range of methods including sum of roots.

(b) i)

$$P(2) = 2^3+3\times 2-14$$
$$= 0$$

By the factor theorem, 2 is a root.

❶ for correct solution

Markers comment

Well done.

ii) $P(x)$ has roots $x=2, \alpha$ and β. Let

$$X = 5, \alpha+3 \text{ and } \beta+3$$
$$\therefore X-3 = 2, \alpha \text{ and } \beta = x, \text{ must be a root of } P(x).$$

$$(X-3)^3+3(X-3)-14$$
$$= X^3-9X^2+27X-27+3X-9-14$$
$$= X^3-9X^2+30X-50$$

❷ for correct solution.

❶ for substituting *X*-3 into $P(x)$.

Markers comment

Generally well done. Students who lost marks usually made arithmetic errors.

(c) i)

Let $P(x) = (x - \alpha)^r Q(x)$,

Then by the product rule

$$P'(x) = Q(x)\, r(x - \alpha)^{r-1} + (x - \alpha)^r Q'(x)$$

$$P'(x) = (x - \alpha)^{r-1}\{rQ(x) + (x - \alpha)\, Q'(x)\}$$

$$P'(x) = (x - \alpha)^{r-1}F(x), \text{ where } F(x) = rQ(x) + (x - \alpha)\, Q'(x)$$

❷ for correct proof.

❶ for $P'(x) = Q(x)\, r(x - c)^{r-1} + (x - c)^r Q'(x)$

Markers comment

Overall not well done. Many students did not define P(x) appropriately and therefore could not complete the proof.

(c) ii)

$$\begin{aligned}
P(x) &= 18x^3 + 3x^2 - 28x + 12 \\
P'(x) &= 54x^2 + 6x - 28 \\
&= 2(27x^2 + 3x - 14) \\
&= 2(3x - 2)(9x + 7)
\end{aligned}$$

$$\begin{aligned}
P\left(\frac{2}{3}\right) &= 18\left(\frac{2}{3}\right)^3 + 3\left(\frac{2}{3}\right)^2 - 28\left(\frac{2}{3}\right) + 12 \\
&= 0
\end{aligned}$$

$$\therefore P\left(\frac{2}{3}\right) = P'\left(\frac{2}{3}\right) = 0$$

and $\frac{2}{3}$ is the double root.

Let the third root be α . Using product of roots,

$$\frac{2}{3} \times \frac{2}{3} \times \alpha = \frac{-12}{18}$$

$$\alpha = -\frac{3}{2}$$

So roots are $\frac{2}{3}, \frac{2}{3}$ and $\frac{-3}{2}$.

❸ for correct solution.

❷ for identifying $\frac{2}{3}$ as the double root

❶ for factorised form of the derivative

Markers comment

Mostly well done. Some students were lazy and just wrote down $P\left(\frac{2}{3}\right) = 0$ without substitution but were not penalised.

(d) i)

$z^6 = -1$

$\because i^6 = \left(i^2\right)^3 = \left(-1\right)^3 = -1$, then i is a root.

All the roots are evenly spaced around a unit circle by $\frac{2\pi}{6} = \frac{\pi}{3}$, so the roots must be

$$z_1 = i = cis\frac{\pi}{2}$$
$$z_2 = cis\left(\frac{\pi}{2} + \frac{\pi}{3}\right) = cis\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} + \frac{i}{2}$$
$$z_3 = cis\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = cis\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{i}{2}$$

and because the co-efficients of $z^6 + 1 = 0$ are real, the conjugates

of z_1, z_2 and z_3 must also be roots.

$$z_4 = \overline{z_1} = -i$$
$$z_5 = \overline{z_2} = -\frac{\sqrt{3}}{2} - \frac{i}{2}$$
$$z_6 = \overline{z_3} = \frac{\sqrt{3}}{2} - \frac{i}{2}$$

❸ for correct roots.

❷ for substantial progress.

❶ for stating roots are evenly distributed around unit circle by $\frac{\pi}{3}$

Markers comment

Many students rushed through this question and used $z = \pm 1$ as a solutinos and spread the other roots around evenly, attaining $\pm\left(\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right)$ instead

(d) ii)

Markers comment

Some students ignored the "hence" instruction, which is usually the easier method!

$z^6+1=0$

$(z^2)^3+1^3=0$

$(z^2+1)(z^4-z^2+1)=0$

$(z+i)(z-i)(z^4-z^2+1)=0$

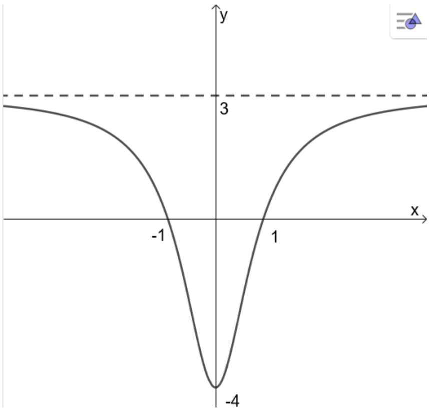
As z^4-z^2+1 is a factor of z^6+1, the roots of z^4-z^2+1 must be amongst those of $z^6+1=0$ and because $\pm i$ are the roots of z^2+1 then the required roots are

$$\pm\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right) \text{ and } \pm\left(-\frac{\sqrt{3}}{2}+\frac{i}{2}\right)$$

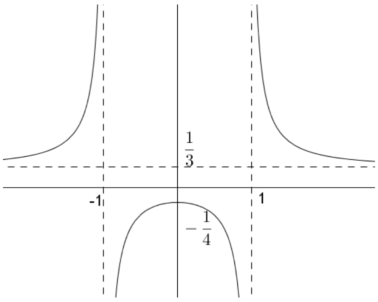
❷ for correct roots.

❶ for correct factorisation i.e. $(z+i)(z-i)(z^4-z^2+1)=0$

Question 13



(a) i)



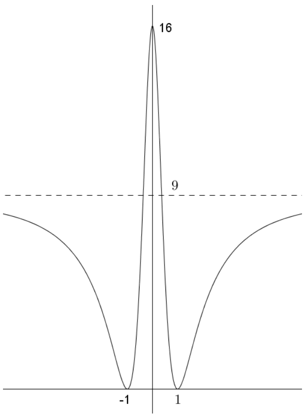
Markers comment

Well done

❷ for correct graph.

❶ for partially correct graph

ii)



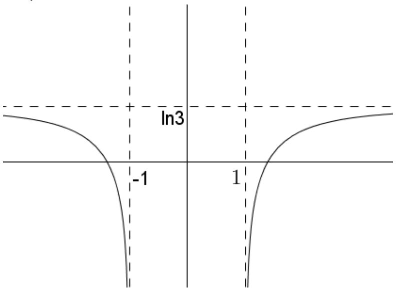
Markers comment

Well done

❷ for correct graph.

❶ for partially correct graph

iii)



❷ for correct graph.

❶ for partially correct graph

Markers comment

Some students sketched the vertical asymptotes incorrectly

(b)

$$x^2 - 3xy + 2y^2 = 3$$

$$2x - \left(y \times 3 + 3x\frac{dy}{dx} \right) + 4y\frac{dy}{dx} = 0$$

$$2x - 3y + \frac{dy}{dx}(4y - 3x) = 0$$

$$\frac{dy}{dx} = \frac{3y - 2x}{4y - 3x}$$

At $(5,2)$,

$$\frac{dy}{dx} = \frac{6-10}{8-15} = \frac{4}{7}$$

The gradient of the curve at (5,2) is $\frac{4}{7}$

❷ for correct gradient.

❶ for correct implicit differentiation.

Markers comment

Some students made errors with the product rule shown in parentheses.

(c)

$$\begin{aligned} pqr &= 100p + 10q + r \\ &= 99p + 9q + p + q + r \\ &= 3(33p + 3q) + 3A \\ &= 3(33p + 3q + A) \end{aligned}$$

Which is divisible by 3

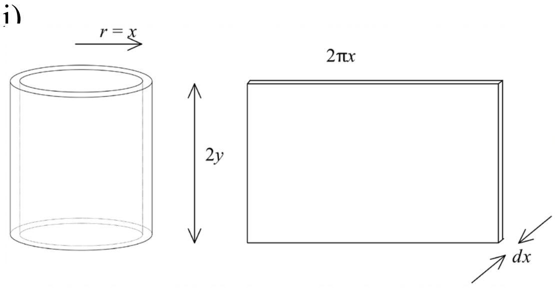
❷ for correct proof.

❶ for $pqr = 99p + 9q + p + q + r$.

Markers comment

Many students either could not prove this or wrote proofs that were way too complicated.

(d) i)



❶ for correct diagram

Markers comment

Well done

(d) ii)

$$dV = 2\pi x \times 2y\,dx$$
$$= 4\pi x\sqrt{r^2 - x^2}\,dx$$

Markers comment

Well done.

$$V = \lim_{dx\to 0}\sum_{x=0}^{r} 4\pi x\sqrt{r^2 - x^2}\,dx$$

$$= 4\pi\int_0^r x\sqrt{r^2 - x^2}\,dx$$

$$= \frac{4\pi}{-2}\int_0^r \sqrt{r^2 - x^2} \times -2x\,dx$$

$$= -2\pi\int_0^r \left(r^2 - x^2\right)^{\frac{1}{2}} \times d\left(r^2 - x^2\right)$$

$$= -2\pi\left[\frac{\left(r^2 - x^2\right)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^r$$

$$= \frac{-4\pi}{3}\left[\left(r^2 - x^2\right)^{\frac{3}{2}}\right]_0^r$$

$$= \frac{-4\pi}{3}\left(0 - \left(r^2\right)^{\frac{3}{2}}\right)$$

$$= \frac{4\pi r^3}{3}$$

❹ for correct proof . ❸ for $\frac{-4\pi}{3}\left[\left(r^2 - x^2\right)^{\frac{3}{2}}\right]_0^r$

❷ for $4\pi\int_0^r x\sqrt{r^2 - x^2}\,dx$ ❶ for $dV = 4\pi x\sqrt{r^2 - x^2}\,dx$

Question 14

(a)

$1)\ I = \int x^2 e^x dx$ $\quad$ let $u = x^2$ and $dv = e^x$

$\therefore du = 2x \quad v = e^x$

$= x^2e^x - \int 2xe^x dx$ $\quad$ let $u = 2x$ and $dv = e^x$

$\therefore du = 2 \quad v = e^x$

$$= x^2e^x - \left(2xe^x - \int 2e^x dx\right)$$

$$= x^2e^x - 2xe^x + 2e^x$$

$$= e^x\left(x^2 - 2x + 2\right) + c$$

Markers comment

Generally well done. Some students accidentally dropped their 2 after the second application of IBP

❷ for correct solution. ❶ for one correct application of IBP.

(b) i)

$$\int_0^{\frac{\pi}{2}} \frac{dx}{1+\cos x+\sin x}$$

Markers comment

Mostly well done

$Let\ t = \tan\frac{x}{2}$

$\tan^{-1} t = \frac{x}{2}$ $\quad$ When $x = \frac{\pi}{2}, t = 1$

$2\tan^{-1} t = x$ $\quad$ When $x = 0, t = 0$

$\frac{2dt}{1+t^2} = dx$

$$I = \int_0^1 \frac{1}{1+\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}} \times \frac{2}{1+t^2} dt$$

$$= \int_0^1 \frac{1}{\frac{1+t^2+1-t^2+2t}{1+t^2}} \times \frac{2}{1+t^2} dt$$

$$= \int_0^1 \frac{2}{2+2t} dt$$

$$= \int_0^1 \frac{1}{1+t} dt$$

$$= \left[\ln|1+t|\right]_0^1$$

$$= \ln 2$$

❸ for correct solution ❷ for $\int_0^1 \frac{1}{1+t} dt$ ❶ for correct change of variable.

(b) ii)

$$I=\int_0^{\frac{\pi}{2}}\frac{x\,dx}{1+\cos x+\sin x}$$

$\text{Let } u=\frac{\pi}{2}-x$

$du=-dx$

$\text{When } x=\frac{\pi}{2}, u=0$

$\text{When } x=0, u=\frac{\pi}{2}$

$$I=-\int_{\frac{\pi}{2}}^{0}\frac{\left(\frac{\pi}{2}-u\right)du}{1+\cos\left(\frac{\pi}{2}-u\right)+\sin\left(\frac{\pi}{2}-u\right)}$$

$$=\int_0^{\frac{\pi}{2}}\frac{\frac{\pi}{2}}{1+\sin u+\cos u}-\frac{u}{1+\sin u+\cos u}du$$

$$I=\int_0^{\frac{\pi}{2}}\frac{\frac{\pi}{2}}{1+\sin u+\cos u}du-\int_0^{\frac{\pi}{2}}\frac{u}{1+\sin u+\cos u}du$$

$$=\frac{\pi}{2}\ln 2-I$$

$$2I=\frac{\pi}{2}\ln 2$$

$$I=\frac{\pi}{4}\ln 2$$

❸ for correct solution

❷ for $I=\int_0^{\frac{\pi}{2}}\frac{\frac{\pi}{2}}{1+\sin u+\cos u}du-\int_0^{\frac{\pi}{2}}\frac{u}{1+\sin u+\cos u}du$

❶ for correct change of variable.

Markers comment

Overall not well done. Students who understood the correct approach generally achieved full marks. Those who did not notice the connection to part (i) were unable to make solid progress

(c)

$$\int\sqrt{\frac{1-x}{1+x}}dx$$

$$=\int\frac{\sqrt{1-x}}{\sqrt{1+x}}\times\frac{\sqrt{1-x}}{\sqrt{1-x}}\,dx$$

$$=\int\frac{1-x}{\sqrt{1-x^2}}\,dx$$

$$=\int\frac{1}{\sqrt{1-x^2}}\,dx+\int\frac{-x}{\sqrt{1-x^2}}\,dx$$

$$=\sin^{-1}x+\frac{1}{2}\int\frac{-2x\,dx}{\sqrt{1-x^2}}$$

$$=\sin^{-1}x+\frac{1}{2}\int\left(1-x^2\right)^{-\frac{1}{2}}d\left(1-x^2\right)$$

$$=\sin^{-1}x+\frac{1}{2}\frac{\left(1-x^2\right)^{\frac{1}{2}}}{\frac{1}{2}}+c$$

$$=\sin^{-1}x+\sqrt{1-x^2}+c$$

❸ for correct solution

❷ for $\sin^{-1}x+\dfrac{1}{2}\displaystyle\int\frac{-2x\,dx}{\sqrt{1-x^2}}$

❶ for correct change of variable.

Markers comment

Students who did not rationalise the numerator struggled to make further progress

(d) i)

Markers comment

Overall well done

$x = a\cos\theta \qquad y = b\sin\theta$

$\frac{dx}{d\theta} = -a\sin\theta \qquad \frac{dy}{d\theta} = b\cos\theta$

By the chain rule

$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$

$= \frac{b\cos\theta}{-a\sin\theta}$

At $\theta = \phi$, the gradient of normal will be $m_N = \frac{a\sin\phi}{b\cos\phi}$

and the equation of the normal will be

$y - b\sin\phi = \frac{a\sin\phi}{b\cos\phi}(x - a\cos\phi)$

$b\cos\phi y - b^2\sin\phi\cos\phi = a\sin\phi x - a^2\sin\phi\cos\phi$

$a\sin\phi x - b\cos\phi y = (a^2 - b^2)\sin\phi\cos\phi$

$\frac{ax}{\cos\phi} - \frac{by}{\sin\phi} = a^2 - b^2$

❷ for correct proof.

❶ for correctly calculating gradient of normal.

(d) ii)

To find co-ordinates of M let $y = 0$.

$$\frac{ax}{\cos\phi} - \frac{b \times 0}{\sin\phi} = a^2 - b^2$$
$$x = \frac{\cos\phi\left(a^2 - b^2\right)}{a}$$
$$\therefore M\left(\frac{\cos\phi\left(a^2 - b^2\right)}{a}, 0\right)$$

To find co-ordinates of N let $x = 0$.

$$\frac{a \times 0}{\cos\phi} - \frac{b \times y}{\sin\phi} = a^2 - b^2$$
$$y = \frac{-\sin\phi\left(a^2 - b^2\right)}{b}$$
$$\therefore N\left(0, \frac{-\sin\phi\left(a^2 - b^2\right)}{b}\right)$$

Using the square of the distance formula

$$\begin{aligned}
PM^2 &= \left(a\cos\phi - \frac{\cos\phi\left(a^2 - b^2\right)}{a}\right)^2 + \left(b\sin\phi - 0\right)^2 \\
&= \left(\frac{a^2\cos\phi - \cos\phi\left(a^2 - b^2\right)}{a}\right)^2 + b^2\sin^2\phi \\
&= \left(\frac{b^2\cos\phi}{a}\right)^2 + b^2\sin^2\phi \\
&= \frac{b^4\cos^2\phi}{a^2} + \frac{a^2b^2\sin^2\phi}{a^2} \\
&= \frac{b^2}{a^2}\left(b^2\cos^2\phi + a^2\sin^2\phi\right)
\end{aligned}$$

Markers comment

Most students had no issues finding the coordinates of M and N. There were some algebraic errors in simplifying $\frac{PM}{PN}$

$$PN^2 = \left(a\cos\phi - 0\right)^2 + \left(b\sin\phi - \frac{\sin\phi\left(b^2 - a^2\right)}{b}\right)^2$$

$$= a^2\cos^2\phi + \left(\frac{b^2\sin\phi - \sin\phi\left(b^2 - a^2\right)}{b}\right)^2$$

$$= a^2\cos^2\phi + \left(\frac{a^2\sin\phi}{b}\right)^2$$

$$= \frac{a^2b^2\cos^2\phi}{b^2} + \frac{a^4\sin^2\phi}{b^2}$$

$$= \frac{a^2}{b^2}\left(b^2\cos^2\phi + a^2\sin^2\phi\right)$$

$$\frac{PM^2}{PN^2} = \frac{\frac{b^2}{a^2}\left(b^2\cos^2\phi + a^2\sin^2\phi\right)}{\frac{a^2}{b^2}\left(b^2\cos^2\phi + a^2\sin^2\phi\right)}$$

$$= \frac{b^4}{a^4}$$

$$\frac{PM}{PN} = \frac{b^2}{a^2}$$

❷ for correct proof

❶ for finding co-ordinates of M and N.

Question 15

(a) i)

$x = cp \qquad y = \dfrac{c}{p}$

$\dfrac{dx}{dp} = c \qquad \dfrac{dy}{dp} = \dfrac{-c}{p^2}$

Markers comment

Well done

$\dfrac{dy}{dx} = \dfrac{dy}{dp} \times \dfrac{dp}{dx} = \dfrac{-1}{p^2}$

So gradient of tangent at P is $\dfrac{-1}{p^2}$

tangent's equation is

$y - \dfrac{c}{p} = \dfrac{-1}{p^2}(x - cp)$

$p^2 y - cp = -x + cp$

$x + p^2 y = 2cp \ldots ①$

❷ for correct proof.

❶ for correctly calculating gradient of tangent.

(a) ii)

Similarly, tangent at Q is

$x + q^2 y = 2cq \ldots ②$

①–②

$\left(p^2 - q^2\right) y = 2c\left(p - q\right)$

$y = \dfrac{2c}{p+q} \ldots ③$

sub ③ in ① $x + p^2 \dfrac{2c}{p+q} = 2cp$

$x = 2cp - \dfrac{2cp^2}{p+q}$

$x = \dfrac{2cp\left(p+q\right) - 2cp^2}{p+q}$

$x = \dfrac{2cpq}{p+q}$

$T\left(\dfrac{2cpq}{p+q}, \dfrac{2c}{p+q}\right)$

❷ for correct proof. ❶ for either co-ordinate of *T*.

Markers comment

Well done. A few students did not simplify the x co-ordinate.

(a) iii)

Markers comment

Well done.

If T lies on second hyperbola, then its co-ordinates satisfy $xy = k^2$

$$\frac{2cpq}{p+q} \times \frac{2c}{p+q} = k^2$$

$$\frac{4c^2 pq}{(p+q)^2} = k^2$$

$$\frac{pq}{(p+q)^2} = \frac{k^2}{4c^2}$$

❶ for correct proof.

(b)

$$dV = 2y \times 2y\,dx$$
$$= 4y^2\,dx$$
$$= 4\left(25 - x^2\right)dx$$
$$V = 4\int_{-5}^{5} 25 - x^2\,dx$$

but $y = 25 - x^2$ is an even function,

$$V = 8\int_{0}^{5} 25 - x^2\,dx$$
$$= 8\left[25x - \frac{x^3}{3}\right]_0^5$$
$$= 8\left(125 - \frac{125}{3}\right)$$
$$= \frac{2000}{3}\,units^3$$

❸ for correct solution

❷ for $V = 8\left[25x - \frac{x^3}{3}\right]_0^5$

❶ for $V = 4\int_{-5}^{5} 25 - x^2\,dx$

Markers comment

Well done. A few students made arithmetic errors when simplifying in the last step.

(c)

Markers comment

Majority of students wrote a complete proof. Those that lost marks usually struggled getting from steps marked # to *

Let $u=\left(1-x^5\right)^n$ $\qquad dv=x$

$du=n\left(1-x^5\right)^{n-1}\times-5x^4dx$ $\qquad v=\dfrac{x^2}{2}$

$$I_n=\left[\frac{x^2}{2}\left(1-x^5\right)^n\right]_0^1-\int_0^1\frac{x^2}{2}\times n\left(1-x^5\right)^{n-1}\times-5x^4dx$$

$$I_n=0+\frac{5n}{2}\int_0^1 x^5\times x\left(1-x^5\right)^{n-1}dx$$

$$I_n=-\frac{5n}{2}\int_0^1 -x^5\times x\left(1-x^5\right)^{n-1}dx \ \#$$

$$I_n=-\frac{5n}{2}\int_0^1\left(1-x^5\right)\times x\left(1-x^5\right)^{n-1}-x\left(1-x^5\right)^{n-1}dx$$

$$I_n=-\frac{5n}{2}\int_0^1 x\left(1-x^5\right)^n dx+\frac{5n}{2}\int_0^1 x\left(1-x^5\right)^{n-1}dx$$

$$I_n=-\frac{5n}{2}I_n+\frac{5n}{2}I_{n-1} \ *$$

$$2I_n=-5nI_n+5nI_{n-1}$$

$$\left(5n+2\right)I_n=5nI_{n-1}$$

$$I_n=\frac{5n}{5n+2}I_{n-1}$$

❹ for correct proof .

❸ for $I_n=-\dfrac{5n}{2}\displaystyle\int_0^1 x\left(1-x^5\right)^n dx+\dfrac{5n}{2}\int_0^1 x\left(1-x^5\right)^{n-1}dx$

❷ for $I_n=0+\dfrac{5n}{2}\displaystyle\int_0^1 x^5\times x\left(1-x^5\right)^{n-1}dx$

❶ for $I_n=\left[\dfrac{x^2}{2}\left(1-x^5\right)^n\right]_0^1-\displaystyle\int_0^1\dfrac{x^2}{2}\times n\left(1-x^5\right)^{n-1}\times-5x^4dx$

(c) ii)

$$I_1 = \frac{5}{7} I_0$$
$$= \frac{5}{7}\int_0^1 x\,dx$$
$$= \frac{5}{7}\left[\frac{x^2}{2}\right]_0^1$$
$$= \frac{5}{7}\times\frac{1}{2}$$
$$I_2 = \frac{10}{12} I_1 = \frac{10}{12}\times\frac{5}{7}\times\frac{1}{2}$$
$$I_3 = \frac{15}{17}\times I_2 = \frac{15}{17}\times\frac{10}{12}\times\frac{5}{7}\times\frac{1}{2}$$

...

$$I_n = \frac{5n}{5n+2}\times I_{n-1}\times\frac{5(n-1)}{5(n-1)+2}\times I_{n-2}\times\frac{5(n-2)}{5(n-2)+2}\times I_{n-3}\times\ldots\times\frac{5\times 3}{17}\times\frac{5\times 2}{12}\times\frac{5\times 1}{7}\times\frac{1}{2}$$
$$I_n = \frac{5^n n!}{2\times 7\times 12\times\ldots\times(5n+2)}$$

❷ for correct proof

❶ for $I_1 = \frac{5}{7}\times\frac{1}{2}$

Markers comment

Some students wrote incomplete proofs in that they failed to establish the last few terms and why

$$I_0 = \frac{1}{2}$$

(c) iii)

$$I_4 = \frac{5^4\times 4!}{2\times 7\times 12\times 17\times 22} = \frac{625}{2618}$$

❶ for correct answer

Markers comment

A lot of arithmetic errors.

Question 16

(a) i)

Terminal velocity is reached when $a=0$ and v becomes V

Markers comment

Well done

$\therefore 0 = g - kV^2$

$kV^2 = g$

$V = \sqrt{\dfrac{g}{k}}$

❶ for correct solution.

(a) ii)

Markers comment

Generally well done

$$\frac{v\,dv}{dx} = g - kv^2$$

$$\frac{dv}{dx} = \frac{g - kv^2}{v}$$

$$\frac{dx}{dv} = \frac{v}{g - kv^2}$$

$$\int dx = \int \frac{v}{g - kv^2}\,dv$$

$$x = -\frac{1}{2k}\int \frac{-2kv}{g - kv^2}\,dv$$

$$x = -\frac{1}{2k}\ln\left(g - kv^2\right) + c$$

When x=0, v=0

$$\therefore\ c = \frac{1}{2k}\ln g$$

$$x = -\frac{1}{2k}\ln|g - kv^2| + \frac{1}{2k}\ln g$$

$$x = \frac{1}{2k}\ln\left|\frac{g}{g - kv^2}\right|$$

$$x = -\frac{1}{2k}\ln\left|\frac{g}{g-kv^2}\right|^{-1}$$

$$x = -\frac{1}{2k}\ln\left|\frac{g-kv^2}{g}\right|$$

$$x = -\frac{1}{2k}\ln\left|1-\frac{kv^2}{g}\right|$$

❸ for correct solution. ❷ for $x = -\frac{1}{2k}\ln\left(g-kv^2\right) + c$ ❶ for $\int dx = \int \frac{v}{g-kv^2}\,dv$

(a) iii) $a = g - kv^2$

Markers comment

Mostly well done.
Students were required to clearly show their working in partial fractions.

$$\frac{dv}{dt} = g - kv^2$$

$$\frac{dt}{dv} = \frac{1}{g-kv^2}$$

$$\int dt = \int \frac{1}{g-kv^2}\,dv$$

$$\int dt = \frac{1}{k}\int \frac{1}{\frac{g}{k}-v^2}\,dv$$

Using partial fractions

$$t = \frac{1}{k}\int \frac{A}{\sqrt{\frac{g}{k}}-v} + \frac{B}{\sqrt{\frac{g}{k}}+v}\,dv$$

$$1 = A\left(\sqrt{\frac{g}{k}}+v\right) + B\left(\sqrt{\frac{g}{k}}-v\right)$$

When $v = \sqrt{\frac{g}{k}}$ When $v = -\sqrt{\frac{g}{k}}$

$$1 = 2A\sqrt{\frac{g}{k}} \qquad 1 = 2B\sqrt{\frac{g}{k}}$$

$$A = \tfrac{1}{2}\sqrt{\frac{k}{g}} \qquad B = \tfrac{1}{2}\sqrt{\frac{k}{g}}$$

$$t=\frac{1}{k}\frac{1}{2}\sqrt{\frac{k}{g}}\int\frac{1}{\sqrt{\frac{g}{k}}-v}+\frac{1}{\sqrt{\frac{g}{k}}+v}dv$$

$$t=\frac{1}{2\sqrt{kg}}\left[-\ln\left(\sqrt{\frac{g}{k}}-v\right)+\ln\left(\sqrt{\frac{g}{k}}+v\right)\right]+c$$

When t=0, v=0

$$0=\frac{1}{2\sqrt{kg}}\left[-\ln\sqrt{\frac{g}{k}}+\ln\sqrt{\frac{g}{k}}\right]+c$$
$$0=0+c$$
$$c=0$$

$$\therefore t=\frac{1}{2\sqrt{kg}}\left[-\ln\left(\sqrt{\frac{g}{k}}-v\right)+\ln\left(\sqrt{\frac{g}{k}}+v\right)\right]$$
$$\therefore t=\frac{1}{2\sqrt{kg}}[-\ln(V-v)+\ln(V+v)]$$
$$t=\frac{1}{2\sqrt{kg}}\ln\left(\frac{V+v}{V-v}\right)$$

❸ for correct solution ❷ for $t=\frac{1}{2\sqrt{kg}}\left[-\ln\left(\sqrt{\frac{g}{k}}-v\right)+\ln\left(\sqrt{\frac{g}{k}}+v\right)\right]+c$

❶ for $\int dt=\frac{1}{k}\int\frac{1}{\frac{g}{k}-v^2}dv$

(a) iv)

Markers comment

Well done

$$x = \frac{-1}{2\times .2}\ln\left(1-\frac{(.8V)^2}{V^2}\right)$$

$$x = \frac{-10}{4}\ln(1-.64)$$

$$x = -2.5\ln(.36)\,metres$$

$$= 2.55m\,(2d.p.)$$

$$t = \frac{1}{2\sqrt{9.8\times .2}}\ln\left(\frac{V+.8V}{V-.8V}\right)$$

$$t = \frac{1}{2\sqrt{1.96}}\ln(9)\,seconds$$

$$t = .78\,seconds\,(2d.p.)$$

❷ for correct solution ❶ for correct value of either x or t.

(b) i)

Markers comment

Well done

$$m_{PA}\times m_{QA} = -1$$

$$\frac{b\sin\theta - 0}{a\cos\theta - a}\times\frac{b\sin\gamma - 0}{a\cos\gamma - a} = -1$$

$$\frac{b\sin\theta}{a(\cos\theta - 1)}\times\frac{b\sin\gamma}{a(\cos\gamma - 1)} = -1$$

❶ for correct proof

(b) ii)

$$\frac{b\sin\theta}{a(\cos\theta-1)}\times\frac{b\sin\gamma}{a(\cos\gamma-1)}=-1$$

$$\frac{-b^2}{a^2}=\frac{(\cos\theta-1)(\cos\gamma-1)}{\sin\theta\sin\gamma}$$

$$\frac{-b^2}{a^2}=\frac{\left(\cos\left[2\left(\frac{\theta}{2}\right)\right]-1\right)\left(\cos\left[2\left(\frac{\gamma}{2}\right)\right]-1\right)}{\sin\left[2\left(\frac{\theta}{2}\right)\right]\sin\left[2\left(\frac{\gamma}{2}\right)\right]}$$

$$\frac{-b^2}{a^2}=\frac{\left(\cos^2\left(\frac{\theta}{2}\right)-\sin^2\left(\frac{\theta}{2}\right)-1\right)\left(\cos^2\left(\frac{\gamma}{2}\right)-\sin^2\left(\frac{\gamma}{2}\right)-1\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\times 2\sin\left(\frac{\gamma}{2}\right)\cos\left(\frac{\gamma}{2}\right)}$$

$$\frac{-b^2}{a^2}=\frac{-2\sin^2\left(\frac{\theta}{2}\right)\times-2\sin^2\left(\frac{\gamma}{2}\right)}{4\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\gamma}{2}\right)\cos\left(\frac{\gamma}{2}\right)}$$

$$\frac{-b^2}{a^2}=\frac{\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\gamma}{2}\right)}{\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\gamma}{2}\right)}$$

$$\frac{-b^2}{a^2}=\tan\left(\frac{\theta}{2}\right)\tan\left(\frac{\gamma}{2}\right)$$

❷ for correct proof ❶ for using a double angle result.

Markers comment

Some students used t-results instead of the solution provided, which was also effective.

(b) iii)

When y=0, $\frac{x}{a}\cos\left(\frac{\theta+\gamma}{2}\right)=\cos\left(\frac{\theta-\gamma}{2}\right)$

$$x=\frac{a\cos\left(\frac{\theta}{2}-\frac{\gamma}{2}\right)}{\cos\left(\frac{\theta}{2}+\frac{\gamma}{2}\right)}$$

$$x=\frac{a\left(\cos\frac{\theta}{2}\cos\frac{\gamma}{2}+\sin\frac{\theta}{2}\sin\frac{\gamma}{2}\right)}{\cos\frac{\theta}{2}\cos\frac{\gamma}{2}-\sin\frac{\theta}{2}\sin\frac{\gamma}{2}}$$

$\div$ every term by $\cos\frac{\theta}{2}\cos\frac{\gamma}{2}$

$$x=\frac{a\left(1+\tan\frac{\theta}{2}\tan\frac{\gamma}{2}\right)}{1-\tan\frac{\theta}{2}\tan\frac{\gamma}{2}}$$

$$x=\frac{a\left(1-\frac{b^2}{a^2}\right)}{1+\frac{b^2}{a^2}}$$

$$x=\frac{a\left(\frac{a^2-b^2}{a^2}\right)}{\frac{a^2+b^2}{a^2}}$$

$$x=a\left(\frac{a^2-b^2}{a^2+b^2}\right)$$

using $b^2=a^2\left(1-e^2\right)$

$b^2=a^2-a^2e^2$

$a^2e^2=a^2-b^2$

$$\therefore x=a\left(\frac{a^2e^2}{a^2+a^2-a^2e^2}\right)$$

$$x=\frac{a^3e^2}{2a^2-a^2e^2}$$

$$x=\frac{ae^2}{2-e^2}$$

❸ for correct solution ❷ for $x=a\left(\frac{a^2-b^2}{a^2+b^2}\right)$ ❶ for $x=\frac{a\left(\cos\frac{\theta}{2}\cos\frac{\gamma}{2}+\sin\frac{\theta}{2}\sin\frac{\gamma}{2}\right)}{\cos\frac{\theta}{2}\cos\frac{\gamma}{2}-\sin\frac{\theta}{2}\sin\frac{\gamma}{2}}$

Markers comment

Some students were confused with all the algebra and did not draw the connection to part (ii). A few students left their answer in terms of b as well.