FRENSHAM

YEAR 12 TRIAL HSC EXAMINATION

2011

MATHEMATICS

EXTENSION 2

Time Allowed 3 hours +5 minutes reading time

INSTRUCTIONS:

- All questions may be attempted
- All questions are of equal value
- Show all necessary working. Marks may be deducted for careless or badly arranged work
- Start each question on a new page
- Board of Studies approved calculators may be used

Student name / number ....

Marks

Question 1 **Begin a new booklet**

(a) Find $\int \frac{x^2+1}{\sqrt{x}}\, dx$. **2**

(b) Find $\int \frac{\cos^3 x}{\sin^2 x}\, dx$ using the substitution $u = \sin x$. **3**

(c) Evaluate $\int_0^{\frac{1}{2}\log_e 3} \frac{1}{e^x + e^{-x}}\, dx$ using the substitution $u = e^x$. **3**

(d) Evaluate in simplest exact form $\int_1^e x^3 \log_e x \; dx$. **3**

(e)(i) Using the substitution $t = \tan\frac{x}{2}$, show that **2**

$$\int_0^{\frac{\pi}{2}} \frac{1}{5+5\sin x - 3\cos x}\, dx = \int_0^1 \frac{1}{4t^2+5t+1}\, dt .$$

(ii) Hence evaluate in simplest exact form $\int_0^{\frac{\pi}{2}} \frac{1}{5+5\sin x - 3\cos x}\, dx$. **2**

Student name / number ....

Question 2 **Begin a new booklet** **Marks**

(a) If $z_1 = 2i$ and $z_2 = 1+3i$, express in the form $a+ib$ (where a and b are real)

(i) $z_1 + \overline{z}_2$. **1**

(ii) $z_1 z_2$. **1**

(iii) $\dfrac{1}{z_2}$. **1**

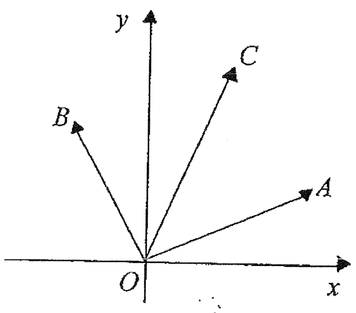
(b)(i) Express $z = 1 + i\sqrt{3}$ in modulus/argument form. **2**

(ii) Hence show that $z^{10} + 512z = 0$. **2**

(c)(i) On an Argand diagram sketch the locus of the point P representing z such that $\left| z - (\sqrt{3}+i) \right| = 1$. **2**

(ii) Find the set of possible values of $|z|$ and the set of possible principal values of $\arg z$. **2**

(d)



In the Argand diagram above, vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ represent the complex numbers z_1, z_2 and $z_1 + z_2$ respectively, where $z_1 = \cos\theta + i\sin\theta$ and $z_1 + z_2 = (1+i)\,z_1$.

(i) Express z_2 in terms of z_1 and show that $OACB$ is a square. **2**

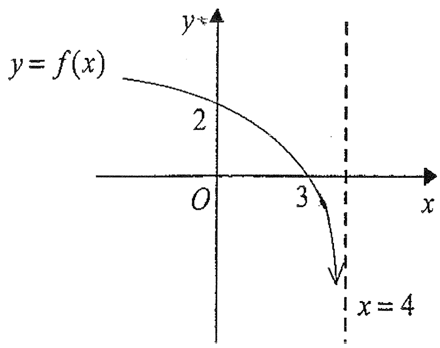
(ii) Show that $(z_1 + z_2)\overline{(z_1 - z_2)} = 2i$. **2**

Student name / number ....

Marks

Question 3 **Begin a new booklet**

(a) The diagram shows the graph of the curve $y = f(x)$. On separate diagrams, sketch the graphs of the curves listed below, showing clearly intercepts on the coordinate axes and the equations of any asymptotes:



(i) $y = \left| f(x) \right|$. **1**

(ii) $y = f\left(|x|\right)$. **1**

(iii) $y = f\left(x^2\right)$. **2**

(iv) $y = \dfrac{1}{f(x)}$. **2**

(b) $P(x)$ is an even polynomial. Show that when $P(x)$ is divided by $\left(x^2 - a^2\right)$, where $a \neq 0$, the remainder is independent of x. **3**

(c) The zeroes of $x^3 + px^2 + qx + r$ are $\alpha, \beta \; and \; \gamma$ (where p, q and r are real numbers).

(i) Find $\alpha\beta + \alpha\gamma + \beta\gamma$. **1**

(ii) Find $\alpha^2 + \beta^2 + \gamma^2$. **1**

(iii) Find a cubic polynomial with integer coefficients whose zeroes are $2\alpha, 2\beta \; and \; 2\gamma$. **2**

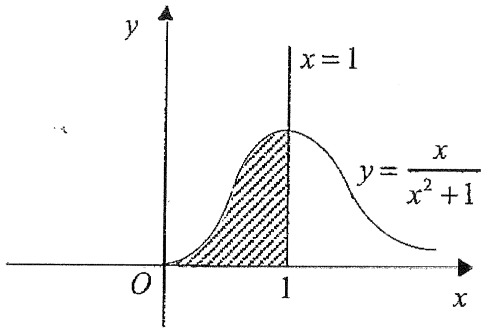
(d) If $p > 0$, and $q > 0$, and $p + q = 1$, show that $\frac{1}{p} + \frac{1}{q} \geq 4$. **2**

Student name / number ....

Marks

Question 4 **Begin a new booklet**

(a)

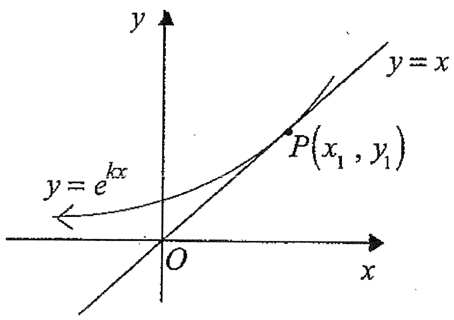


The region bounded by the curve $y = \frac{x}{x^2+1}$ and the x-axis between $x = 0$ and $x = 1$ is rotated through one complete revolution about the y-axis.

(i) Use the method of cylindrical shells to show that the volume V of the solid formed is given by $V = 2\pi \int_0^1 \frac{x^2}{x^2+1}\, dx$ **1**

(ii) Hence find the value of V in simplest exact form. **3**

(b)



The line $y = x$ is tangent to the curve $y = e^{kx}$ (where $k > 0$) at the point $P(x_1, y_1)$ on the curve. By considering the gradient of OP show that $k = \frac{1}{e}$. **3**

Question 4 continued

(c) The Hyperbola H has the equation $\frac{x^2}{25} - \frac{y^2}{9} = 1$.

(i) Find the eccentricity of H. **1**

(ii) Find the co-ordinates of the foci of H. **1**

(iii) Draw a neat one third of a page sketch of H. **2**

(iv) The line $x = 6$ cuts H at A and B. Find the coordinates of A and B if A is in the first quadrant. **2**

(v) Derive the equation of the tangent to H at A. **2**

Student name / number ....

Marks

Question 5 **Begin a new booklet**

(a) A lifebelt mould is made by rotating the circle $x^2 + y^2 = 64$
Through one complete revolution about the line $x = 28$, where
All the measurements are in centimetres.

(i) Use the method of slicing to show that the volume, $V\ cm^3$ of the lifebelt is given by **5**

$V = 112\pi \int_{-8}^{8} \sqrt{64 - y^2\, dy}.$

(ii) Find the exact volume of the lifebelt. **2**

(b) (i) Show that the tangent to the ellipse $\frac{x^2}{12} + \frac{y^2}{4} = 1$ at the point **8**

P(3, 1) has equation $x + y = 4$.

ii) If this tangent cuts the directrix in the fourth quadrant at the point T,

and S is the corresponding focus, show that SP and ST are perpendicular.

Student name / number ....

Marks

Question 6 **Begin a new booklet**

(a) i) Show that $\tan(A + \frac{\pi}{2}) = -\cot A$. **2**

ii) Use the method of Mathematical Induction, and the result in (i), to show that **4**

$$\tan\left\{(2n+1)\frac{\pi}{4}\right\} = (-1)^n \qquad \text{for all integers } n \geq 1.$$

(b) Given the equation $y^2 + xy + x^2 = 1$

i) Make y the subject. **2**

ii) Hence, or otherwise, find $\frac{dy}{dx}$ **2**

(c) Given that $z = \cos\theta + i\sin\theta$ and $z^n + z^{-n} = 2\cos n\theta$, show that **4**

$$\cos^4\theta = \tfrac{1}{8}\cos 4\theta + \tfrac{1}{2}\cos 2\theta + \tfrac{3}{8}$$

(d) Show that $\cos\left(x + \frac{\pi}{2}\right) = -\sin x$. **1**

Student name / number ....

Marks

Question 7 **Begin a new booklet**

(a)(i) Show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. **2**

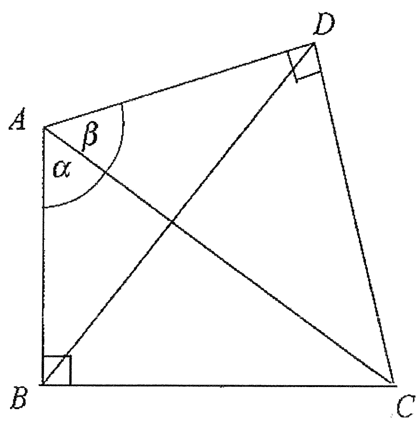
(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sqrt{1+(x-\frac{\pi}{4})^2}}\,dx$. **3**

(b) Let $I_n = \int_0^1 (1-x^r)^n\,dx$, where $r>0$, for $n = 0, 1, 2, \ldots$.

(i) Show that $I_n = \frac{nr}{nr+1} I_{n-1}$ for $n = 1, 2, 3, \ldots$. **3**

(ii) Hence evaluate $\int_0^1 (1-x^{\frac{3}{2}})^3\,dx$. **2**

(c)



$ABCD$ is a quadrilateral in which $\angle ABC = \angle ADC = \frac{\pi}{2}$, $\angle CAB = \alpha$, $\angle CAD = \beta$ and $AC = 1$.

(i) Show that $\angle BDC = \alpha$. **2**

(ii) Hence show that $BD = \sin(\alpha+\beta)$. **3**

Student name / number ....

Marks

Question 8 **Begin a new booklet**

a) i) Write the general solution to $\tan 4\theta = 1$ **1**

ii) Use De Moivre's Theorem and the binomial theorem to show that $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$ **3**

iii) Hence find the roots of $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$ in the form $x = \tan\theta$. **3**

iv) Hence prove that : **2**

$$\tan^2\frac{\pi}{16} + \tan^2\frac{3\pi}{16} + \tan^2\frac{5\pi}{16} + \tan^2\frac{7\pi}{16} = 28$$

(Hint: Let the roots be α, β, γ and δ).

(b) (i) Use a diagram to explain why **1**

$$\int_0^b \sin x\,dx = \lim_{n\to\infty}\left(\sin\frac{b}{n} + \sin\frac{2b}{n} + \ldots + \sin\frac{nb}{n}\right)\cdot\frac{b}{n}$$

for $b = \dfrac{\pi}{2}$.

(ii) Given that $2\sin\theta\sin\alpha = \cos(\theta - \alpha) - \cos(\theta + \alpha)$, show that **2**

$$\sum_{k=1}^{n}\sin\left(\frac{kb}{n}\right) = \frac{\cos\left(\frac{b}{2n}\right) - \cos\left(b + \frac{b}{2n}\right)}{2\sin\left(\frac{b}{2n}\right)}$$

(iii) Hence show that $\int_0^b \sin x\,dx = 1 - \cos b$. **3**

END OF EXAM

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

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QUESTION 1

(a)

$$\int \frac{x^2+1}{x^{\frac{1}{2}}}\,dx$$

$$= \int x^{3/2} + x^{-1/2}\,dx$$

$$= \frac{2x^{5/2}}{5} + 2x^{\frac{1}{2}} + c$$

$$= \frac{2\sqrt{x^5}}{5} + 2\sqrt{x} + c$$

(b)

$$\int \frac{\cos^2 x \,.\, \cos x}{\sin^2 x}\,dx$$

$$= \int \frac{(1-\sin^2 x)\cos x\,dx}{\sin^2 x}$$

Let $u = \sin x$
$du = \cos x\,dx$

$$= \int \frac{1-u^2}{u^2}\,du$$

$$= \int u^{-2} - 1\,du$$

$$= -u^{-1} - u + c$$

$$= \frac{-1}{\sin x} - \sin x + c \quad \text{OR} \quad -\operatorname{cosec} x - \sin x + c$$

(c)

$$\int_0^{\frac{1}{2}\ln 3} \frac{1}{e^x + e^{-x}}\,dx \times \frac{e^x}{e^x}$$

Let $u = e^x$
$du = e^x\,dx$

when $x = 0$, $u = 1$
$x = \ln\sqrt{3}$, $u = \sqrt{3}$

$$= \int_0^{\ln\sqrt{3}} \frac{e^x\,dx}{e^{2x}+1}$$

$$= \int_1^{\sqrt{3}} \frac{du}{u^2+1} = \Big[\tan^{-1} u\Big]_1^{\sqrt{3}} = \tan^{-1}\sqrt{3} - \tan^{-1} 1$$

$$= \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

(d) $\int_1^e x^3 \ln x \, dx$ Let $u = \ln x$ $v = x^4/4$

$u' = 1/x$ $v' = x^3$

$= \left.\frac{x^4}{4}\ln x\right]_1^e - \int_1^e \frac{x^4}{4}\cdot\frac{1}{x}\,dx$

$= \frac{e^4}{4} - \int_1^e \frac{x^3}{4}\,dx$

$= \frac{e^4}{4} - \left.\frac{x^4}{16}\right]_1^e$

$= \frac{e^4}{4} - \left(\frac{e^4}{16} - \frac{1}{16}\right)$

$= \frac{3e^4+1}{16}$

(e) i) let $\tan\frac{x}{2} = t$ $\sin x = \frac{2t}{1+t^2}$ $\cos x = \frac{1-t^2}{1+t^2}$

$\int_0^{\pi/2} \frac{dx}{5 + 5\sin x - 3\cos x}$ $x = 0,\ t = 0$ $x = \pi/2 \quad t = 1$ $dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx$

$dx = \frac{2}{1+t^2}\,dt$

$= \int_0^1 \frac{\frac{2}{1+t^2}\,dt}{5 + \frac{10t}{1+t^2} - \frac{3-3t^2}{1+t^2}}$

$= \int_0^1 \frac{2\,dt}{5(1+t^2) + 10t - 3 + 3t^2}$

$= \int_0^1 \frac{2\,dt}{2 + 8t^2 + 10t}$

$= \int_0^1 \frac{dt}{1 + 5t + 4t^2}$

$= \int_0^1 \frac{dt}{(4t+1)(t+1)}$

$$= \int_0^1 \frac{dt}{(4t+1)(t+1)}$$

using partial fractions Let $\frac{1}{(4t+1)(t+1)} = \frac{A}{(4t+1)} + \frac{B}{t+1}$

$A(t+1) + B(4t+1) = 1$

$At + A + 4Bt + B = 1$

$\therefore At + 4Bt = 0$

$A + 4B = 0 \ldots (1)$

$A + B = 1 \ldots (2)$

$3B = -1 \quad (1) - (2)$

$B = -\frac{1}{3}, \quad A = \frac{4}{3}$

$$= \int_0^1 \frac{4/3}{4t+1} - \frac{1/3}{t+1} \, dt$$

$$= \frac{1}{3} \int_0^1 \frac{4}{4t+1} - \frac{1}{t+1} \, dt$$

$$= \frac{1}{3} \Big[\ln(4t+1) - \ln(t+1) \Big]_0^1$$

$$= \frac{1}{3} \ln\left(\frac{4t+1}{t+1}\right)\Bigg]_0^1$$

$$= \frac{1}{3} \left[\ln\left(\frac{5}{2}\right) - \ln 1 \right]$$

$$= \frac{1}{3} \ln \frac{5}{2}$$

(4)

QUESTION 2

a) $z_1 = 2i, \quad z_2 = 1+3i, \quad \overline{z_2} = 1-3i$

i) $z_1 + \overline{z_2} = 2i + 1 - 3i$
$= 1 - i$ — 1

ii) $z_1 z_2 = 2i(1+3i)$
$= 2i - 6$
$= -6 + 2i$ — 1

iii) $\frac{1}{z_2} = \frac{1}{1+3i} \times \frac{1-3i}{1-3i}$

$= \frac{1-3i}{1+9}$

$= \frac{1-3i}{10} = \frac{1}{10} - \frac{3}{10}i$ — 1

b) $z = 1 + i\sqrt{3} \qquad |z| = \sqrt{1^2 + \sqrt{3}^2} = 2$
$\theta = \tan^{-1}\sqrt{3} = \pi/3$

$z = 2(\cos \pi/3 + i \sin \pi/3)$ — 1 Modulus, 1 argument

ii) $z^{10} + 512z = 2^{10}\left(\cos \frac{10\pi}{3} + i \sin \frac{10\pi}{3}\right) + 2^{10}\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$

(1)

$= 2^{10}\left(\cos\left(2\pi + \frac{4\pi}{3}\right) + i \sin\left(2\pi + \frac{4\pi}{3}\right) + \cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right)\right)$

$= 2^{10}\left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} + \cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$

(1)

$= 2^{10}\left(-\cos \frac{\pi}{3} - i \sin \frac{\pi}{3} + \cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$

$= 0$

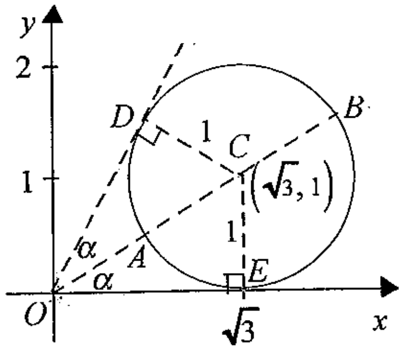
c. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • sketches a circle with correct centre	1
• sketches a circle with correct radius	1
ii • states set of values for $\|z\|$	1
• states set of values for $\arg z$	1

Answer

i. $\left| z-(\sqrt{3}+i) \right| = 1$



$\alpha = \tan^{-1} \frac{1}{\sqrt{3}}$, $\text{Arg } z = 2\alpha = \frac{2\pi}{6} = \frac{\pi}{3}$

ii. $OC = 2$ and $\alpha = \frac{\pi}{6}$

$OA \le |z| \le OB$ $\quad \therefore 1 \le |z| \le 3$

$0 \le \text{Arg} z \le \angle EOD$ $\quad \therefore 0 \le \text{Arg} z \le \frac{\pi}{3}$

d. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • expresses z_2 in terms of z_1	1
• explains why $OACB$ is a square	1
ii • uses properties of a square to deduce $z_1 + z_2 = i(z_1 - z_2)$	1
• uses the side and diagonal lengths of the square to complete the proof	1

Answer $\quad z_2 = z_1 + iz_1 - z_1$

i. $z_1 + z_2 = (1+i)\, z_1$ $\quad \therefore z_2 = i\, z_1$ $\quad$ Hence $\overrightarrow{OB}$ is the rotation of $\overrightarrow{OA}$ anticlockwise by 90°. Hence $OACB$ is a parallelogram in which $OA = OB$ and $\angle AOB = 90°$. $\therefore OACB$ is a square.

ii. The diagonals of a square are equal and meet at right angles.

$\therefore \overrightarrow{OC}$ is the anticlockwise rotation of $\overrightarrow{BA}$ by 90°. Hence

$$z_1 + z_2 = i(z_1 - z_2)$$

$$(z_1 + z_2)\overline{(z_1 - z_2)} = i|z_1 - z_2|^2$$

But $BA^2 = OA^2 + OB^2 = 1 + 1 \Rightarrow |z_1 - z_2|^2 = 2$.

$$\therefore (z_1 + z_2)\overline{(z_1 - z_2)} = 2i$$

Question 3

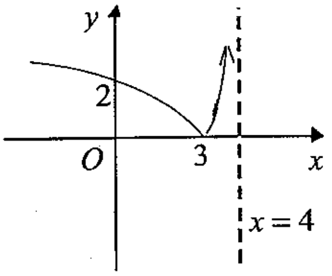
a. Outcomes assessed : E6

Marking Guidelines

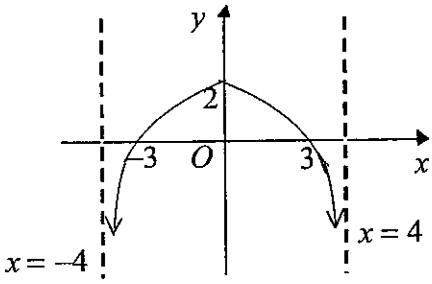
Criteria	Marks
i • copies curve for $x \le 3$ and reflects section of curve for $x > 3$ in x-axis	1
ii • copies curve for $x \ge 0$ and includes reflection of this section of curve in the y-axis	1
iii • sketches curve that is concave down, symmetric in the y-axis, with turning point $(0,2)$	1
• shows asymptotes and x-intercepts	1
iv • shows vertical asymptote $x = 3$ and sketches left hand branch correctly	1
• sketches right hand branch correctly showing nature at $x = 4$	1

Answer

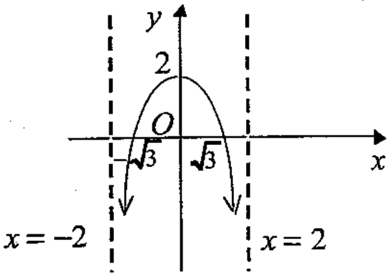
i. $y = |f(x)|$



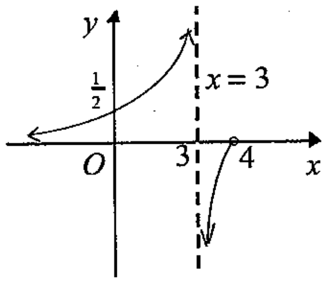
ii. $y = f(|x|)$



iii. $y = f(x^2)$



iv. $y = \dfrac{1}{f(x)}$



b. Outcomes assessed : E4

Marking Guidelines

Criteria	Marks
• states remainder on division by $(x^2 - a^2)$ is $(cx + d)$ for some constants c and d	1
• uses definition of an even function to deduce $ca + d = -ca + d$	1
• completes proof by showing $c = 0$	1

Answer

$P(x) \equiv (x^2 - a^2)Q(x) + cx + d$ for constants c, d where $cx + d$ is the remainder on division by $x^2 - a^2$.

$P(x)$ even $\Rightarrow P(-a) = P(a)$ $\quad \therefore ca + d = -ca + d$

$2ca = 0$ $\quad$ But $a \ne 0$ $\quad \therefore c = 0$.

Hence remainder is some constant d, which is independent of x.

Q3)

c) $x^3 + px^2 + qx + r$ has zeroes α, β, γ

i) $\alpha\beta + \alpha\gamma + \beta\gamma = c/a = q$ ①

ii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= (-p)^2 - 2(q)$

$= p^2 - 2q$ ①

(iii) Let $x = 2y$

$y = \frac{x}{2}$ sub into Poly

$$\left(\frac{x}{2}\right)^3 + p\left(\frac{x}{2}\right)^2 + q\left(\frac{x}{2}\right) + r = 0$$ ①

$$\frac{x^3}{8} + \frac{px^2}{4} + \frac{qx}{2} + r = 0$$

$$x^3 + 2px^2 + 4qx + 8r = 0$$ ①

(d) $p > 0$, $q > 0$, $p + q = 1$.

$\frac{p+q}{2} \geq \sqrt{pq}$ (arithmetic mean ≥ geometric mean)

$p + q \geq 2\sqrt{pq}$ Let $p = \frac{1}{p}$ & $q = \frac{1}{q}$

$\frac{1}{p} + \frac{1}{q} \geq 2\sqrt{\frac{1}{pq}}$ ---- ✱

$$\frac{p+q}{pq} \geq \frac{2}{\sqrt{pq}}$$

$\frac{pq}{p+q} \leq \frac{\sqrt{pq}}{2}$ divide both sides by $\sqrt{pq}$ & multiply by $(p+q)$

$\frac{\sqrt{pq}}{\sqrt{pq}} \times \frac{pq}{\sqrt{pq}} \leq \frac{p+q}{2}$

$\sqrt{pq} \leq \frac{p+q}{2}$

But $p+q = 1$

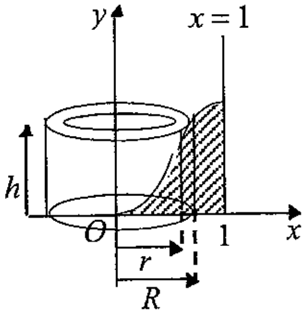
$\sqrt{pq} \leq \frac{1}{2}$ so $\frac{1}{\sqrt{pq}} \geq 2$

So $\frac{1}{p} + \frac{1}{q} \geq 4$ from ✱

⑦

Answer

i.



$h = \dfrac{x}{x^2+1}$

$r = x$

$R = x + \delta x$

$$\delta V = \pi\left(R^2 - r^2\right)h$$
$$= \pi\left(R+r\right)\left(R-r\right)h$$
$$= \pi(2x+\delta x)(\delta x)\frac{x}{x^2+1}$$

Ignoring terms in $(\delta x)^2$,

$$V = \lim_{\delta x \to 0} \sum_{x=0}^{x=1} \pi \frac{2x^2}{x^2+1}\delta x$$
$$= 2\pi \int_0^1 \frac{x^2}{x^2+1}\,dx$$

ii. $V = 2\pi \displaystyle\int_0^1 \left(1 - \frac{1}{x^2+1}\right)dx = 2\pi\left[x - \tan^{-1} x\right]_0^1$ $\qquad \therefore V = \dfrac{\pi}{2}(4-\pi)$

b. Outcomes assessed : E6

Marking Guidelines

Criteria	Marks
• differentiates to obtain gradient of tangent at P	1
• uses gradient of OP is 1 to deduce $x_1 = y_1 = \frac{1}{k}$	1
• substitutes in equation of curve to find k.	1

Answer

$y = e^{kx}$ $\quad \therefore \dfrac{dy}{dx} = ke^{kx}$. Hence tangent at P has gradient $ke^{kx_1} = ky_1$, since $y_1 = e^{kx_1}$.

But gradient of OP is 1 (since P lies on line $y = x$) $\quad \therefore ky_1 = 1$ and hence $x_1 = y_1 = \dfrac{1}{k}$.

Then since P lies on $y = e^{kx}$, $\dfrac{1}{k} = e^{k \cdot \frac{1}{k}}$ $\quad \therefore k = \dfrac{1}{e}$.

c. Outcomes assessed : E3, E4

Marking Guidelines

Criteria	Marks
i • uses O, P, Q collinear to deduce result	1
ii • writes the coordinates of two of the points	1
• writes the coordinates of the remaining two points	1
iii • deduces that $XYUV$ is a rhombus	1
• expresses the area of the rhombus in terms of its diagonal lengths to obtain the result	1
iv • compares the areas of the quadrilateral and ellipse to deduce that $\lvert\sin 2\theta\rvert = 1$	1
• states the four values of θ	1
• sketches the ellipse inscribed in the quadrilateral giving the required detail.	1

9

4(c) i) $\frac{x^2}{25} - \frac{y^2}{9} = 1$

$a = 5, b = 3$

$b^2 = a^2(e^2 - 1)$

$9 = 25(e^2 - 1)$

$e^2 - 1 = 9/25$

$e^2 = 34/25$

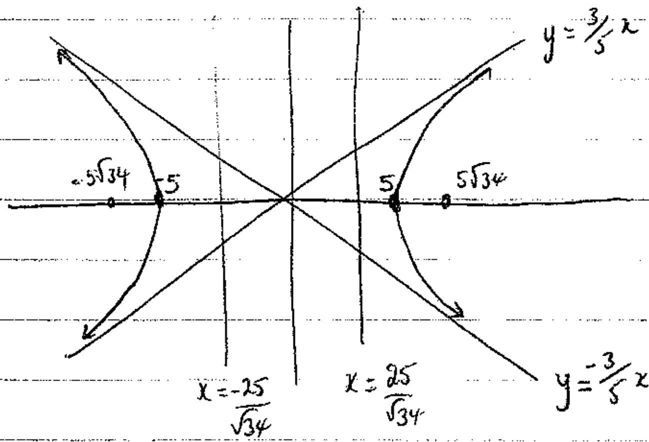
$e = \sqrt{34}/5$ (1)

ii) Foci $(\pm ae, 0)$

$\left(\pm 5 \times \frac{\sqrt{34}}{5}, 0\right)$

$(\pm\sqrt{34}, 0)$ (1)

iii)



iv) $\frac{6^2}{25} - \frac{y^2}{9} = 1$

$324 - 25y^2 = 225$

$25y^2 = 99$

$y^2 = \frac{99}{25}$ $\therefore y = \frac{\pm\sqrt{99}}{5}$

$\therefore$ A is $\left(6, \frac{\sqrt{99}}{5}\right)$, B is $\left(6, \frac{-\sqrt{99}}{5}\right)$

v) $\frac{x^2}{25} - \frac{y^2}{9} = 1$

$\frac{2x}{25} - \frac{2y \cdot \frac{dy}{dx}}{9} = 0$

$$\therefore \frac{dy}{dx} = \frac{-2x}{25} \cdot \frac{-9}{2y}$$

$$= \frac{9x}{25y}$$

at $\left(6, \frac{\sqrt{99}}{5}\right)$ $\quad \frac{dy}{dx} = \frac{9}{25} \cdot 6 \cdot \frac{5}{\sqrt{99}}$

$$= \frac{54}{5\sqrt{99}}$$

$$\therefore y - \frac{\sqrt{99}}{5} = \frac{54}{5\sqrt{99}}(x-6)$$

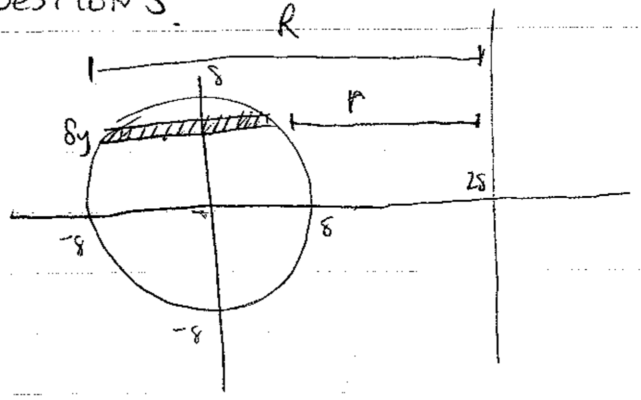
$$5\sqrt{99}\,y - 99 = 54x - 324$$

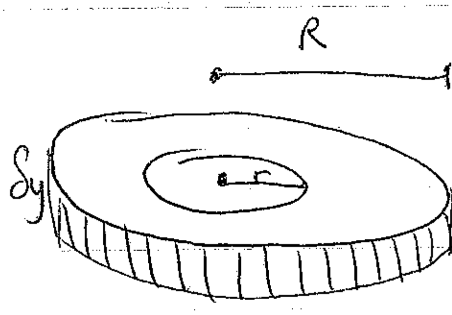
$$54x - 5\sqrt{99}\,y - 225 = 0$$

QUESTION 5.

(11)

(a) i)





The volume of the Slice is $\delta V = \pi (R^2 - r^2)\, \delta y$

$= \pi (R+r)(R-r)\, \delta y$

$= \pi (56)\, 2\sqrt{64-y^2}\, \delta y$

$V = \lim_{\delta y \to 0} \sum_{y=-8}^{8} 112\pi \sqrt{64-y^2}\, \delta y$

(5)

$= 112\pi \int_{-8}^{8} \sqrt{64-y^2}\, dy$

ii) $\int_{-8}^{8} \sqrt{64-y^2}\, dy$ is a Semi Circle Radius 8

$\therefore$ ~~Area~~ Vol $= \frac{1}{2} \times \pi \times r^2$

$= \frac{1}{2} \times \pi \times 8^2 = 32\pi$

$\therefore V = 112\pi \times 32\pi$

$= 3584\pi^2 \text{ cm}^3.$

(12)

(b) $\frac{x^2}{12} + \frac{y^2}{4} = 1$

$\frac{2x}{12} + \frac{2y\frac{dy}{dx}}{4} = 0$ using implicit diff.

$\frac{x}{6} + \frac{y \cdot \frac{dy}{dx}}{2} = 0$

$\frac{x}{6} = -\frac{y \cdot dy/dx}{2}$

$2x = -6y\frac{dy}{dx}$

$\frac{dy}{dx} = -\frac{x}{3y}$ at $P(3,1)$

$m = \frac{dy}{dx} = \frac{-3}{3} = -1$

Tangent is $y - y_1 = m(x - x_1)$

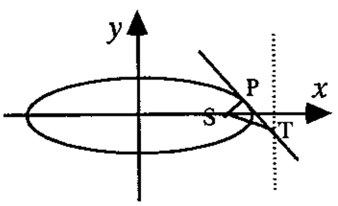
$y - 1 = -1(x - 3)$

$y - 1 = -x + 3$

$x + y - 4 = 0$

$x + y = 4$

(ii) If this tangent cuts the directrix in the fourth quadrant at the point T, and S is the corresponding focus, show that SP and ST are perpendicular.



$\frac{b^2}{a^2} = 1 - e^2 \qquad \therefore\ e = \sqrt{\frac{2}{3}}$

$\therefore$ directrix is $x = \frac{a}{e}$ -> $\underline{x = 3\sqrt{2}}$

and focus at $x = ae$ -> $\underline{S(2\sqrt{2}, 0)}$

Putting $x = 3\sqrt{2}$ in $x + y = 4$ gives...

$\underline{T(3\sqrt{2},\ 4 - 3\sqrt{2})}$

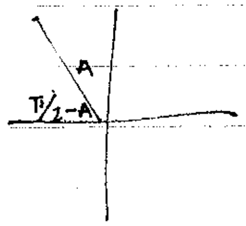
Now, $m_{SP} = \frac{1}{3 - 2\sqrt{2}}$ and $m_{ST} = \frac{4 - 3\sqrt{2}}{\sqrt{2}}$

$m_{SP} \times m_{ST} = \frac{1}{3 - 2\sqrt{2}} \times \frac{4 - 3\sqrt{2}}{\sqrt{2}} = \frac{4 - 3\sqrt{2}}{3\sqrt{2} - 4} = -1 \qquad \therefore$ PS $\perp$ ST.

Question 6

a) i) $\tan(A + \pi/2)$

$= \tan(\pi - (\pi/2 - A))$



$= -\tan(\pi/2 - A)$

$= -\cot A$

ii) Aim to prove $\tan\left\{(2n+1)\frac{\pi}{4}\right\} = (-1)^n \quad \forall \mathbb{Z}^+$

Step 1 Prove true for $n = 1$

LHS $= \tan\left\{(2(1)+1)\frac{\pi}{4}\right\}$ RHS $= (-1)^1$

$= -1$

$= \tan\frac{3\pi}{4}$

$= -\tan\pi/4$

$= -1$ $\therefore$ LHS = RHS True for $n = 1$

Step 2 Assume true for $n = k$

Assume $\tan\left\{(2k+1)\frac{\pi}{4}\right\} = (-1)^k$ - - - - (✱)

Step 3 Prove true for $n = k+1$

Prove $\tan\left\{(2(k+1)+1)\frac{\pi}{4}\right\} = (-1)^{k+1}$

LHS $= \tan\left\{(2(k+1)+1)\pi/4\right\}$

$= \tan\left\{(2k+3)\pi/4\right\}$

$= \tan\left(2k \cdot \frac{\pi}{4} + \frac{3\pi}{4}\right)$

$\frac{3\pi}{4} = \frac{\pi + \pi/2}{2}$

$= \pi/2 + \pi/4$

$= \tan\left(\frac{k\pi}{2} + \frac{\pi}{2} + \frac{\pi}{4}\right)$

$= \tan\left(\left(\frac{k\pi}{2} + \frac{\pi}{2}\right) + \pi/2\right)$

$$= -\cot\left(\frac{k\pi}{2} + \frac{\pi}{4}\right) \quad \text{using (i)}$$

$$= -\cot\left(\frac{2k\pi + \pi}{4}\right)$$

$$= -\cot\left\{(2k+1)\frac{\pi}{4}\right\}$$

$$= \frac{-1}{\tan\{(2k+1)(\pi/4)\}}$$

$$= \frac{-1}{(-1)^k} \quad \text{using ---- *}$$

$$= (-1)^1 \div (-1)^k$$

$$= (-1)^{1-k}$$

$$= (-1) \times (-1)^{-k}$$

$$= (-1) \times (-1)^k$$

$$= (-1)^{k+1} = RHS$$

∴ Statement is true for $n = k+1$.

<u>STEP 4</u> We have proven Statement is true for $n=1$. Assuming it is true for $n=k$, we have proven it true for $n=k+1$, ∴ it is true for $n=1+1=2$, $n=2+1=3$, and so on for all integral $n \geq 1$.

Q6

b) $y^2 + xy + x^2 = 1$

$y^2 + xy = 1 - x^2$

Complete the square

$y^2 + xy + \left(\frac{x}{2}\right)^2 = 1 - x^2 + \left(\frac{x}{2}\right)^2$

$\left(y + \frac{x}{2}\right)^2 = 1 - x^2 + \frac{x^2}{4}$

$\left(y + \frac{x}{2}\right)^2 = 1 - \frac{3x^2}{4}$

$y = -\frac{x}{2} \pm \sqrt{1 - \frac{3x^2}{4}}$

$y = -x/2 \pm \frac{\sqrt{4 - 3x^2}}{2}$

$y = \frac{-x \pm \sqrt{4 - 3x^2}}{2}$

(ii) $y^2 + xy + x^2 = 1$

$2y\frac{dy}{dx} + x\frac{dy}{dx} + y + 2x = 0$

$\frac{dy}{dx}(2y + x) + y + 2x = 0$

$\frac{dy}{dx}(2y + x) = -(y + 2x)$

$\frac{dy}{dx} = \frac{-(y + 2x)}{2y + x}$

Q6 c) $z = \cos\theta + i\sin\theta$ $\quad z^n + z^{-n} = 2\cos n\theta$

$z^1 + z^{-1} = 2\cos\theta \quad \therefore (z^1 + z^{-1})^4 = 16\cos^4\theta$

Also $(z^1 + z^{-1})^4 = z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4}$

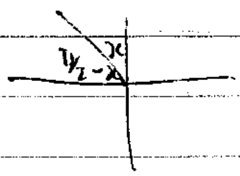
$= (z^4 + z^{-4}) + 4(z^2 + z^{-2}) + 6$

$= 2\cos 4\theta + 8\cos 2\theta + 6$

$\therefore 16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$

$\therefore \cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$.

6d) $\cos\left(x + \frac{\pi}{2}\right)$



$= \cos\left(\pi - \left(\frac{\pi}{2} - x\right)\right)$

$= -\cos\left(\frac{\pi}{2} - x\right)$

$= -\sin x$

Question 7

a. Outcomes assessed : E8

Marking Guidelines

Criteria	Marks
i • makes the substitution $u = a - x$	1
• uses property that value of a definite integral does not depend on the variable of integration	1
ii • uses the result from (i) to write the given definite integral with $\cos^2 x$ replacing $\sin^2 x$	1
• uses the table of standard integrals to find the primitive of twice the given integral	1
• evaluates the given integral by substitution of the limits and rearranging	1

Answer

i. Let $u = a - x$

Then $du = -dx$

and

$x = 0 \Rightarrow u = a$

$x = a \Rightarrow u = 0$

$$\int_0^a f(x)\ dx = \int_a^0 f(a-u).-du = \int_0^a f(a-u)\ du = \int_0^a f(a-x)\ dx$$

ii. Let $a = \frac{\pi}{2}$, $f(x) = \dfrac{\sin^2 x}{\sqrt{1+(x-\frac{\pi}{4})^2}}$. Then $f(\frac{\pi}{2}-x) = \dfrac{\sin^2(\frac{\pi}{2}-x)}{\sqrt{1+(\frac{\pi}{2}-x-\frac{\pi}{4})^2}} = \dfrac{\cos^2 x}{\sqrt{1+(x-\frac{\pi}{4})^2}}$.

Using (i), if $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sqrt{1+(x-\frac{\pi}{4})^2}}\, dx$, then $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\sqrt{1+(x-\frac{\pi}{4})^2}}\, dx$.

$$2I = \int_0^{\frac{\pi}{2}} \left\{ \frac{\sin^2 x}{\sqrt{1+(x-\frac{\pi}{4})^2}} + \frac{\cos^2 x}{\sqrt{1+(x-\frac{\pi}{4})^2}} \right\} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1+(x-\frac{\pi}{4})^2}}\, dx$$

$$= \left[\ln\left\{ (x-\tfrac{\pi}{4}) + \sqrt{1+(x-\tfrac{\pi}{4})^2} \right\} \right]_0^{\frac{\pi}{2}}$$

$$= \ln\left\{ \frac{\frac{\pi}{4} + \sqrt{1+(\frac{\pi}{4})^2}}{-\frac{\pi}{4} + \sqrt{1+(\frac{\pi}{4})^2}} \right\}$$

$$\therefore I = \tfrac{1}{2}\ln\left\{ \frac{\pi + \sqrt{16+\pi^2}}{-\pi + \sqrt{16+\pi^2}} \right\}$$

$$= \tfrac{1}{2}\ln\left\{ \frac{\left(\pi + \sqrt{16+\pi^2}\right)^2}{\left(16+\pi^2\right) - \pi^2} \right\}$$

$$= \tfrac{1}{2}\ln\left\{ \tfrac{1}{4}\left(\pi + \sqrt{16+\pi^2}\right) \right\}^2$$

$$= \ln\left\{ \tfrac{1}{4}\left(\pi + \sqrt{16+\pi^2}\right) \right\}$$

b. Outcomes assessed : E8

Marking Guidelines

Criteria	Marks
i • applies integration by parts	1
• evaluates the first part and rearranges the second integrand	1
• expresses the second integral in terms of I_n, I_{n-1} then rearranges to obtain result	1
ii • uses the recurrence formula to express I_3 in terms of I_0	1
• evaluates I_0 and hence evaluates I_3	1

Answer

i. $I_n = \int_0^1 (1-x^r)^n \, dx$, $\quad n = 0, 1, 2, \ldots \quad$ where $r > 0$

For $n = 1, 2, 3, \ldots$

$$I_n = \left[x(1-x^r)^n \right]_0^1 - n\int_0^1 x.(1-x^r)^{n-1}.(-rx^{r-1}) \, dx$$

$$= 0 - nr\int_0^1 \left\{(1-x^r)-1\right\}(1-x^r)^{n-1} \, dx$$

$$= nr\left\{-I_n + I_{n-1}\right\}$$

$$\therefore (nr+1)I_n = nrI_{n-1}$$

$$I_n = \frac{nr}{nr+1} I_{n-1}$$

ii. For $r = \frac{3}{2}$, $\quad I_3 = \frac{\left(3\times\frac{3}{2}\right)}{\left(3\times\frac{3}{2}+1\right)} \cdot \frac{\left(2\times\frac{3}{2}\right)}{\left(2\times\frac{3}{2}+1\right)} \cdot \frac{\left(1\times\frac{3}{2}\right)}{\left(1\times\frac{3}{2}+1\right)} I_0 = \frac{9}{11}\cdot\frac{3}{4}\cdot\frac{3}{5} I_0$

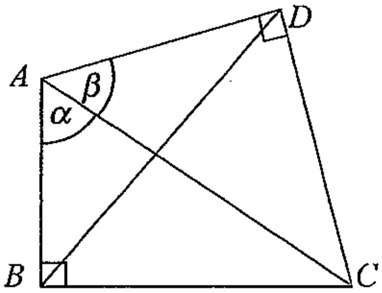
But $I_0 = \int_0^1 1 \, dx = 1$. $\quad$ Hence $I_3 = \frac{81}{220}$.

c. Outcomes assessed : PE2, PE3

Marking Guidelines

Criteria	**Marks**
i • explains why ABCD is a cyclic quadrilateral	1
• uses 'angles in the same segment' to deduce result	1
ii • explains why $\sin\angle BCD = \sin(\alpha+\beta)$	1
• explains why $BC = \sin\alpha$	1
• uses the sine rule in ΔBCD to obtain required result	1

Answer



i. $ABCD$ is a cyclic quadrilateral $\quad$ (*opposite angles ABC and ADC are supplementary*)
$\therefore \angle BDC = \angle BAC = \alpha$ $\quad$ (*in circle ABCD, angles subtended at circumference by same arc BC are equal*)

ii. $\angle BCD = \pi - (\alpha+\beta)$ $\quad$ (*opposite angles of a cyclic quadrilateral are supplementary*)

$\therefore \sin\angle BCD = \sin\left\{\pi - (\alpha+\beta)\right\} = \sin(\alpha+\beta)$

Also in ΔABC, $BC = AC\sin\alpha = \sin\alpha$ $\quad$ (*given* $AC = 1$)

Hence in ΔBCD, $\quad \frac{BD}{\sin\angle BCD} = \frac{BC}{\sin\angle BDC} \Rightarrow \frac{BD}{\sin(\alpha+\beta)} = \frac{\sin\alpha}{\sin\alpha} = 1$. $\qquad \therefore BD = \sin(\alpha+\beta)$

Question 8

ai) $\tan 4\theta = 1$

$\tan 4\theta = \tan \pi/4$

$4\theta = \pi n + \pi/4$

$\theta = \dfrac{\pi n + \pi/4}{4}$

$= \dfrac{4\pi n + \pi}{16}$ ✓

ii) $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$ (de Moivre) ✓

By the Binomial Theorem

$(\cos\theta + i\sin\theta)^4 = \cos^4\theta + 4\cos^3\theta\, i\sin\theta + 6\cos^2\theta\, i^2\sin^2\theta$
$+ 4\cos\theta\, i^3\sin^3\theta + i^4\sin^4\theta$

Equating Real + imaginary Coefficients.

$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

$\therefore \tan 4\theta = \dfrac{\sin 4\theta}{\cos 4\theta} = \dfrac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}$

dividing by $\dfrac{\cos^4\theta}{\cos^4\theta}$

$= \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$

(iii) $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$

$x^4 - 6x^2 + 1 = 4x - 4x^3$

$\dfrac{4x - 4x^3}{x^4 - 6x^2 + 1} = 1$

Let $x = \tan\theta$ then $\dfrac{4\tan\theta - 4\tan^3\theta}{\tan^4\theta - 6\tan^2\theta + 1} = 1$

ie $\tan 4\theta = 1$

$\therefore \theta = \frac{\pi n}{4} + \frac{\pi}{16}, \ n \in \mathbb{Z}$ from (i)

Consider $n = 0, \pm 1, \pm 2$

ie $x = \tan \pi/16, \tan 5\pi/16, -\tan \frac{3\pi}{16}$ (or $\tan {13\pi}/16$)

and $-\tan \frac{7\pi}{16}$ (or $\tan 9\pi/16$)

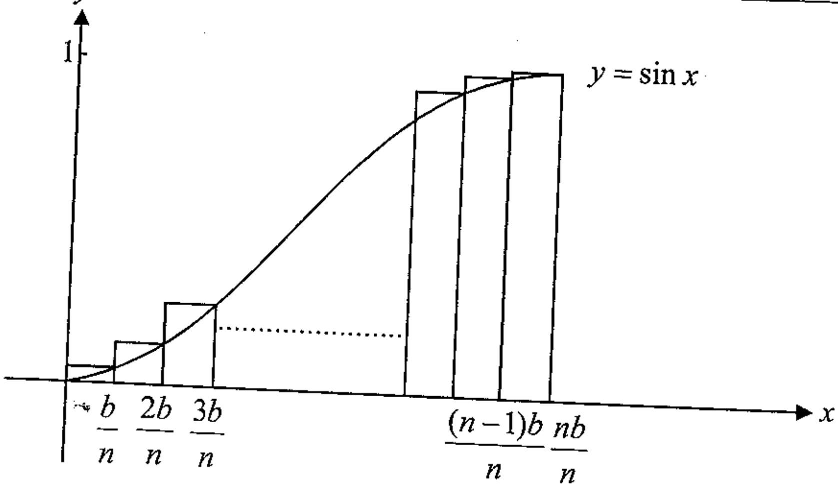
(iv) $\tan^2 \pi/16 + \tan^2 3\pi/16 + \tan^2 5\pi/16 + \tan^2 7\pi/16$

$= (\alpha+\beta+\gamma+\delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$

$= (-4)^2 - 2(-6)$

$= 28$ as required.

(b) (i)



The diagram shows a series of upper rectangles each of width $\frac{b}{n}$ and of height $\sin\frac{b}{n}, \sin\frac{2b}{n}, \sin\frac{3b}{n}, \ldots, \sin\frac{nb}{n}$ respectively as you move from left to right.

The sum of the area of the rectangles is $\left(\sin\frac{b}{n} + \sin\frac{2b}{n} + \ldots + \sin\frac{nb}{n}\right)\cdot\frac{b}{n}$.

The area under the graph of $y = \sin x$ between $x = 0$ and $x = b$ where $b = \frac{\pi}{2}$

is therefore given by $\lim_{n\to\infty}\left(\sin\frac{b}{n} + \sin\frac{2b}{n} + \ldots + \sin\frac{nb}{n}\right)\cdot\frac{b}{n}$.

1 mark	Explanation including diagram

Question 8 (cont'd)

(ii) Now,

$$2\sin\left(\frac{b}{2n}\right)\left(\sin\left(\frac{b}{n}\right)+\sin\left(\frac{2b}{n}\right)+....+\sin\left(\frac{nb}{n}\right)\right)$$

$$=\cos\left(\frac{b}{2n}-\frac{b}{n}\right)-\cos\left(\frac{b}{2n}+\frac{b}{n}\right)$$

$$+\cos\left(\frac{b}{2n}-\frac{2b}{n}\right)-\cos\left(\frac{b}{2n}+\frac{2b}{n}\right)$$

$$+\cos\left(\frac{b}{2n}-\frac{3b}{n}\right)-\cos\left(\frac{b}{2n}+\frac{3b}{n}\right)+$$

.

.

.

$$+\cos\left(\frac{b}{2n}-\frac{nb}{n}\right)-\cos\left(\frac{b}{2n}+\frac{nb}{n}\right)$$

$$=\cos\left(\frac{b}{2n}\right)-\cos\left(\frac{3b}{2n}\right)$$

$$+\cos\left(\frac{3b}{2n}\right)-\cos\left(\frac{5b}{2n}\right)$$

$$+\cos\left(\frac{5b}{2n}\right)-\cos\left(\frac{7b}{2n}\right)+$$

.

.

.

$$+\cos\left(\frac{b}{2n}-\frac{nb}{n}\right)-\cos\left(\frac{b}{2n}+\frac{nb}{n}\right)$$

$$=\cos\left(\frac{b}{2n}\right)-\cos\left(b+\frac{b}{2n}\right)$$

So, $$2\sin\left(\frac{b}{2n}\right)\sum_{k=1}^{n}\sin\left(\frac{kb}{n}\right)=\cos\left(\frac{b}{2n}\right)-\cos\left(b+\frac{b}{2n}\right)$$

So, $$\sum_{k=1}^{n}\sin\left(\frac{kb}{n}\right)=\frac{\cos\left(\frac{b}{2n}\right)-\cos\left(b+\frac{b}{2n}\right)}{2\sin\left(\frac{b}{2n}\right)}$$

as required

2 marks	Multiplying $\sum_{k=1}^{n}\sin\left(\frac{kb}{n}\right)$ by $2\sin\left(\frac{b}{2n}\right)$ and obtaining correct expression
1 mark	First part only

Question 8 (cont'd)

(iii) We must use what has already been found.

$$\int_0^b \sin x\,dx = \lim_{n\to\infty}\left(\sin\left(\frac{b}{n}\right)+\sin\left(\frac{2b}{n}\right)+\ldots+\sin\left(\frac{nb}{n}\right)\right)\cdot\frac{b}{n}$$

$$=\lim_{n\to\infty}\frac{\cos\left(\frac{b}{2n}\right)-\cos\left(b+\frac{b}{2n}\right)}{2\sin\left(\frac{b}{2n}\right)}\cdot\frac{b}{n}$$

$$=\lim_{n\to\infty}\left(\cos\left(\frac{b}{2n}\right)-\left(\cos b\cos\left(\frac{b}{2n}\right)-\sin b\sin\left(\frac{b}{2n}\right)\right)\right)\times\frac{b}{2n}\times\frac{1}{\sin\left(\frac{b}{2n}\right)}$$

$$=\lim_{n\to\infty}\left(\cos\left(\frac{b}{2n}\right)-\cos b\cos\left(\frac{b}{2n}\right)+\sin b\sin\left(\frac{b}{2n}\right)\right)\times\lim_{n\to\infty}\frac{b}{2n}\times\frac{1}{\sin\left(\frac{b}{2n}\right)}$$

$$=(1-\cos b+0)\times 1 \quad \text{since } \lim_{n\to\infty}\frac{\theta}{\sin\theta}=1, \text{and } \lim_{n\to\infty}\cos\frac{b}{2n}=1, \text{and}$$

$$\lim_{n\to\infty}\sin\frac{b}{2n}=0 \quad \text{since } \lim_{n\to\infty}\frac{b}{2n}=0.$$

$$=1-\cos b$$

as required.

3 marks	Obtaining line marked (*) AND taking each of the two limits correctly to obtain the correct expression
2 marks	Obtaining line marked (*) AND taking one of the limits correctly
1 mark	Obtaining line marked (*)