GIRRAWEEN HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2002

MATHEMATICS

EXTENSION 2

Time allowed - Three hours
(Plus 5 minutes' reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are supplied
- Board-approved calculators may be used.
- Each question attempted is to be returned on a *separate* piece of paper clearly marked Question 1 , Question 2 , etc. Each piece of paper must show your name.
- You may ask for extra pieces of paper if you need them.

QUESTION 1

a) Find $\int \frac{dx}{\sqrt{x^2+2x+5}}$ 2

b) (i) Find real numbers a, b and c such that

$$\frac{3x}{(x+1)(x^2+2x+4)} \equiv \frac{a}{x+1} + \frac{bx+c}{x^2+2x+4}$$ 2

(ii) Find $\int \frac{3x}{(x+1)(x^2+2x+4)}dx$ 2

c) Use integration by parts to find

$$\int_0^1 \tan^{-1} x \, dx$$ 3

d) Find $\int_0^{\frac{2\pi}{3}} \frac{1}{5+4\cos x}dx$ 3

e) Find $\int \sin^3 x \cos^2 x \, dx$ 3

QUESTION 2

(a) Given the two complex numbers $z = 3-4i$ and $w = 4+3i$,
find zw and $\frac{1}{w}$ in the form $x+iy$. 2

b) On separate argand diagrams draw a neat sketch of the locus specified by

(i) $z^2 - \bar{z}^2 = 4i$ 2

(ii) $\arg(z-2) = \arg z + \frac{\pi}{2}$ 2

c) If $z = \sqrt{3}+i$

(i) Find the exact value of mod z and arg z. 2

(ii) By using De Moivres theorem write $\frac{1}{z^5}$ in form x+iy. 2

d) Let P, Q, R represent the complex numbers z_1, z_2, z_3 respectively.
What geometric properties characterize triangle PQR if $z_2 - z_1 = i(z_3 - z_1)$?
Give reasons for your answer. 3

e) The polynomial $z^3 - 3z^2 + 7z - 5$ has one root equal to 1-2i.
Factorize this polynomial 2

QUESTION 3

(a) An ellipse has the equation $\frac{x^2}{25}+\frac{y^2}{16}=1$

(i) Find the eccentricity. 1

(ii) Find coordinates of the foci S, S' and equation of directrices. 2

(iii) Sketch the ellipse showing all the above features and where it crosses the coordinate axes. 1

(iv) If P is a point on the ellipse show that PS + PS' is independent of the position of P. 2

(b) Consider the hyperbola with equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

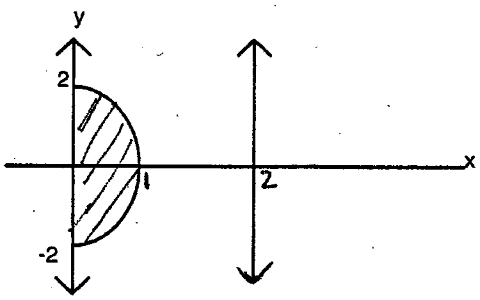
(i) Show that the equation of the tangent at the point P $(a\sec\theta, b\tan\theta)$ has the equation $bx\sec\theta - ay\tan\theta = ab$. 2

(ii) Deduce the equation of the normal at P. 2

(iii) Find A and B where the tangent and normal respectively cut the y-axis. 2

(iv) Show that AB is the diameter of the circle that passes through the foci of the hyperbola. 3

QUESTION 4



(a) A solid S is formed by rotating the region bounded by the parabola $y^2 = 4(1-x)$ and the y axis $360°$ about the line $x = 2$.

(i) By slicing perpendicular to the axis of rotation, find the exact volume of S. **4**

(ii) (*a*) Use the method of cylindrical shells to show that the volume of S is also given by $\int_0^1 8\pi(2-x)\sqrt{1-x}\,dx$. **2**

(*b*) Confirm your answer to part (i) by calculating this definite integral using the substitution $u = 1-x$. **3**

(b) A dome has a circular base of radius 10 metres. Each cross section of the dome perpendicular to the x-axis is a parabola, whose height is the same as the base width.

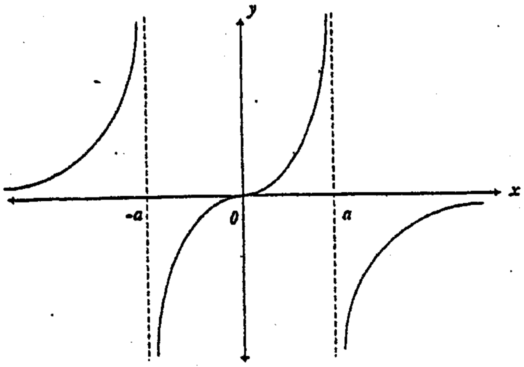
(i) Why would Simpson's rule give the exact area of the parabolic cross section? **1**

(ii) Show that the area of the parabolic cross-section is $\frac{8y^2}{3}$ square metres. **2**

(iii) Find the volume of the dome. **3**

QUESTION 5

(a) The graph of y=f(x) is shown below



Draw sketches of the following

(i) $y=f(x-a)$ 1

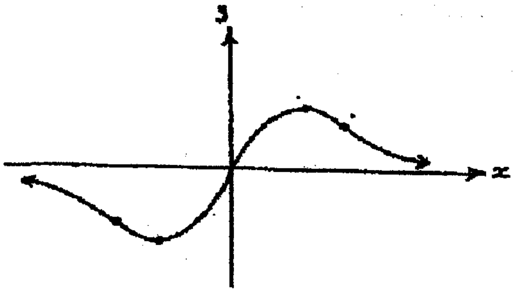
(ii) $y=f'(x)$ 2

(iii) $y=\frac{1}{f(x)}$ 2

(iv) $y=f(x)^2$ 2

(b) Find integers a and b such that $(x+1)^2$ is a factor of x^3+4x^2+ax+b 3

(c)



The curve $y=\frac{2x}{1+x^2}$ is sketched in the diagram above

(i) Show that the equation $kx^3+(k-2)x=0$ can be written in the form $\frac{2x}{1+x^2}=kx$ 2

(ii) Using a graphical approach based on the curve $y=\frac{2x}{1+x^2}$, or otherwise, find the real values of k for which the equation $kx^3+(k-2)x=0$ has exactly 1 solution. 3

QUESTION 6

(a) A particle of mass m is projected vertically upwards under gravity

The air resistance to the motion is $-\frac{1}{100}mgv^2$ where v is the speed of the particle .

(i) Show that during the upward motion of the particle, if x is the upward vertical displacement of the particle from its projection point at time t then

$$\ddot{x} = -\frac{1}{100}g(100+v^2).$$ **2**

(ii) If the speed of projection is u show that the greatest height (above the point of projection) reached by the particle is

$$\frac{50}{g}\ln\left(\frac{100+u^2}{100}\right).$$ **4**

(b) Let ω be a non-real cube root of unity .

(i) Show that $1+\omega+\omega^2=0$. **1**

(ii) Hence simplify $(1+\omega)^2$. **1**

(iii) Show that $(1+\omega)^3 = -1$.. **1**

(iv) Use part iii) to simplify $(1+\omega)^{3n}$ and hence show that

$$^{3n}C_0 - \frac{1}{2}(^{3n}C_1 + {}^{3n}C_2) + {}^{3n}C_3 - \frac{1}{2}(^{3n}C_4 + {}^{3n}C_5) + {}^{3n}C_6 - \ldots\ldots\ldots {}^{3n}C_{3n} = (-1)^n.$$ **3**

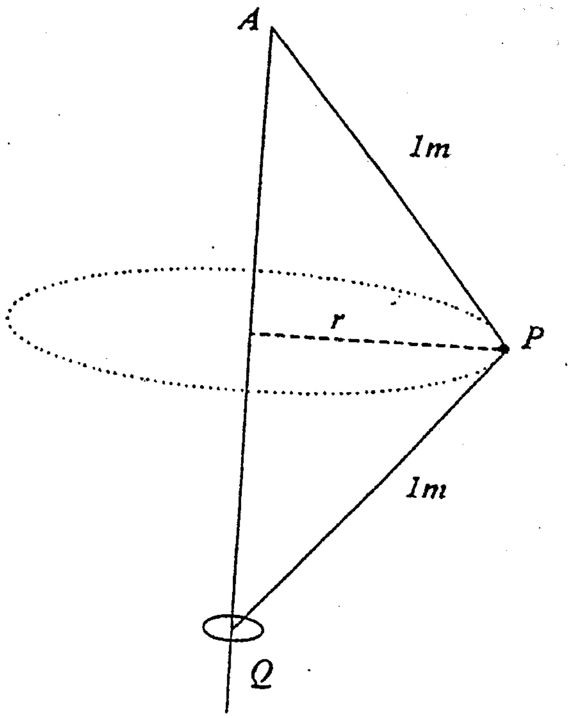
(HINT: You may assume $\text{Re}(\omega) = -\frac{1}{2}$ and that $\text{Re}(\omega^2) = -\frac{1}{2}$)

(c) (i) Show that for a>0 and $n \neq 0, \log_{a^n} x = \frac{1}{n}\log_a x$ **1**

(ii) Hence evaluate $\log_2 3 + \log_4 3 + \log_{16} 3 + \log_{256} 3 + \ldots\ldots\ldots$ **2**

QUESTION 7

(a) A particle P, of mass 2kg, is attached by a light inelastic string of length 1m to a fixed point A as shown in the diagram below. Another string of equal length attaches P to a smooth ring Q, of mass 3kg which is free to slide on a vertical wire that passes through A. The particle P is rotating in a horizontal circle of radius r, about the vertical wire with a constant angular velocity of 2π radians per second



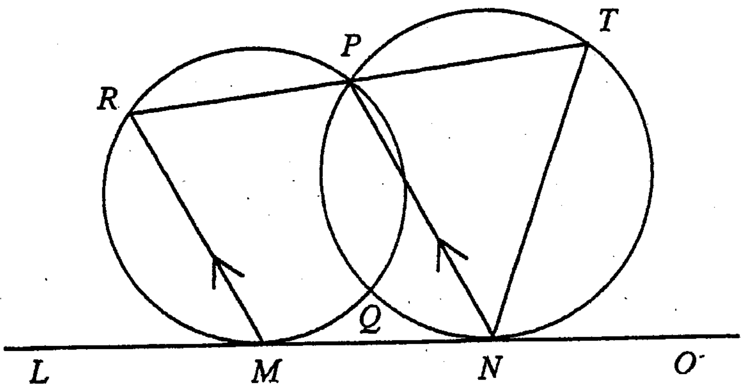
Let T_1 represent the tension in the string PQ, T_2 the tension in the string AP and θ the angle of inclination of AP to the vertical wire.

(i) Copy the above diagram onto your paper and clearly indicate on your sketch all the forces acting on P and Q. **1**

(ii) Write down the equations expressing the vertical and horizontal equilibrium of forces at points P and Q. **3**

(iii) By using the equations in (ii) evaluate $\tan\theta$ in terms of r.
Hence calculate the vertical distance h of P below A ($g=9.8ms^{-2}$) **4**

QUESTION 7 (cont)



(b) In the diagram the two circles intersect at P and Q. LMNO is a common tangent to the two circles. R is a point on one circle such that MR || NP. RP produced meets the other circle at T.

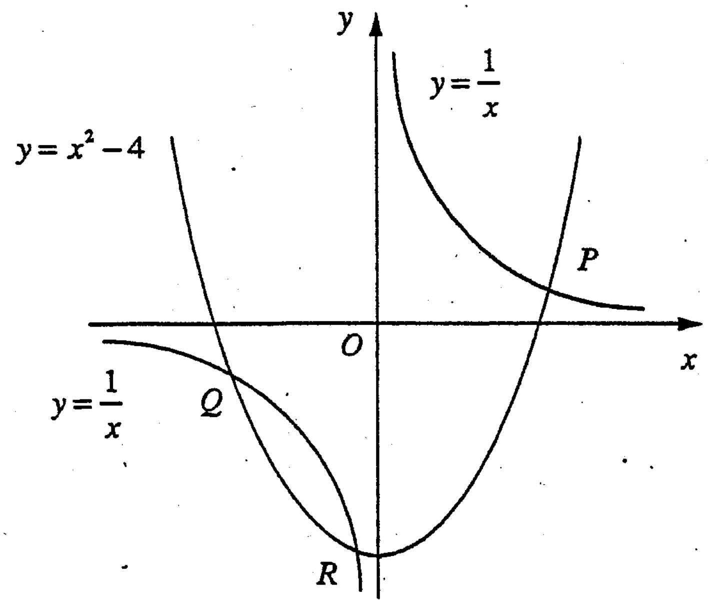
(i) Copy the diagram.

(ii) Show that MNTR is a cyclic quadrilateral. **4**

(iii) G is the point of intersection of MT and NR. The circle through the points T,R,and G is drawn. Show that the tangent to this circle at G is parallel to MN. **3**

QUESTION 8

(a)



The curves $y = x^2 - 4$ and $y = \frac{1}{x}$ intersect at the points P,Q,R where $x=\alpha, x = \beta, x = \gamma$

(i) Show that α, β, γ are the roots of the equation $x^3 - 4x - 1 = 0$ **1**

(ii) Find a polynomial equation with integer coefficients which has roots $\alpha^2, \beta^2, \gamma^2$. **2**

(iii) Find a polynomial equation with integer coefficients which has roots $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$. **2**

(iv) Hence find the numerical value of $OP^2 + OQ^2 + OR^2$. **2**

(b) Newton's Method can be used to determine numerical approximations to the real roots of the equation $x^3 = 4$.
Let $x_1 = 2, x_2, x_3, \ldots\ldots\ldots x_n, \ldots\ldots$ be a series of estimates obtained by iterative applications of Newton's method.

(i) Show that $x_{n+1} = \frac{2}{3}(x_n + \frac{2}{x_n^{\ 2}})$. **2**

(ii) Show algebraically that $x_{n+1} - \sqrt[3]{4} = \frac{(x_n - \sqrt[3]{4})^2 (2x_n + \sqrt[3]{4})}{3x_n^2}$. **3**

(iii) Given that $x_n > \sqrt[3]{4}$ show that $x_{n+1} - \sqrt[3]{4} < (x_n - \sqrt[3]{4})^2$. **2**

(iv) Show that x_6 is accurate to 12 decimal places. **1**

Question 1 Girraween HS Trial HSC Extn 2

(a) $\int \frac{dx}{\sqrt{(x+1)^2+4}}$ (2)

$= \ln(x+1+\sqrt{(x+1)^2+4}) + c$

(b)(i) $3x = a(x^2+2x+4) + (bx+c)(x+1)$

$x=-1$

$-3 = 3a$

$\therefore a = -1$

$x=0$

$0 = 4a + c$

$\therefore c = 4$ (2)

$x=1$

$3 = 7a + 2(b+c)$

$3 = -7 + 2b + 8$

$\therefore 2b = 2$

$b = 1$

(ii) $\int \frac{3x}{(x+1)(x^2+2x+4)}dx = \int \frac{-1}{x+1} + \frac{x+4}{x^2+2x+4}dx$

$= -\ln(x+1) + \int \frac{\frac{1}{2}(2x+2)}{x^2+2x+4} + \frac{3}{x^2+2x+4}dx$

$= -\ln(x+1) + \frac{1}{2}\ln(x^2+2x+4) + 3\int \frac{dx}{(x+1)^2+3}$

$= -\ln(x+1) + \frac{1}{2}\ln(x^2+2x+4) + \frac{3}{\sqrt{3}}\tan^{-1}\frac{x+1}{\sqrt{3}} + c$

(2)

c) $\int \tan^{-1}x \frac{d}{dx}(x)\,dx$

$= \left[x\tan^{-1}x\right]_0^1 - \int_0^1 \frac{x}{x^2+1}dx$

(c) $\frac{\pi}{4} - \frac{1}{2}\int_0^1 \frac{2x}{x^2+1}dx$

$= \frac{\pi}{4} - \left[\frac{1}{2}\ln(x^2+1)\right]_0^1$

$= \frac{\pi}{4} - \frac{1}{2}\ln 2$ (3)

(d) Let $t = \tan\frac{x}{2}$

$dt = \frac{2}{1+t^2}$

when $x=0$ $t=0$

$x=\frac{2\pi}{3}$ $t=\sqrt{3}$

$\int_0^{\sqrt{3}} \frac{\frac{2}{1+t^2}}{5+4\left(\frac{1-t^2}{1+t^2}\right)}dt$ (3)

$= \int_0^{\sqrt{3}} \frac{2}{5+5t^2+4-4t^2}dt$

$= \int_0^{\sqrt{3}} \frac{2}{9+t^2}$

$= \left[\frac{2}{3}\tan^{-1}\frac{t}{3}\right]_0^{\sqrt{3}} = \frac{\pi}{9}$

(e) Let $u = \cos x$ $\frac{du}{dx} = -\sin x$

$\int (1-\cos^2 x)\cos^2 x \sin x\,dx$

$= \int (1-u^2)u^2 - du$ (3)

$= \int -u^2 + u^4\,du$

$= -\frac{u^3}{3} + \frac{u^5}{5}$

$= -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + c$

QUESTION 2

(a) $(3-4i)(4+3i)$

$= 12+9i-16i+12$

$= 24-7i$ (1)

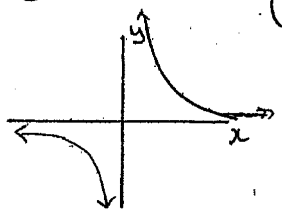
$\frac{1}{4+3i} \times \frac{4-3i}{4-3i} = \frac{4-3i}{25}$ (1)

(b)(i) $(x+iy)^2 - (x-iy)^2 = 4i$

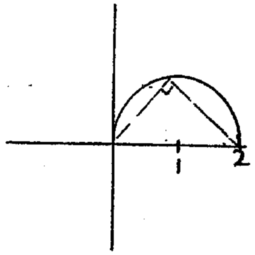
$x^2+2xiy+y^2-(x^2-2xiy+y^2) = 4i$

$4xiy = 4i$

$\therefore xy = 1$ (2)



(ii) $\arg(z-2) - \arg z = \frac{\pi}{2}$ (2)



(c)(i) $\mod z = \sqrt{4}$

$= 2$ (2)

$\arg z = \tan^{-1}\frac{1}{\sqrt{3}}$

$=$

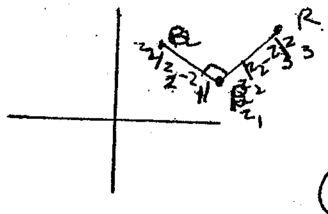
(ii) $2^{-5}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^{-5}$

$= \frac{1}{32}\left(\cos\frac{-5\pi}{6} + i\sin\frac{-5\pi}{6}\right)$

$= \frac{1}{32}\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$

$= -\frac{\sqrt{3}}{64} - \frac{i}{64}$ (2)

(d)



(3)

ΔPQR is a right angled isosceles Δ $PQ = PR$

$|z_2 - z_1| = |i(w_3 - w_1)|$

$= |w_3 - w_1|$

multiplication by i rotates $90°$ anticlockwise but does not change the length.

(e) Roots $1-2i$, $1+2i$ third

sum of roots $1-2i+1+2i+\alpha = 3$

$\therefore \alpha = 1$ (2)

$(x-1+2i)(x-1-2i)(x-1)$

(a)(i) $a^2 = 25 \quad b^2 = 16$

$16 = 25(1-e^2)$

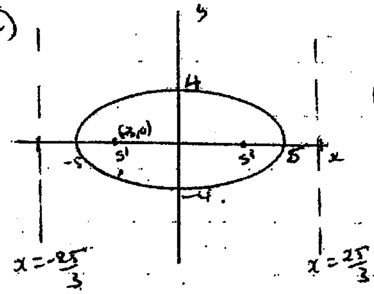
$e^2 = 1 - \frac{16}{25}$

$e = 3/5$ (1)

(ii) $S(3,0) \quad S'(-3,0)$

$x = \pm \frac{25}{3}$ (2)

(iii)



(1)

(iv) By definition

$PS = e\,PM$ and $PS' = e\,PM'$

$SP + S'P = e(PM + PM')$
$= e(M'M)$
$= e \cdot \frac{2a}{e}$
$= 2a$

which is independent of P
or in this case

$SP + SP' = 10.$

(b)(i) $\frac{dx}{d\theta} = a\sec\theta\tan\theta$

$\frac{dy}{d\theta} = b\sec^2\theta$

$\frac{dy}{dx} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta}$

$= \frac{b\sec\theta}{a\tan\theta}$

$y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$

$a\tan\theta y - ab\tan^2\theta = b\sec\theta x - ab\sec^2\theta$

$-a\tan\theta y + bx\sec\theta = ab(\sec^2\theta - \tan^2\theta)$

$\sec^2\theta - \tan^2\theta = 1$ (2)

$bx\sec\theta - ay\tan\theta = ab$

(ii) $\frac{dy}{dx} = -\frac{a\tan\theta}{b\sec\theta}$

$y - b\tan\theta = \frac{-a\tan\theta}{b\sec\theta}(x - a\sec\theta)$

$b\sec\theta y - b^2\tan\theta\sec\theta = -ax\tan\theta + \frac{a^2\tan\theta}{\sec\theta}$

$\div \tan\theta\sec\theta$

$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$ (2)

(iii) when $x = 0$

$A\left(0, -\frac{b}{\tan\theta}\right)$ tangent

$B\left(0, \frac{(a^2+b^2)\tan\theta}{b}\right)$ normal

(iv) focus of ~~pa~~ hyperbola

$S(ae, 0)$

If AB diameter $A\hat{B}S$ is a right angle

$m_{AS} = \frac{~~ab\tan\theta~~}{ae\tan\theta}$

$m_{BS} = \frac{(a^2+b^2)\tan\theta}{abe}$

$m_{AS} \times m_{BS} = \frac{b}{ae\tan\theta} \times \frac{(a^2+b^2)\tan\theta}{abe}$

$= \frac{(a^2+b^2)}{a^2e^2}$

$e^2 = 1 + \frac{b^2}{a^2}$

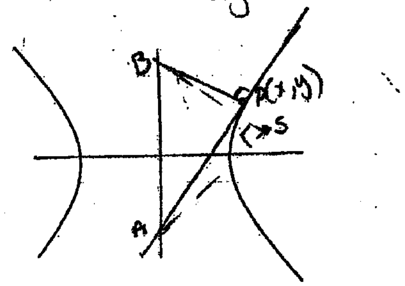
$e^2 = \frac{b^2}{a^2} + 1$

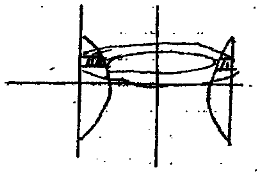
$= \frac{b^2+a^2}{a^2}$ (3)

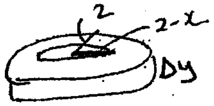
$\therefore m_{AS} \times m_{BS} = \frac{-a^2}{a^2+b^2} \times \frac{a^2+b^2}{a^2}$

$= -1$

so AB is diameter of circle passes through foci.







Now $y^2=4-4x$

$4x=4-y^2$

$x=\frac{4-y^2}{4}$

$\Delta V = \left[\pi 2^2 - \pi(2-x)^2\right]\Delta y$

$V = \lim_{\Delta y \to 0} \sum_{y=-2}^{2} 4\pi - \pi\left(2-\frac{4-y^2}{4}\right)^2 \Delta y$

$= \int_{-2}^{2} 4\pi - \pi\left(2-1+\frac{y^2}{4}\right)^2 dy$ (4)

$= \pi\int_{-2}^{2} 4 - \left(1+\frac{y^2}{2}+\frac{y^4}{16}\right)dy$

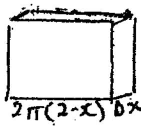
$= 2\pi\int_0^2 3 - \frac{y^2}{2} - \frac{y^4}{16}\, dy$

$= 2\pi\left[3y - \frac{y^3}{6} - \frac{y^5}{80}\right]_0^2$

$= 2\pi\left[6 - \frac{8}{6} + \frac{32}{80}\right]$

$= \underline{8\tfrac{8}{15}\pi}$

(ii) a)



$\Delta y = 2y \times 2\pi(2-x)\,\Delta x$

$V = \lim_{\Delta x \to 0} \sum_{x=0}^{x=1} 4\pi y(2-x)\,\Delta x$

$V = \int_0^1 4\pi\sqrt{4-4x}\,(2-x)\,dx$

$= 8\pi\int_0^1 \sqrt{1-x}\,(2-x)\,dx$ (2)

b) $u = 1-x \quad \frac{du}{dx} = -1$

$x=0\ u=1$ when $x=1\ u=0$

$8\pi\int_1^0 (u+1)\sqrt{u}\ -du$

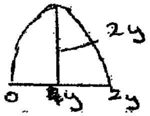
$= 8\pi\int_0^1 u^{3/2} + u^{1/2}\, du$

$= 8\pi\left[\frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right]_0^1$

$= 8\pi\left[\frac{2}{5}+\frac{2}{3}\right]$ (3)

$= \underline{8\tfrac{8}{15}\pi}$

(b)



(i) Simpson's rule is derived by fitting a parabolic arc to a set of 3 points. Hence it will give exact area for a parabola. (1)

(ii) $A = \frac{y}{3}\left[f(0) + 4f(y) + f(2y)\right]$

$= \frac{y}{3}\left[0 + 8y + 0\right]$

$= \frac{8y^2}{3}$ (2)

(iii) $\Delta V = \frac{8y^2}{3}\Delta x$

$x^2+y^2=100$

$y^2 = 100-x^2$

$\int_{-10}^{10} 8\frac{(100-x^2)}{3}\,dx$

$2\int_0^{10} \frac{8}{3}(100-x^2)\,dx$

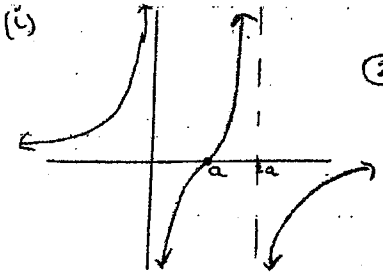
$= \frac{16}{3}\left[100x - \frac{x^3}{3}\right]_0^{10}$

$= \frac{16}{3}\left[100\times10 - \frac{1000}{3}\right]$ (3)

$= 3555\tfrac{5}{9}$

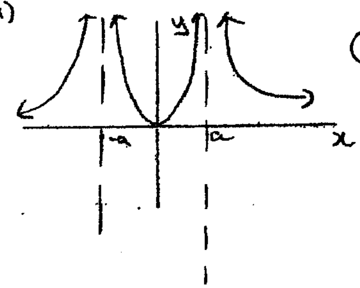
QUESTION 5

a) (i)



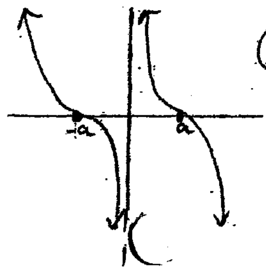
(2)

(ii)



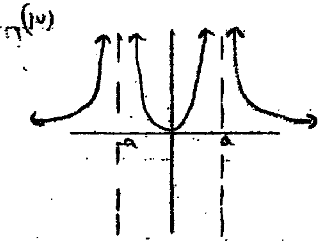
(2)

(iii)



(2)

(iv)



(2)

(b) when $x=-1$

$-1+4-a+b=0$

$\underline{a-b=3}$ (3)

$f'(x) = 3x^2+8x+a$

$x=-1 \quad f'(-1) = 3-8+a = 0$

$\therefore \quad \underline{a=5 \quad b=2}$

(c) (i) $kx^3 + kx - 2x = 0$

$2x = kx^3 + kx$

$2x = kx(x^2+1)$ (2)

$\frac{2x}{x^2+1} = kx$

(ii) Look at intersection of $y_1 = kx$ with $y_2 = \frac{2x}{x^2+1}$

If k negative only goes through (0, 0)

$\frac{dy_2}{dx} = \frac{(1+x^2)2 - (2x)(2x)}{(1+x^2)^2}$

when $x=0 \quad \frac{dy}{dx} = 2$

from $k=2$ to y-axis only touches at one point

$k \geq 2$ and $k \leq 0$

(3)

(a) $m\ddot{x} = -\frac{1}{100} mgv^2 - mg$

~~[illegible]~~

↑ ↓mg ↓ $\frac{1}{100} mgv^2$

$\ddot{x} = -\frac{1}{100} gv^2 - g$ (2)

$= -\frac{1}{100} g(100+v^2)$

$v\frac{dv}{dx} = -\frac{1}{100} g(100+v^2)$

$\frac{dv}{dx} = -\frac{g(100+v^2)}{100v}$

$\frac{dx}{dv} = -\frac{100v}{g(100+v^2)}$

$x = -\frac{50}{g}\ln(100+v^2) + c$

when $x=0$ $v=u$

$0 = -\frac{50}{g}\ln(100+u^2) + c$

$c = \frac{50}{g}\ln(100+u^2)$ (4)

$x = \frac{50}{g}\ln\left(\frac{100+u^2}{100+v^2}\right)$

when $v=0$ greatest height

$x = \frac{50}{g}\ln\left(\frac{100+u^2}{100}\right)$

(b)(i) since $z^3 = 1$

Roots are $1, \omega, \omega^2$

sum of roots are $1+\omega+\omega^2 = 0$

$(-\omega^2)^2 = \omega^4$
$= (\omega^3)\omega$
$= \omega$ (1)

(iii) $(1+\omega)^3 = (-\omega^2)^3$
$= -\omega^6$
$= -(\omega^3)^2$
$= -1$ (1)

(iv) $(1+\omega)^{3n} = (-1)^n$

$(1+\omega)^{3n} = {}^{3n}C_0 + {}^{3n}C_1\omega + \dots + {}^{3n}C_{3n}\omega^{3n}$

using $\omega^3 = 1$

${}^{3n}C_0 + {}^{3n}C_1\omega + {}^{3n}C_2\omega^2 + {}^{3n}C_3 + \dots {}^{3n}C_{3n}$

Take Re part of both sides

${}^{3n}C_0 + {}^{3n}C_1 Re(\omega) + {}^{3n}C_2 Re(\omega^2) + \dots {}^{3n}C_{3n} = (-1)^n$

${}^{3n}C_0 - \frac{1}{2}({}^{3n}C_1 + {}^{3n}C_2) + {}^{3n}C_3 + \dots {}^{3n}C_{3n} = (-1)^n$

using hint. (3)

c(i) $\log_{a^n} x = \frac{\log_a x}{\log_a a^n}$
$= \frac{\log_a x}{n}$ (1)

(ii) $\log_2 3 + \log_4 3 + \log_{16} 3 + \dots$

$= \log_2 3 + \frac{1}{2}\log_2 3 + \frac{1}{4}\log_2 3 + \frac{1}{8}\log_2 3 + \dots$

$= \log_2 3\left[1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots\right]$

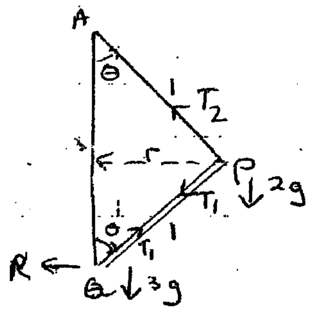
$1 + \frac{1}{2} + \frac{1}{4} + \dots$

S_∞ of G.P $= \frac{1}{1-\frac{1}{2}}$
$= 2$ (2)

$\therefore = 2\log_2 3$

QUESTION 7

(c)



(1)

(ii) A+Q $T_1\sin\theta = R$

$T_1\cos\theta = 3g$ — A
vertical

A+P vertical

$T_1\cos\theta + 2g = T_2\cos\theta$ — B

horizontal

$T_1\sin\theta + T_2\sin\theta = 2r\omega^2$
$= 2r(2\pi)^2$

$T_1\sin\theta + T_2\sin\theta = 8\pi^2 r$ — C (3)

(iii) $T_1 = \frac{3g}{\cos\theta}$ from A

sub in B

$3g + 2g = T_2\cos\theta$

$\therefore T_2 = \frac{5g}{\cos\theta}$ (4)

sub in C

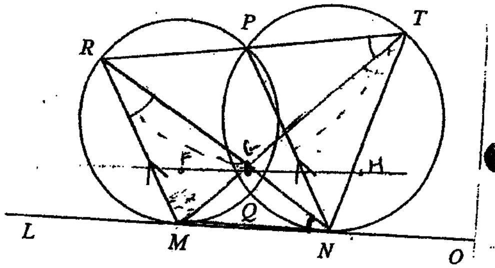
$\frac{3g}{\cos\theta}\sin\theta + \frac{5g}{\cos\theta}\sin\theta = 8\pi^2 r$

$\therefore 8g\tan\theta = 8\pi^2 r$

$\tan\theta = \frac{\pi^2 r}{9.8}$

Now $\frac{r}{h} = \tan\theta$

$\therefore h = r \div \frac{\pi^2 r}{9.8}$



(i) Let $R\hat{M}N = \theta$

~~$R\hat{N}M = 180 - \theta$ (cointerior angles MR ∥ ...)~~

$P\hat{N}M = 180 - \theta$ (cointerior angles MR ∥ NP)

$P\hat{T}N = 180 - \theta$ (alternate segment theorem)

$R\hat{M}N + P\hat{T}N = 180$ (4)

∴ RTMN cyclic quad opposite angles supplementary

(ii) Let tangent be FGH.

$R\hat{G}F = R\hat{T}M$ (alternate segment theorem)

$R\hat{T}M = R\hat{N}M$ (angles in same segment)

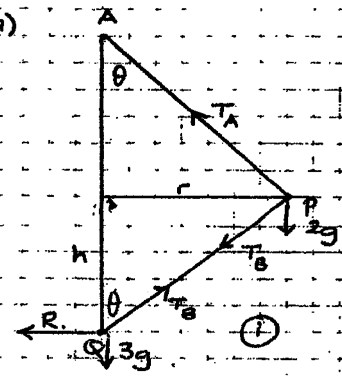
(MNTR cyclic quad)

$\therefore R\hat{G}F = R\hat{N}M$

∴ FG ∥ MN (pair of equal corresponding angles) (3)

Question 7

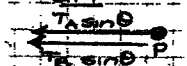
a)



①

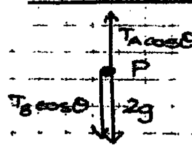
(ii) horizontal $= mr\omega^2$

at P



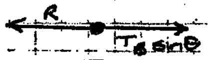
$T_A \sin\theta + T_B \sin\theta = mr\omega^2$ ①

vertical = 0



$T_A \cos\theta - T_B\cos\theta - 2g = 0$ ①

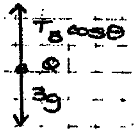
at Q



$T_B \sin\theta - R = mr\omega^2$

but $r = 0$

$\therefore T_B \sin\theta - R = 0$



$T_B \cos\theta - 3g = 0$ ①

(iii) $T_B \cos\theta = 3g$

$T_B = \frac{3g}{\cos\theta}$

$T_A \cos\theta - T_B \cos\theta = 2g$

$T_A \cos\theta - 3g = 2g$

$T_A \cos\theta = 5g$

$T_A = \frac{5g}{\cos\theta}$

$T_A \sin\theta + T_B \sin\theta = mr\omega^2$

$\frac{5g\sin\theta}{\cos\theta} + \frac{3g\sin\theta}{\cos\theta} = 2(r)(2\pi)^2$

$8g\tan\theta = 8\pi^2 r$

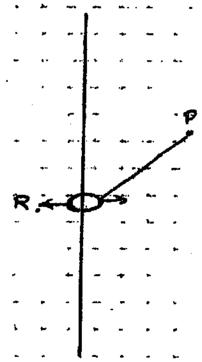
$\tan\theta = \frac{\pi^2 r}{g}$ ③

$\frac{r}{h} = \tan\theta$

$= \frac{\pi^2 r}{g}$

$h = \frac{g}{\pi^2}$

$= 0.99$ ②



as P moves it pulls on the ring, the ring thus exerts a horizontal force on the wire. By Newton's 3rd Law the wire exerts an equivalent force on the ring, call it R.

QUESTION 8

(i) $\frac{1}{x} = x^2 - 4$

$1 = x^3 - 4x$

$x^3 - 4x - 1 = 0$ ①

(ii) let $y = x^2$

$x = \sqrt{y}$

$(\sqrt{y})^3 - 4\sqrt{y} - 1 = 0$

$y\sqrt{y} - 4\sqrt{y} = 1$

$\sqrt{y}(y-4) = 1$

$y(y-4)^2 = 1$

$y(y^2 - 8y + 16) = 1$

$y^3 - 8y^2 + 16y - 1 = 0$ ②

(iii) let $y = 1/x$

$\left(\frac{1}{x}\right)^3 - 8\left(\frac{1}{x}\right)^2 + 16\left(\frac{1}{x}\right) - 1 = 0$

$1 - 8x + 16x^2 - x^3 = 0$ ②

$x^3 - 16x^2 + 8x - 1 = 0$

(iv) $P\left(\alpha, \frac{1}{\alpha}\right)$ $Q\left(\beta, \frac{1}{\beta}\right)$ $R\left(\gamma, \frac{1}{\gamma}\right)$

$OP^2 + OQ^2 + OR^2$

$\therefore \alpha^2 + \beta^2 + \gamma^2 + \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$

$= 8 + 16$

$= 24$ ②

(b)(i) $f(x) = x^3 - 4$

$f'(x) = 3x^2$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$x_{n+1} = x_n - \frac{x_n^3 - 4}{3x_n^2}$ ②

$= x_n - \frac{x_n}{3} + \frac{4}{3x_n^2}$

$= \frac{2}{3}\left[x_n + \frac{2}{x_n^2}\right]$

(ii) $\frac{(x_n - \sqrt[3]{4})^2(2x_n + \sqrt[3]{4})}{3x_n^2}$

$= \frac{[x_n^2 - 2\sqrt[3]{4}x_n + (\sqrt[3]{4})^2][2x_n + \sqrt[3]{4}]}{3x_n^2}$

$= \frac{2x_n^3 + \sqrt[3]{4}x_n^2 - 4\sqrt[3]{4}x_n^2 - 2(\sqrt[3]{4})^2x_n + 2(\sqrt[3]{4})^2x_n + (\sqrt[3]{4})^3}{3x_n^2}$

$= \frac{2x_n^3 - 3\sqrt[3]{4}x_n^2 + 4}{3x_n^2}$

$= \frac{2}{3}\left[x_n + \frac{2}{x_n^2}\right] - \sqrt[3]{4}$

$= x_{n+1} - \sqrt[3]{4}$ ③

(iii) Need to prove

$\frac{2x_n + \sqrt[3]{4}}{3x_n^2} < 1$

since $x_n > \sqrt[3]{4}$

$\frac{2x_n + \sqrt[3]{4}}{3x_n^2} < \frac{2x_n + x_n}{3x_n^2}$

$< \frac{1}{x_n}$

Since $x_n > \sqrt[3]{4}$

> 1

$\therefore \frac{1}{x_n} < 1$ ②

$x_{n+1} - \sqrt[3]{4} < (x_n - \sqrt[3]{4})^2$

(iv) $x_6 - \sqrt[3]{4} < (x_5 - \sqrt[3]{4})^2$

$< (x_4 - \sqrt[3]{4})^4$

$< (x_3 - \sqrt[3]{4})^8$

$< (2 - \sqrt[3]{4})^{32}$

$< 4.98 \times 10^{-13}$

∴ accurate to 12 decimal places has had 12 zeroes.

(1)