2010

TRIAL

HIGHER SCHOOL CERTIFICATE

GIRRAWEEN HIGH SCHOOL

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board – approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

Attempt Questions 1 – 8

All questions are of equal value

Total marks-120
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE piece of paper clearly marked Question 1, Question 2, etc. Each piece of paper must show your name.

Question 1 (15 Marks)

a) $\int \frac{dx}{\sqrt{4x^2-9}}$ **2**

b) $\int \frac{xdx}{\sqrt{4x^2-9}}$ **2**

c) (i) Find real numbers ***a, b*** and ***c*** such that $\frac{4x^2+7x+11}{(x+3)(x^2+4)} = \frac{a}{x+3} + \frac{bx+c}{x^2+4}$ **2**

(ii) Hence show $\int_0^2 \frac{4x^2+7x+11}{(x+3)(x^2+4)} = \ln\frac{50}{9} + \frac{\pi}{8}$ **2**

d) Use $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x}$ **2**

e) Find $\int \sqrt{x}\ln x dx$ **2**

f) Evaluate $\int_2^6 \frac{dx}{x\sqrt{2x-3}}$ **3**

Question 2 (15 marks)

a) (i) Find real numbers ***a*** and ***b*** such that $\sqrt{9-40i} = a+ib$ 2

(ii) Hence find the solutions to $z^2 - 3z + 10i = 0$ 2

b) (i) Sketch the graph of $|z-4i| = 2$ 2

(ii) Hence find the greatest and least values of $arg\, z$ 2

c) If $\omega = \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5}$ where k is an integer and $\omega \neq 1$

(i) Show that $\omega^n + \omega^{-n} = 2\cos\frac{2nk\pi}{5}$ 2

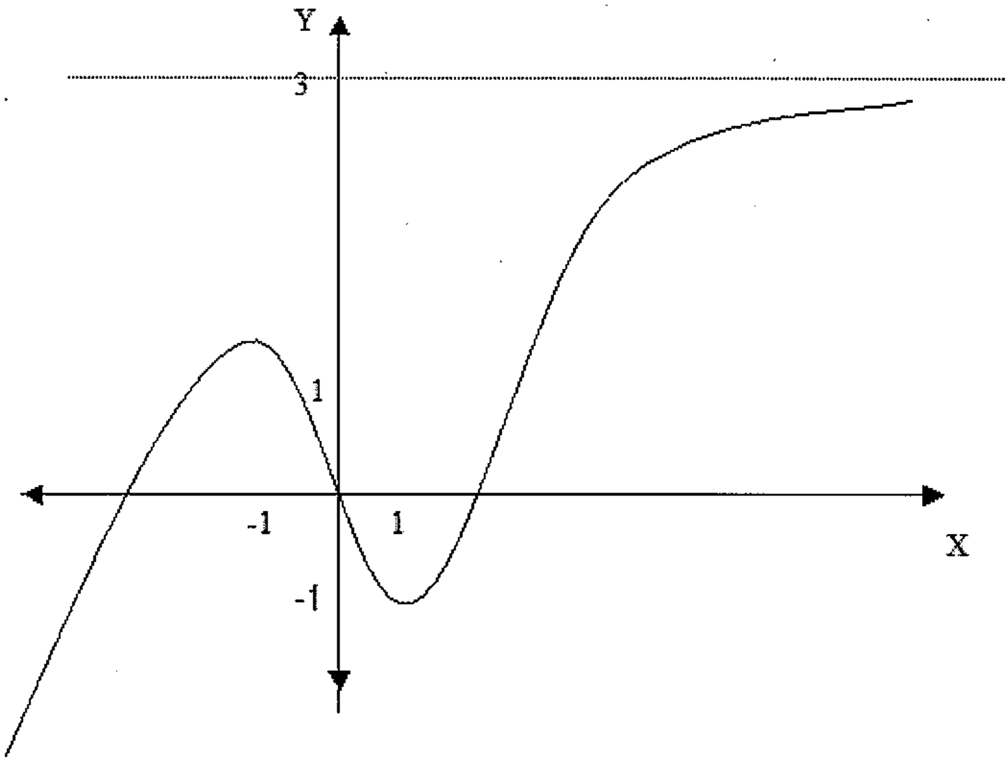
(ii) Show that $\omega^5 = 1$ 1

(iii) Show that $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$ 1

(iv) Hence or otherwise show that $(\omega + \omega^{-1})^2 + (\omega^2 + \omega^{-2})^2 = 3$ 2

(v) Deduce that $(\cos\frac{2k\pi}{5})^2 + (\cos\frac{4k\pi}{5})^2 = \frac{3}{4}$ 1

Question 3 (15 Marks)



a) The graph of $y = f(x)$ is shown above. It has a local maximum at $x = -1$ and a local minimum at $x = 1$ the curve asymptotes to $y = 3$. Draw neat sketches of the following.

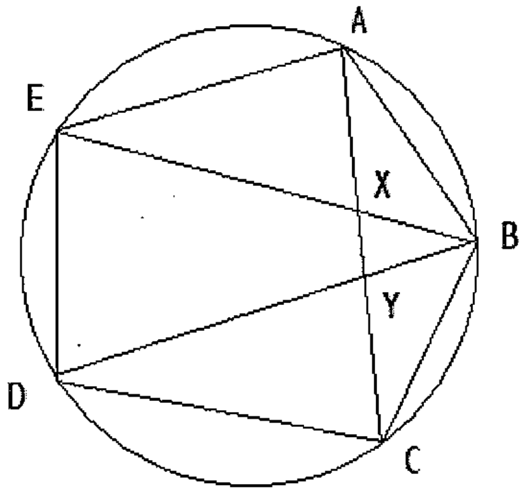
(i) $y = \ln f(x)$ 2

(ii) $y = e^{f(x)}$ 2

(iii) $y = f'(x)$ 2

(iv) $y = f(\frac{1}{x})$ 2

Question 3 (continued)



b) The pentagon ABCDE is inscribed inside the circle, with BA = BC. The diagonal AC meets the diagonals BE and BD at X and Y respectively.

(i) Show that $\angle BCA = \angle BEC$ 1

(ii) Prove that EDYX is a cyclic quadrilateral 3

c) The ellipse E has equation $\frac{x^2}{16}+\frac{y^2}{9}=1$ The point $P(x_1, y_1)$ lies on the ellipse

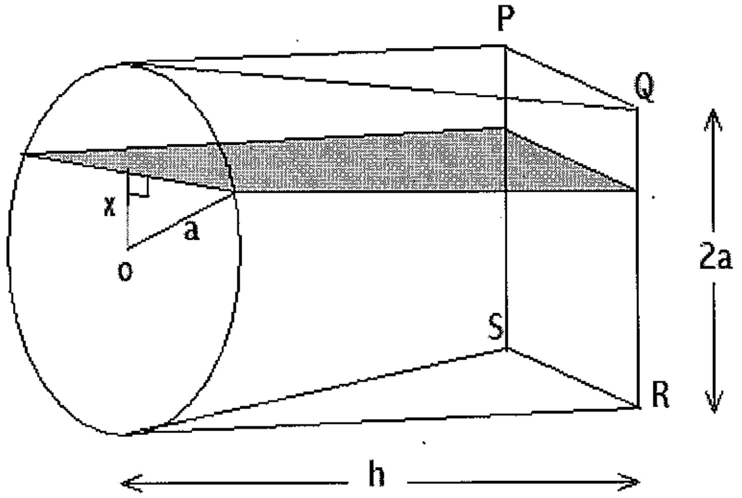
(i) Find the equation of the tangent at P to E. 2

(ii) The chord of contact to the ellipse has equation $\frac{x_0 x}{16}+\frac{y_0 y}{9}=1$

(there is no need to derive this). Show that if the chord passes through the focus, (x_0, y_0) lies on the directrix. 1

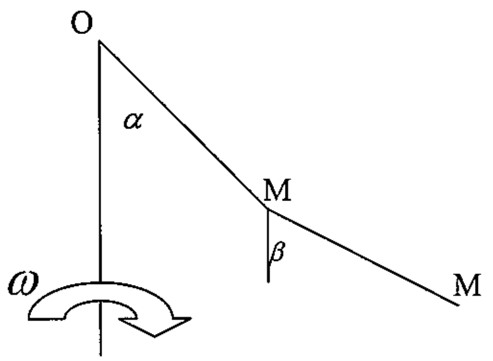
Question 4 (15 Marks)

a) The diagram below shows a solid of length h. It has a circular end of radius a units and the other end is the square PQRS of side $2a$. Horizontal cross-sections parallel to the base of the solid are taken at x as marked on the diagram.



(i)	Find the area of the slice at x.	2
(ii)	Express the volume of the solid as an integral	1
(iii)	Find the volume of the solid in (ii)	3

b) A particle hangs by a light inextensible string of length $\boldsymbol{a}$ from a fixed point O and a second particle of equal mass hangs from the first by an equal string. The whole system moves with with constant angular speed ω about the vertical through O, the upper and lower strings making constant angles α and β respectively with the vertical.



(i) Resolve forces vertically and horizontally for both masses m 4

(ii) Show that $\tan\beta = p(\sin\alpha + \sin\beta)$ 3

(iii) Show that $\tan\alpha = p(\sin\alpha + 0.5\sin\beta)$ 2

Where $p = \dfrac{a\omega^2}{g}$

Question 5 (15 Marks)

a) Given the locus $\frac{x^2}{9-k}+\frac{y^2}{4-k}=1$, with k < 4 Find:

(i) The eccentricity. 2

(ii) The coordinates of the foci. 1

(iii) The equation of the directrices. 1

b) Given the locus $\frac{x^2}{9-k}+\frac{y^2}{4-k}=1$, with 4 < k < 9 Find:

(i) The eccentricity. 2

(ii) The coordinates of the foci. 1

(iii) The equation of the directrices. 1

c) A vehicle rounds a banked track of radius 600m, inclined at an angle of θ to the horizontal. When the car travels at a speed of 10m/s the friction force up the track is equal to the friction force down the track when the vehicle travels at 20m/s. Gravity g = 9.8m/s.

(i) Resolve the forces in mutually perpendicular directions at 10m/s. 2

(ii) Resolve the forces in mutually perpendicular directions at 20m/s. 2

(iii) Find the angle θ at which the track is banked. 2

(iv) Find the speed the car travels to experience no friction force. 1

Hint the mutually perpendicular directions may be parallel and perpendicular to the track or perpendicular and horizontal.

Question 6. (15 Marks)

a) If α, β and γ are the roots of the cubic equation $x^3 - px^2 + qx - r = 0$. Find in terms of p , q and r

(i) $\alpha + \beta + \gamma$ 1

(ii) $\alpha^2 + \beta^2 + \gamma^2$ 2

(iii) $\alpha^3 + \beta^3 + \gamma^3$ 2

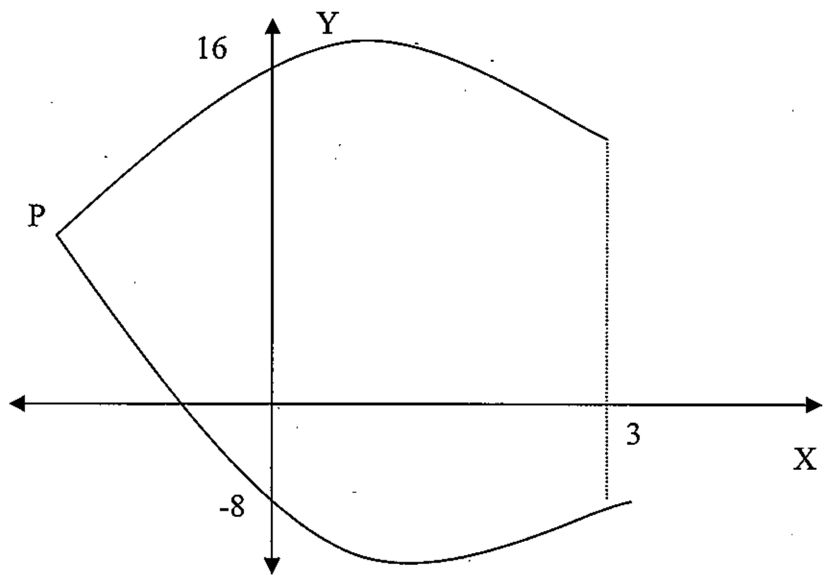
(iv) Hence find a solution to the set of equations 4

$X + Y + Z = -1$

$X^2 + Y^2 + Z^2 = 5$

$X^3 + Y^3 + Z^3 = -7$

b)



The region bounded by the curves $y = 16 - x^2$ and $y = x^2 - 2x - 8$ and the line x = 3 is rotated about the line x = 3. The point P is the point of intersection of the curves $y = 16 - x^2$ and $y = x^2 - 2x - 8$ in the second quadrant.

(i) Find the coordinates of P. 1

(ii) Use the method of cylindrical shells to express the volume of the resulting solid of revolution as an integral 3

(iii) Evaluate the integral in (ii) 2

Question 7 (15 Marks)

a) The sequence of numbers $u_1, u_2, u_3, u_4, \ldots\ldots u_n$ is defined as follows

$u_1 = 1,\ u_2 = 1$ and $u_n = u_{n-1} + u_{n-2}$ for $n > 3$

Prove that for every positive integer n, $u_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$

Where α and β $(\alpha > \beta)$ are the roots of $x^2 - x - 1 = 0$

(Hint in step 1. prove true for n = 1 and n = 2) 4

b) (i) Show that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ 2

(ii) Hence find the value of $\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\sin x + \cos x}\, dx$ 2

c) If n is a positive integer prove $\left(\frac{1+i\tan\theta}{1-i\tan\theta}\right)^n = \frac{1+i\tan n\theta}{1-i\tan n\theta}$ 2

d) The ellipse E has equation $2y^2 - 3xy + 2x^2 = 14$

(i) Using implicit differentiation or otherwise find an expression for the first derivative. 1

(ii) Find the coordinates of any turning points of E 2

(iii) Find the coordinates of any vertical tangents to E 2

Question 8 (15 Marks)

a) (i) Prove that $\dfrac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}=\sin\theta+i\cos\theta$ 2

(ii) Hence find the smallest value of θ such that

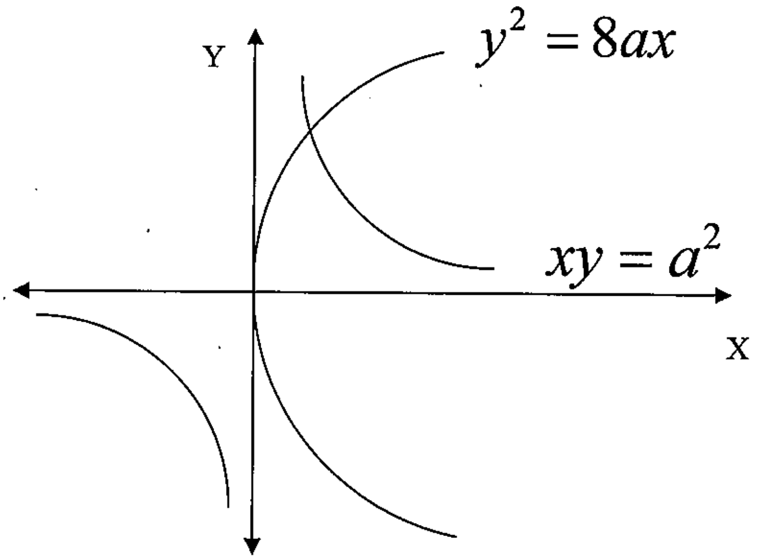
$(1+\sin\theta+i\cos\theta)^5+i(1+\sin\theta-i\cos\theta)^5=0$ 2

b) Let $I_n=\int_0^1 x(1-x)^n\,dx$ n = 0,1,2,3 …

(i) Show that $I_n=\dfrac{n}{n+2}I_{n-1}$ 3

(ii) Show that $I_n=\dfrac{1}{2\left({}^{n+2}C_2\right)}$ 2

c)



Given the hyperbola $xy=a^2$ (H)

and the parabola $y^2=8ax$ (P)

(i) Find the coordinates of A the point of intersection of H and P 1

(ii) If x + y + k = 0 is the common tangent to H and P, find k 2

(iii) Find the points of contact B on P and C on H 2

(iv) Show that AB is a tangent to H at A 1

SOLUTIONS EXT 2 2010

Q1 a) $\displaystyle\int \frac{dx}{\sqrt{4x^2-9}} = \frac{1}{2}\int \frac{dx}{\sqrt{x^2-\left(\frac{3}{2}\right)^2}}$ (2)

$= \frac{1}{2}\ln\left(x + \sqrt{x^2 - \frac{9}{4}}\right) + C$

b) $\displaystyle\int \frac{x\,dx}{\sqrt{4x^2-9}} = I$

let $u = 4x^2 - 9$

$du = 8x\,dx$

$I = \frac{1}{8}\displaystyle\int \frac{8x\,dx}{\sqrt{4x^2-9}}$

$= \frac{1}{8}\displaystyle\int \frac{du}{\sqrt{u}}$

$= \frac{1}{4}\sqrt{u} + C$

$= \frac{1}{4}\sqrt{4x^2-9} + C$ (2)

c) (i) $\dfrac{4x^2+7x+11}{(x+3)(x^2+4)} = \dfrac{a}{x+3} + \dfrac{bx+c}{x^2+4}$

$4x^2+7x+11 = a(x^2+4) + (x+3)(bx+c)$

IF $x=-3$.

$36-21+11 = 13a$

$\boxed{a = 2}$.

$x = 0$

$11 = 8 + 3c$

$\boxed{c = 1}$ (2)

$x = 1$

$22 = 10 + 4b + 4$

$\boxed{b = 2}$.

c) (ii) $\displaystyle\int_0^2 \frac{4x^2+7x+11}{(x+3)(x^2+4)}$

$= \displaystyle\int_0^2 \frac{2}{x+3} + \frac{2x}{x^2+4} + \frac{1}{x^2+4}\,dx$

$= \left[2\ln(x+3) + \ln(x^2+4) + \frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^2$

$= 2\ln 5 + \ln 8 + \frac{\pi}{8} - 2\ln 3 + \ln 4$

$= \ln\frac{25\times 8}{9\times 4} + \frac{\pi}{8}$

$= \ln\frac{50}{9} + \pi/8$ (2)

d) $t = \tan\frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$= \frac{1}{2}\left(\tan^2\frac{x}{2} + 1\right)$

$\frac{dt}{dx} = \frac{1}{2}(t^2+1)$

$\frac{2dt}{1+t^2} = dx$

$\sin x = \frac{2t}{1+t^2}$

$\displaystyle\int_0^{\pi/2} \frac{dx}{1+\sin x} = \int_0^1 \frac{\frac{2dt}{1+t^2}}{1+\frac{2t}{1+t^2}}$

$= \displaystyle\int_0^1 \frac{2\,dt}{1+2t+t^2} = \int_0^1 \frac{2dt}{(1+t)^2}$

$= \left[\frac{-2}{1+t}\right]_0^1$

$= 1$ (2)

e) $\int \sqrt{x} \ln x \, dx$

let $u = \ln x \quad dv = x^{1/2}$

$du = \frac{1}{x} \quad v = \frac{2}{3} x^{3/2}$

$\int u\,dv = uv - \int v\,du$ (2)

$= \frac{2}{3} x^{3/2} \ln x - \frac{2}{3}\int x^{3/2} \cdot \frac{1}{x} dx$

$= \frac{2}{3} x\sqrt{x} \ln x - \frac{4}{9} x^{3/2} + C$

f) $\int_2^6 \frac{dx}{x\sqrt{2x-3}} = I$

let $u = (2x-3)^{\frac{1}{2}}$

$\frac{du}{dx} = \frac{1}{(2x-3)^{1/2}} \quad du = \frac{dx}{\sqrt{2x-3}}$.

Now $2x - 3 = u^2$

$2x = u^2 + 3 \quad x = \frac{1}{2}(u^2+3)$

when $x = 2 \quad u = 1$, $x = 6 \quad u = 3$.

$I \Rightarrow \int_1^3 \frac{2\,du}{u^2+3}$ (3)

$= \left[\frac{2}{\sqrt{3}} \tan^{-1} \frac{u}{\sqrt{3}}\right]_1^3 = \frac{\pi}{3\sqrt{3}}$

Question 2

a) i) $\sqrt{9-40i} = a + bi$

$9 - 40i = a^2 - b^2 + 2abi$

$9 = a^2 - b^2$

$-20 = ab$

$a = \pm 5 \quad b = \mp 4$

$\pm(5-4i)$ (2)

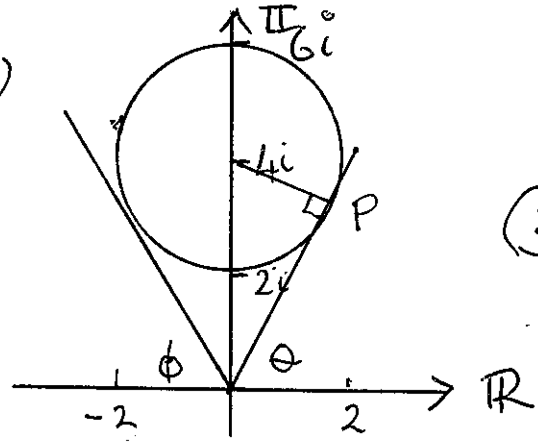
ii) $z^2 - 3z + 10i = 0$

$z = \frac{+3 \pm \sqrt{9-40i}}{2}$

$z = \frac{+3 \pm (5-4i)}{2}$

$z = 4 - 2i, \; -1 + 2i$ (2)

b) i)



(2)

ii) FROM DIAGRAM TANGENT AT P

$\sin(90° - \theta) = \frac{2}{4}$

$90 - \theta = 30$

$\theta = 60°$ (2)

$\therefore$ BY SYMMETRY $\phi = 60°$

$\therefore$ MIN ARG $= 60°$ MAX ARG $= 120°$.

c) $w^n + w^{-n}$

DE MOIVRE THEOREM

$= \cos\frac{2kn\pi}{5} + i\sin\frac{2kn\pi}{5}$

$+ \cos -\frac{2kn\pi}{5} + i\sin -\frac{2kn\pi}{5}$

$\cos\theta$ is an ~~odd~~ even function

$\sin\theta$ is an odd function

$= \cos\frac{2kn\pi}{5} + i\sin\frac{2kn\pi}{5}$

$+ \cos\frac{2kn\pi}{5} - i\sin\frac{2kn\pi}{5}$

$= 2\cos\frac{2kn\pi}{5}$. (2)

Q2 continued

c)(ii) $w^5 = \cos\frac{2k\pi 5}{5} + i\sin\frac{2k\pi 5}{5}$

$= \cos 2k\pi + i\sin 2k\pi$

$= 1.$ (1)

(iii) $w^5 = 1$

$w^5 - 1 = 0$ (1)

$(w-1)(1+w+w^2+w^3+w^4) = 0$

$w \neq 1 \therefore 1+w+w^2+w^3+w^4 = 0.$

(iv) $(w+w^{-1})^2 + (w^2+w^{-2})^2$

$= w^2 + 2 + w^{-2} + w^4 + 2 + w^{-4}$

$= 4 + w^{-4}w^5 + w^2 + w^{-2}w^5 + w^4$

as $w^5 = 1$

$= 4 + w + w^2 + w^3 + w^4$

$= 3 + 1 + w + w^2 + w^3 + w^4$

$= 3 + 0 = 3.$ (2)

v) $w + w^{-1} = 2\cos\frac{2k\pi}{5}$

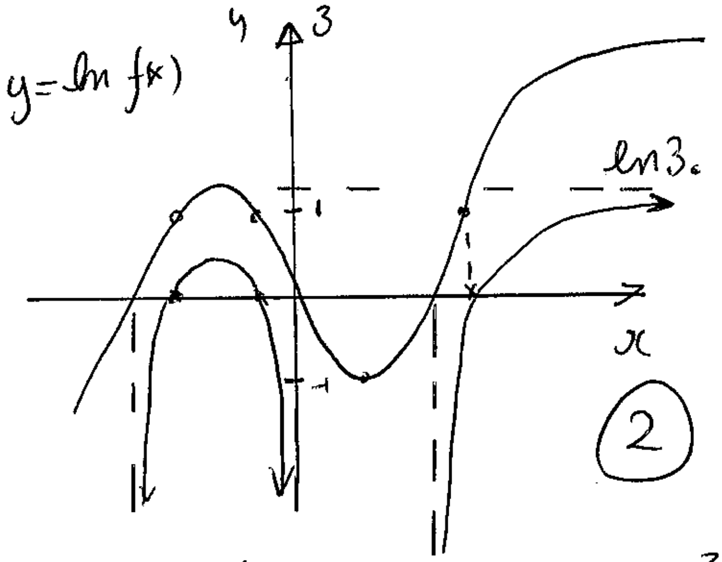
$w^2 + w^{-2} = 2\cos\frac{4k\pi}{5}$

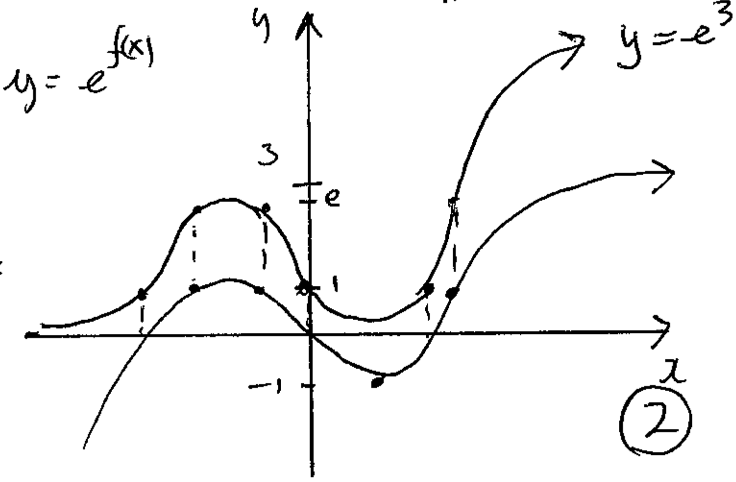
$\therefore \left(2\cos\frac{2k\pi}{5}\right)^2 + \left(2\cos\frac{4k\pi}{5}\right)^2 = 3$ part (iv)

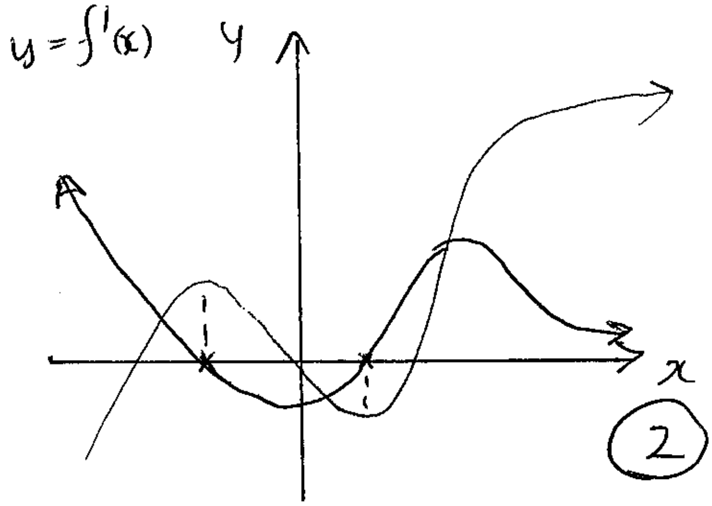
$\therefore 4\cos^2\frac{2k\pi}{5} + 4\cos^2\frac{4k\pi}{5} = 3$

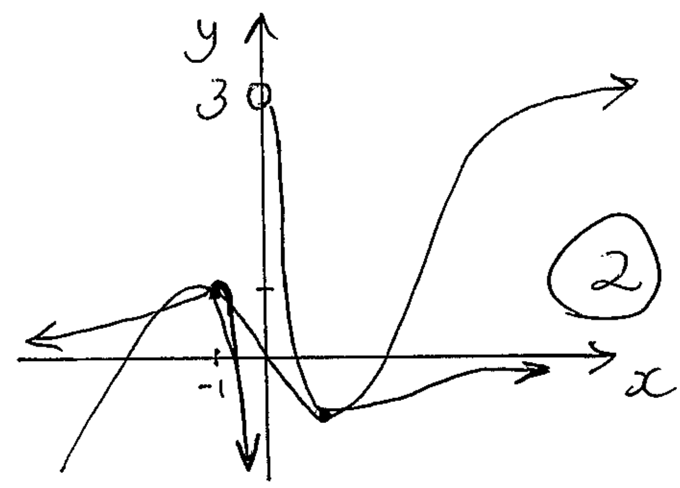
$\cos^2\frac{2k\pi}{5} + \cos^2\frac{4k\pi}{5} = \frac{3}{4}$ (1)

Question 3.









Question 3 (cont.)

$\angle BCA = \angle BEC$ Part (i)

$\angle CBD = \angle CED$ ($\angle$'s subtended at the circumference by same arc.

$\therefore \angle BCA + \angle CBD = \angle BEC + \angle CED$

But $\angle AYB = \angle BCA + \angle CBD$

(EXT $\angle$ IS EQUAL TO SUM INTERIOR OPPOSITE $\angle$'s)

$\therefore \angle AYB = \angle BEC + \angle CED$

$\angle AYB = \angle BED$ (ADDITION OF ADJACENT $\angle$'s) 3

$\therefore$ EDYX IS A CYCLIC QUAD (INT $\angle$ EQUAL EXT OPPOSITE $\angle$).

c) (i) $\frac{x^2}{16} + \frac{y^2}{9} = 1$

$\frac{2x}{16} + \frac{2y}{9}\frac{dy}{dx} = 0$

$\frac{2y}{9}\frac{dy}{dx} = \frac{-2x}{16}$

$\frac{dy}{dx} = \frac{-9x}{16y}$

EQ^N OF TANGENT

$Y - Y_1 = \frac{-9X_1}{16Y_1}(X - X_1)$

$\frac{Y_1Y - Y_1^2}{9} = \frac{X_1^2 - X_1X}{16}$

$\frac{X_1X}{16} + \frac{Y_1Y}{9} = \frac{X_1^2}{16} + \frac{Y_1^2}{9}$

But (X_1, Y_1) LIES ON E

$\therefore \frac{X_1X}{16} + \frac{Y_1Y}{9} = 1.$ 2

(ii) FOCUS HAS CO-ORDINATES

$S(ae, 0) = S(\sqrt{7}, 0)$

$\frac{x_0x}{16} + \frac{y_0y}{9} = 1$

$\frac{\sqrt{7}x_0}{16} + \frac{0(y_0)}{9} = 1$

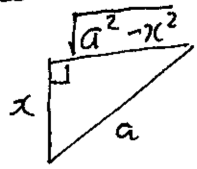
$x_0 = \frac{16}{\sqrt{7}}$

equation of directrix

$x = \frac{a}{e} = \frac{4}{\frac{\sqrt{7}}{4}} = \frac{16}{\sqrt{7}}$

(x_0, y_0) LIES ON DIRECTRIX 1.

Question 4.

a) 1) 

$\therefore$ length $= 2\sqrt{a^2-x^2}$ 2

$\text{Area}_{\text{TRAP}} = \frac{1}{2}(a+b)h$
$= \frac{1}{2}\left(2\sqrt{a^2-x^2} + 2a\right)h.$
$= \left(\sqrt{a^2-x^2} + a\right)h$

Q4 (a) (cont.)

(ii) $\delta Vol = \left(\sqrt{a^2-x^2} + a\right) h\,\delta x$

$$Vol = \lim_{\delta x \to 0} 2 \sum_{x=0}^{a} \left(\sqrt{a^2-x^2} + a\right) h\,\delta x$$

$$V = 2h \int_0^a \sqrt{a^2-x^2} + a \, dx \qquad (1)$$

(iii) $$V = 2h \int_0^a \sqrt{a^2-x^2}\, dx + 2h \int_0^a a \, dx$$

Now $\int_0^a \sqrt{a^2-x^2}\, dx$ IS THE 1/4 CIRCLE $= \frac{\pi a^2}{4}$

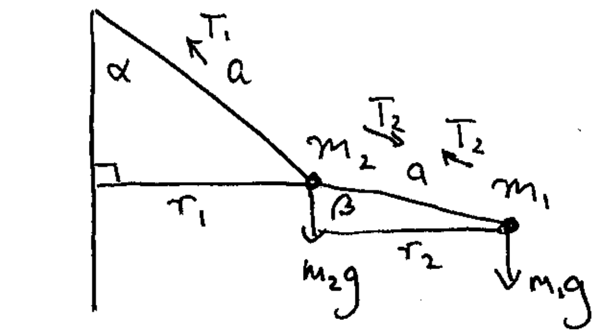
$$V = 2h \frac{\pi a^2}{4} + 2h \left[ax\right]_0^a$$

$$V = 2h \frac{\pi a^2}{4} + 2h a^2$$

$$V = 2ha^2\left(\frac{\pi}{4} + 1\right) u^3 \qquad (3)$$

(b) at m_1 (LOWER mass)

Vertically



$T_2 \cos\beta = mg$ (A) (1)

Horizontally

$T_2 \sin\beta = m r \omega^2$ (1)

$= ma(\sin\alpha + \sin\beta)\omega^2$ (B)

at m_2 (upper mass)

Vertically (1)

$T_1 \cos\alpha - T_2 \cos\beta = mg$ (C)

HORIZONTALLY

$T_1 \sin\alpha - T_2 \sin\beta = m r \omega^2$

$T_1 \sin\alpha - T_2 \sin\beta = ma \sin\alpha\, \omega^2$ (1) (D)

(ii) $\frac{(B)}{(A)}$ $\quad \frac{T_2 \sin\beta}{T_2 \cos\beta} = \frac{ma\omega^2(\sin\alpha + \sin\beta)}{mg}$

$$\tan\beta = \frac{a\omega^2}{g}(\sin\alpha + \sin\beta)$$

$$\tan\beta = p(\sin\alpha + \sin\beta) \qquad (3)$$

(iii) FROM (A)

$T_2 \cos\beta = mg$

$T_2 = \frac{mg}{\cos\beta}$

SUBST INTO (C)

$T_1 \cos\alpha - mg = mg$

$T_1 \cos\alpha = 2mg$

$T_1 = \frac{2mg}{\cos\alpha}$

SUBST INTO (D)

$2mg\tan\alpha - mg\tan\beta = ma\sin\alpha\,\omega^2$

$2\tan\alpha = \frac{a\omega^2}{g}\sin\alpha + \tan\beta$

$2\tan\alpha = \frac{a\omega^2}{g}\sin\alpha + p(\sin\alpha + \sin\beta)$

$2\tan\alpha = 2p\sin\alpha + p\sin\beta$ (2)

$\tan\alpha = p(\sin\alpha + 0.5\sin\beta)$

Question 5

(a) (i) $b^2 = a^2(1-e^2)$ <u>ellipse</u>

$$e^2 = \frac{a^2-b^2}{a^2}$$

$$e^2 = \frac{(9-K)-(4-K)}{9-K}$$

$$e^2 = \frac{5}{9-K} \qquad e = \frac{\sqrt{5}}{\sqrt{9-K}} \quad 2$$

(ii) Foci $(\pm ae, 0)$

$$\left(\pm \sqrt{9-K}\,\frac{\sqrt{5}}{\sqrt{9-K}}, 0\right) = \pm(\sqrt{5}, 0) \quad 1$$

(iii) DIRECTRICES $y = \dfrac{\pm a}{e}$

$$y = \frac{\pm\sqrt{9-K}}{\frac{\sqrt{5}}{\sqrt{9-K}}} = \frac{\pm(9-K)}{\sqrt{5}} \quad 1$$

(b) (i) $b^2 = a^2(e^2-1)$ hyperbola

$$e^2 = \frac{a^2+b^2}{a^2}$$

$$e^2 = \frac{9-K+(4-K)}{9-K}$$

$$e^2 = \frac{5}{9-K} \qquad e = \frac{\sqrt{5}}{\sqrt{9-K}} \quad 2$$

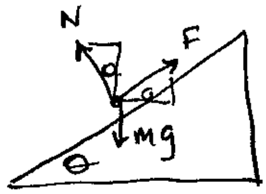
(ii) Foci $(\pm ae, 0)$

$$= \pm(\sqrt{5}, 0) \quad 1$$

(iii) DIRECTRICES $y = \dfrac{\pm a}{e}$

$$y = \pm\frac{9-K}{\sqrt{5}} \quad 1$$

(c)

(i) 

10M/S

Vertically

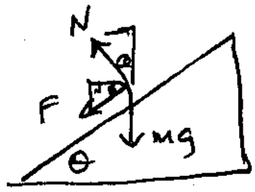
$$N\cos\theta + F\sin\theta = mg \quad A$$

horizontally

$$N\sin\theta - F\cos\theta = \frac{MV^2}{r}$$

$$N\sin\theta - F\cos\theta = M\frac{100}{600} \quad B$$

2

(ii) 

Vertically

$$N\cos\theta - F\sin\theta = mg \quad C$$

Horizontally

$$N\sin + F\cos\theta = \frac{MV^2}{r} \quad 2$$

$$N\sin\theta + F\cos\theta = M\frac{400}{600} \quad D$$

(iii) $A \times \sin\theta - B \times \cos\theta$

$$N\cos\theta\sin\theta + F\sin^2\theta = mg\sin\theta$$

$$-\; N\sin\theta\cos\theta - F\cos^2\theta = \frac{M}{6}\cos\theta$$

$$F = M\left(g\sin\theta - \frac{\cos\theta}{6}\right)$$

$D \times \cos\theta - C \times \sin\theta$

$$N\sin\theta\cos\theta + F\cos^2\theta = \frac{2M}{3}\cos\theta$$

$$-\; N\sin\theta\cos\theta - F\sin^2\theta = Mg\sin\theta$$

$$F = M\left(\frac{2\cos\theta}{3} - g\sin\theta\right)$$

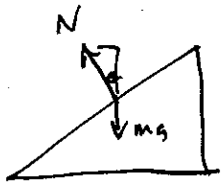
$$\therefore\ g\sin\theta - \frac{\cos\theta}{6} = \frac{2}{3}\cos\theta - g\sin\theta$$

$$2g\sin\theta = \tfrac{5}{6}\cos\theta$$

$$\tan\theta = \tfrac{5}{12g}$$

$\theta = 2.435^\circ$. (2)

(IV) WITHOUT FRICTION



$N\cos\theta = mg$

$N\sin\theta = \dfrac{mv^2}{r}$

$\tan 2.435 = \dfrac{v^2}{rg}$ (1)

$v^2 = 250$

$V = \sqrt{250} = 15.81\,m/s$

Question 6

(a) $x^3 - px^2 + qx - r = 0$

(i) $\alpha + \beta + \gamma = -b/a = p$ (1)

(ii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$= p^2 - 2q$ (2)

(iii) $\alpha^3 - p\alpha^2 + q\alpha - r = 0$

$\alpha^3 = p\alpha^2 - q\alpha + r$

$\beta^3 = p\beta^2 - q\beta + r$

$\gamma^3 = p\gamma^3 - q\gamma + r$

$\alpha^3 + \beta^3 + \gamma^3 = p(\alpha^2+\beta^2+\gamma^2) - q(\alpha+\beta+\gamma) + 3r$

$= p^3 - 2pq - qp + 3r$

$= p^3 - 3pq + 3r$ (2)

(iv) let $p = -1$

$p^2 - 2q = 5$

$p^3 - 3pq + 3r = -7$

$\therefore p = -1 \quad q = -2$

and $r = 0$.

x, y, z are the roots of

$x^3 + x^2 - 2x = 0$

$x(x^2 + x - 2) = 0$

$x(x+2)(x-1) = 0$

$\therefore x, y, z = 0, 1, -2$ (4)

IN ANY ORDER

b) (ii) $\delta V = \pi[R^2 - r^2]h$

$= \pi\left[(3-x)^2 - (3-(x+\delta x))^2\right] \cdot \left[(16-x^2) - (x^2 - 2x - 8)\right]$

$= \pi\left[\cancel{9} - \cancel{6x} + \cancel{x^2} - (\cancel{9} - \cancel{6x} - 6\delta x + \cancel{x^2} + 2x\delta x + \delta x^2)\right] \cdot \left[24 + 2x - 2x^2\right]$

$= \pi\left[6\delta x - 2x\delta x\right]\left[24 + 2x - 2x^2\right]$

$= 4\pi\left[3 - x\right]\left[12 + x - x^2\right]\delta x$

$V = \lim_{\delta x \to 0} 4\pi \sum_{-3}^{3} [3-x][12+x-x^2]\delta x$ (3)

$V = 4\pi \int_{-3}^{3} 36 - 9x - 4x^2 + x^3 \, dx$

(i) FOR P

$y = 16 - x^2$

$y = x^2 - 2x - 8$

$16 - x^2 = x^2 - 2x - 8$

$0 = 2x^2 - 2x - 24$

$0 = 2(x-4)(x+3)$

$P\ (-3, 7)$ (1)

Question 6 (cont)

b) (iii) $V = 4\pi \int_{-3}^{3} 36 - 9x - 4x^2 + x^3 \, dx$

$= 4\pi \left[36x - \frac{9x^2}{2} - \frac{4x^3}{3} + \frac{x^4}{4} \right]_{-3}^{3}$

$= 4\pi \left[\left(108 - \cancel{\frac{81}{2}} - 36 + \cancel{\frac{81}{4}}\right) - \left(-108 - \cancel{\frac{81}{2}} + 36 + \cancel{\frac{81}{4}}\right) \right.$

$= 4\pi \left[144 \right] = 576\pi \ u^3$ (2)

Question 7.

a) $u_1 = 1 \quad u_2 = 1$

$u_n = u_{n-1} + u_{n-2} \quad n > 3$

$u_n = \frac{1}{\sqrt{5}}\left(\alpha^n - \beta^n\right)$

where $\alpha > \beta$ roots $x^2 - x - 1 = 0$

$\alpha, \beta = \frac{1 \pm \sqrt{5}}{2}$

$\alpha = \frac{1+\sqrt{5}}{2} \quad \beta = \frac{1-\sqrt{5}}{2}$

Step 1 Prove true $n = 1, 2$

$u_1 = 1 \quad u_1 = \frac{1}{\sqrt{5}}(\alpha - \beta)$

$u_1 = \frac{1}{\sqrt{5}}(\sqrt{5}) = 1$

TRUE FOR $n = 1$

$u_2 = 1 \quad u_2 = \frac{1}{\sqrt{5}}(\alpha^2 - \beta^2)$

$= \frac{1}{\sqrt{5}}(\alpha - \beta)(\alpha + \beta)$

$= \frac{1}{\sqrt{5}}(\sqrt{5})(1)$

$(\alpha + \beta = -b/a)$

Step 2. Assume true for $n = k-1, \ k-2$

Now $u_k = \frac{1}{\sqrt{5}}\left(\alpha^k - \beta^k\right)$

$u_k = u_{k-1} + u_{k-2}$

$u_k = \frac{1}{\sqrt{5}}\left(\alpha^{k-1} - \beta^{k-1} + \alpha^{k-2} - \beta^{k-2}\right)$

$= \frac{1}{\sqrt{5}}\left[\alpha^{k-2}(\alpha + 1) - \beta^{k-2}(\beta + 1)\right]$

But $\alpha^2 - \alpha - 1 = 0$

$\therefore \alpha^2 = \alpha + 1 \quad \beta^2 = \beta + 1$

$= \frac{1}{\sqrt{5}}\left[\alpha^{k-2}(\alpha^2) - \beta^{k-2}(\beta^2)\right]$

$= \frac{1}{\sqrt{5}}\left[\alpha^k - \beta^k\right]$ as req^d

Step 3 By the principle of Mathematical Induction true for all n. (4)

b) $I = \int_0^a f(x) \, dx$

Let $x = a - u$

$\frac{dx}{du} = -1$

when $x = 0 \quad u = a$

when $x = a \quad u = 0$

$f(x) = f(a-u)$

$I = \int_a^0 f(a-u) \, -du$

$= -\int_a^0 f(a-u) \, du$

$= \int_0^a f(a-u) \, du$

By Change of Variable

$$= \int_0^a f(a-x)\,dx \quad (2)$$

(ii) $$I = \int_0^{\pi/2} \frac{\sin^3 x}{\cos x + \sin x}\,dx$$

$$= \int_0^{\pi/2} \frac{\sin^3(\pi/2 - x)}{\cos(\pi/2 - x) + \sin(\pi/2 - x)}\,dx$$

$$= \int_0^{\pi/2} \frac{\cos^3 x}{\cos x + \sin x}\,dx$$

$$\therefore 2I = \int_0^{\pi/2} \frac{\cos^3 x + \sin^3 x}{\cos x + \sin x}\,dx$$

$$I = \frac{1}{2}\int_0^{\pi/2} \frac{\cancel{\cos x + \sin x}\,(\cos^2 x + \cos x \sin x + \sin^2 x)}{\cancel{\cos x + \sin x}}$$

$$I = \frac{1}{2}\int_0^{\pi/2} 1 + \cos x \sin x\,dx$$

$$= \frac{1}{2}\int_0^{\pi/2} 1 + \frac{1}{2}\sin 2x\,dx$$

$$= \frac{1}{2}\left[x + \frac{1}{4}\cos 2x\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\left(\pi/2 - \frac{1}{4}\cos\pi\right) - \left(0 - \frac{1}{4}\cos 0\right)\right]$$

$$= \frac{1}{2}\left[\pi/2 + 1/4 + 1/4\right]$$

$$= \frac{1}{4}\left[\pi + 2\right]. \quad (2)$$

(c) $$\left(\frac{1 + i\tan\theta}{1 - i\tan\theta}\right)^n = \left[\frac{\frac{\cos\theta + i\sin\theta}{\cos\theta}}{\frac{\cos\theta - i\sin\theta}{\cos\theta}}\right]^n$$

$$= \left[\frac{\cos\theta + i\sin\theta}{\cos\theta + i\sin -\theta}\right]^n$$

DE MOIVRE THEOREM

$$= \frac{\cos n\theta + i\sin n\theta}{\cos -n\theta + i\sin -n\theta}$$

$$= \frac{\cos n\theta + i\sin n\theta}{\cos n\theta - i\sin n\theta}$$

$\cos\theta$ is an even fn.
$\sin\theta$ is an odd fn.

$$= \frac{(1 + i\tan n\theta)\cos n\theta}{(1 - i\tan n\theta)\cos n\theta}$$

$$= \frac{1 + i\tan n\theta}{1 - i\tan n\theta} \quad (2)$$

d) (i) E $\quad 2y^2 - 3xy + 2x^2 = 14$

$$4y\frac{dy}{dx} - 3y - 3x\frac{dy}{dx} + 4x = 0$$

$$4y - 3x\frac{dy}{dx} = 3y - 4x$$

$$\frac{dy}{dx} = \frac{3y - 4x}{4y - 3x} \quad (1)$$

(ii) FOR TURNING POINTS

$$\frac{dy}{dx} = 0 \qquad 0 = 3y - 4x$$

$$y = 4/3\,x$$

$$2\left(\frac{4}{3}x\right)^2 - 3x\left(\frac{4}{3}x\right) + 2x^2 = 14$$

$$\frac{32x^2}{9} - 4x^2 + 2x^2 = 14$$

$$\frac{14x^2}{9} = 14 \quad (2)$$

$$x^2 = 9$$

$$x = \pm 3 \quad y = \pm 4.$$

Question 7 (cont)

(iii) FOR VERTICAL TANGENTS

$\frac{dy}{dx} \to 0 \quad \therefore 4y - 3x = 0$

$4y = 3x$

$y = \frac{3x}{4}$

$2\left(\frac{3x}{4}\right)^2 - 3x\left(\frac{3x}{4}\right) + 2x^2 = 14$

$\frac{18x^2}{16} - \frac{36x^2}{16} + \frac{32x^2}{16} = 14$

$\frac{14x^2}{16} = 14$ (2)

$x^2 = 16 \quad x = \pm 4 \quad y = \pm 3$

Question 8.

(a) (i) $\frac{1 + \sin\theta + i\cos\theta}{1 + \sin\theta - i\cos\theta} \times \frac{1 + \sin\theta + i\cos\theta}{1 + \sin\theta + i\cos\theta}$

$= \frac{(1+\sin\theta)^2 + 2(1+\sin\theta)i\cos\theta - \cos^2\theta}{(1+\sin\theta)^2 + \cos^2\theta}$

$= \frac{1 + 2\sin\theta + \sin^2\theta + 2(1+\sin\theta)i\cos\theta - (1 - \sin^2\theta)}{1 + 2\sin\theta + \sin^2\theta + \cos^2\theta}$

$= \frac{2\sin\theta + 2\sin^2\theta + 2(1+\sin\theta)i\cos\theta}{2 + 2\sin\theta}$

$= \frac{2(1+\sin\theta)\sin\theta + 2(1+\sin\theta)i\cos\theta}{2(1+\sin\theta)}$

$= \sin\theta + i\cos\theta$. (2)

(ii) $(1 + \sin\theta + i\cos\theta)^5 + i(1 + \sin\theta - i\cos\theta)^5 = 0$

$\therefore \left(\frac{1 + \sin\theta + i\cos\theta}{1 + \sin\theta - i\cos\theta}\right)^5 + i = 0$

$(\sin\theta + i\cos\theta)^5 + i = 0$

$\left[\cos(\pi/2 - \theta) + i\sin(\pi/2 - \theta)\right]^5 + i = 0$

$\cos(5\pi/2 - 5\theta) + i\sin(5\pi/2 - 5\theta) = -i$

BY DE MOIVRE

$\cos(\pi/2 - 5\theta) + i\sin(\pi/2 - 5\theta) = -i$

$\sin(-5\theta) + i\cos(-5\theta) = -i$

$-\sin 5\theta + i\cos 5\theta = -i$

$\therefore \sin 5\theta = 0$

$\cos 5\theta = -1$

$\therefore 5\theta = \pi$

$\theta = \pi/5$ (2)

(b) (i) $I_n = \int_0^1 x(1-x)^n\,dx$

let $dv = x \qquad u = (1-x)^n$

$v = \frac{x^2}{2} \qquad du = -n(1-x)^{n-1}$

$I_n = \left[\frac{x^2}{2}(1-x)^n\right]_0^1 - \int_0^1 \frac{x^2}{2} \cdot -n(1-x)^{n-1}\,dx$

$= 0 - \frac{n}{2}\int_0^1 [(1-x) - 1]\,x(1-x)^{n-1}\,dx$

$= -\frac{n}{2}(I_n - I_{n-1})$

$\therefore (n+2)I_n = nI_{n-1}$

$I_n = \frac{n}{n+2} I_{n-1}$. (3)

(ii) $I_n = \frac{n}{n+2} I_{n-1}$

$I_n = \frac{n}{n+2} \cdot \frac{n-1}{n+1} \cdot \frac{n-2}{n} \cdots \frac{1}{3} I_0$

$I_n = \frac{n!\,2!}{(n+2)!} I_0$

$I_n = \frac{1}{^nC_2} I_0$

$I_0 = \int_0^1 x\,dx = \left[\frac{1}{2}x^2\right]_0^1$

$= 1/2$

$\therefore I_n = \frac{1}{2(^nC_2)}$ (2)

(c) $xy = a^2 \quad (H)$

$y^2 = 8ax \quad (P)$

FROM H $\quad y = \frac{a^2}{x}$

$\left(\frac{a^2}{x}\right)^2 = 8ax$

$\frac{a^4}{x^2} = 8ax$

Question 8 (cont)

$$\frac{a^3}{8} = x^3$$

$$x = \frac{a}{2} \qquad y = 2a.$$

$$A\left(\frac{a}{2}, 2a\right) \qquad 1$$

(ii) $x + y + k = 0$ (T)

$y^2 = 8ax$ (P)

$y = -(x+k)$ FROM (T)

SUBST P $(x+k)^2 = 8ax$

$$x^2 + (2k - 8a)x + k^2 = 0$$

IF TANG $\Delta = 0$

$$(2k - 8a)^2 - 4k^2 = 0$$

$$4k^2 - 32ka + 64a^2 - 4k^2 = 0$$

$$64a^2 = 32ka$$

$$k = 2a$$

TEST WITH (H)

$x + y + 2a = 0$ (T)

$xy = a^2$ (H)

$$y = -(x+2a)$$

$$-x(x+2a) = a^2$$

$$0 = x^2 - 2ax + a^2$$

$$0 = (x-a)^2 \qquad 2$$

$\therefore$ TANGENT.

(iii) Point of Contact

B on P

$y^2 = 8ax$ (P)

$x + y + 2a = 0$ (T)

$$y = -(x+2a)$$

$$(x+2a)^2 = 8ax$$

$$x^2 + 4ax + 4a = 8ax$$

$$x^2 - \cancel{4}ax + 4a = 0$$

$$(x - 2a)^2 = 0$$

$x = 2a \quad \therefore y = -4a$

$$B(2a, -4a) \qquad 1$$

$$C(-a, -a) \qquad 1$$

(iv) $M_{AB} = \dfrac{2a + 4a}{\frac{a}{2} - 2a}$

$A\left(\frac{a}{2}, 2a\right) \qquad B(2a, -4a)$

$$= -4.$$

GRADIENT of H at $a/2$

$$\frac{dy}{dx} = \frac{-a^2}{x^2}$$

$$= \frac{-a^2}{a^2/4}$$

$$= -4.$$

$\therefore$ AB IS TANGENT. 1