2012

TRIAL

HIGHER SCHOOL CERTIFICATE

EXAMINATION

GIRRAWEEN HIGH SCHOOL

MATHEMATICS EXTENSION 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board – approved calculators may be used
- A table of standard integrals is provided
- Show all necessary working in Questions 11-16

Total marks - 100

Section 1 pages 2-3

10 marks

- Attempt Questions 1-10
- Allow about 20 minutes for this section

Section 2 pages 4 - 11

- Attempt Questions 11 - 16
- Allow about 2 hours 40 minutes for this section

SECTION 1

Multiple Choice (10 marks) Circle your answer on the question paper.

1. If z_1 and z_2 are any two non-zero complex numbers such that $|z_1+z_2|=|z_1|+|z_2|$ then $\arg(z_1)-\arg(z_2)$ is

(A) $-\frac{\pi}{2}$ (B) 0 (C) $\frac{\pi}{2}$ (D) $\frac{\pi}{4}$

2. If z is a complex number such that $|z-3-4i|+|z+3+4i|=10$, then the locus of z is

(A) An ellipse (B) a circle (C) a hyperbola (D) a straight line

3. The real values of x and y if

$\sqrt{x}(i+\sqrt{y})-15=i(8-\sqrt{y})$

(A) 36, 225 (B) 25, 9 (C) 25, 225 (D) 9, 25

4. If α and β are the roots of $x^2-2x+4=0$, then the value of $\alpha^3+\beta^3$ is

(A) $\frac{5}{2}$ (B) -128 (C) -16 (D) 64

5. If $\int x^4\sin(6x^5)dx=\frac{\lambda}{6}\cos(6x^5)+C,\ x\neq 0$ then the value of λ is

(A) 5 (B) $\frac{1}{5}$ (C) -5 (D) $-\frac{1}{5}$

6. If $\phi(x)=\int_0^x t\sin t\,dt$, then $\phi'(x)$ is

(A) $x\cos x$ (B) $x\sin x$ (C) $\cos x+x\sin x$ (D) $\frac{x^2}{2}$

7. The value of $\int_0^2 |x-1|dx$ is

(A) -1 (B) 1 (C) 2 (D) 3

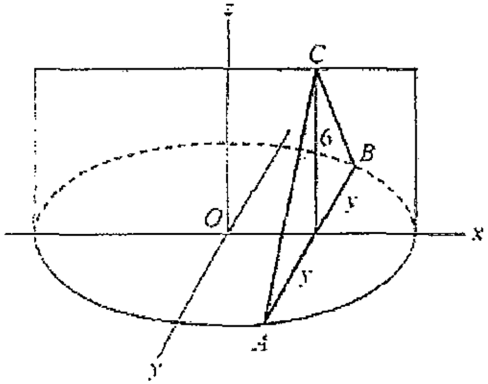
8. The coordinates of a focus of the ellipse $4x^2+9y^2=1$ are

(A) $\left(-\frac{\sqrt{5}}{6},0\right)$ (B) $\left(0,\frac{\sqrt{5}}{6}\right)$ (C) $\left(\frac{\sqrt{5}}{3},0\right)$ (D) $\left(-\frac{\sqrt{5}}{3},0\right)$

9. The equations of the directrices of the hyperbola $3x^2-6y^2=-18$ are

(A) $x=\pm1$ (B) $y=\pm1$ (C) $x=\pm2$ (D) $y=\pm2$

10. A solid has a base in the form of an ellipse with major axis 10 and minor axis 8. Every cross-section perpendicular to the major axis is an isosceles triangle with altitude 6. Which one of the following is the correct expression for the volume of the solid.



(A) $V=\frac{24}{5}\int_{-4}^{4}\sqrt{25-y^2}\,dy$ (C) $V=\frac{24}{5}\int_{-5}^{5}\sqrt{25-x^2}\,dx$

(B) $V=4\int_0^4\sqrt{25-y^2}\,dy$ (D) $V=4\int_0^5\sqrt{25-x^2}\,dx$

Question 11 (15 marks) **Marks**

Evaluate:

(a) $\int \frac{e^x-e^{-x}}{\left(e^x+e^{-x}\right)^2}dx$ 2

(b) $\int x\tan^{-1}x\,dx$ 2

(c) $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}dx$ 3

(d) (i) Find the real numbers A, B and C such that

$$\frac{3x+1}{(x-2)^2(x+2)}=\frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{x+2}$$ 2

(ii) Hence evaluate $\int \frac{3x+1}{(x-2)^2(x+2)}dx$ 2

(e) Use the result $\int_0^a f(x)dx=\int_0^a f(a-x)dx$ to evaluate $\int_0^{\frac{\pi}{4}}\log(1+\tan x)dx$ 4

Question 12 (15 marks)

(a) (i) Prove that if $x=\alpha$ is a root of multiplicity k of the real polynomial equation $P(x)=0$, then $x=\alpha$ is also a root of the equation $\frac{dP}{dx}=0$ of multiplicity $k-1$. 2

(ii) Solve $P(x)=x^4-11x^3+42x^2-68x+40$, given that $P(x)=0$ has a root of multiplicity 3. 2

(b) Let α,β,γ be the roots of the cubic equation $x^3+px^2+q=0$, where p,q are real. The equation $x^3+ax^2+bx+c=0$ has roots $\alpha^2,\beta^2,\gamma^2$. Find a,b,c in terms of p,q. 2

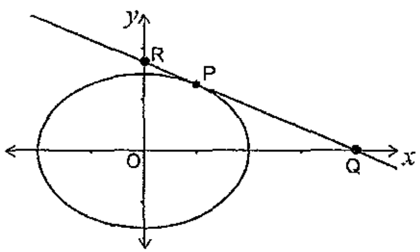
(c) A particle of mass m kilograms starts falling from rest having been initially projected vertically upwards from the ground. It experiences air resistance of magnitude mkv^2 on both the upward and downward motion of the journey where k is a positive constant and v is the velocity of the particle at any instant.

(i) Show that the terminal velocity, V_0, of the particle is given by $V_0 = \sqrt{\frac{g}{k}}$. 1

(ii) If W is the velocity of the particle when it hits the ground, show that the distance, S, fallen is given by $\frac{1}{2k}\ln\left(\frac{g}{g-kW^2}\right)$. 2

(iii) The maximum height attained by the particle is given by $H = \frac{1}{2k}\ln\left(\frac{g+kU^2}{g}\right)$ where U is the initial velocity of projection. Show that $\frac{1}{W^2} = \frac{1}{U^2} + \frac{1}{V_0^2}$. 2

(d) The point P $(4\cos\theta, 3\sin\theta)$ lies on the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.



(i) Find the equation of the tangent to the ellipse at P. 2

(ii) The tangent at P cuts the x-axis at Q and the y-axis at R. Show that the area of ΔORQ is $\frac{12}{\sin 2\theta}$. 1

(iii) Find the coordinates of P so that area of ΔORQ is a minimum. 1

Question 13 (15 marks)

(a) z is a complex number such that $|z| = 4, \arg z = \frac{5\pi}{6}$. Express z in the form $a+ib$ where a and b are real. 1

(b) $z_1 = 1+i\sqrt{3}$ and $z_2 = 1-i$ are two complex numbers.

(i) Express z_1, z_2 and $z_1 z_2$ in modulus/argument form. 2

(ii) Find the smallest positive integer such that $z_1^n z_2^n$ is purely imaginary. For this value of n, write the value of $z_1^n z_2^n$ in the form bi where b is a real number. 3

(c) Sketch the locus of the following:

(i) $\arg(z-1-2i) = \frac{\pi}{4}$ 1

(ii) $z\bar{z} - 3(z+\bar{z}) \le 0$ 2

(d) (i) Find the seven seventh roots of -1 2

(ii) Factorise z^7+1 over the real field R. 2

(iii) Prove that $\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} = \frac{1}{2}$ 2

Question 14 (15 marks)

(a) Find the volume of the solid generated by revolving the region bounded by

$x^2 - y^2 = 16$, and $x = 8$ about the y – axis.(see Figure 1) 3

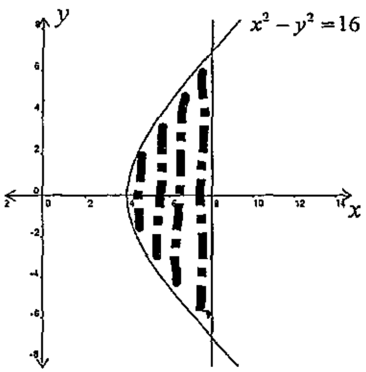


Figure 1

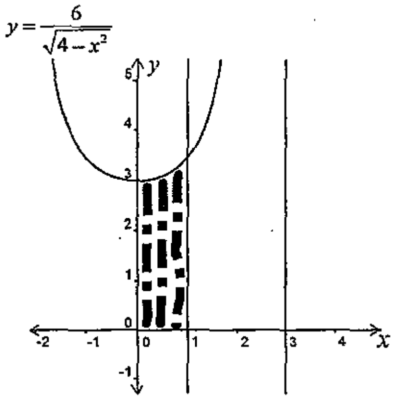


Figure 2

(b) A mould for a section of concrete piping is made by rotating the region bounded by the curve $y = \dfrac{6}{\sqrt{4-x^2}}$ and the x – axis between the lines $x = 0$ and $x = 1$ through one complete revolution about the line $x = 3$. All measurements are in metres.(see Figure 2)

(i) By condidering strips of width Δx parallel to the axis of rotation, show that the volume V m^3 of the concrete used in the piping is given by

$$V = 12\pi \int_0^1 \frac{3-x}{\sqrt{4-x^2}}\,dx$$ 2

(ii) Hence find the volume of the concrete used in the piping, giving your answer correct to the nearest cubic metre. 3

(c) $P\left(p, \dfrac{1}{p}\right)$ and $Q\left(q, \dfrac{1}{q}\right)$ are two points on the rectangular hyperbola $xy = 1$. M is the midpoint of the chord PQ.

(i) Show that the chord PQ has equation $x + pqy - (p+q) = 0$ 2

(ii) If P and Q move on the rectangular hyperbola such that the perpendicular distance of the chord PQ from the origin O (0,0) is always $\sqrt{2}$, show that

$(p+q)^2 = 2(1 + p^2q^2)$ 1

(iii) Hence find the equation of the locus of M, stating any restriction on its domain and range. 4

Question 15 (15 marks)

(a) A body is projected vertically upwards from the surface of the Earth with initial speed u. The acceleration due to gravity at any point on its path is inversely proportional to the square of its distance from the centre of the Earth.

(i) Prove that the speed at any position x is given by

$$v^2 = u^2 + 2gR^2\left(\frac{1}{x} - \frac{1}{R}\right)$$ 2

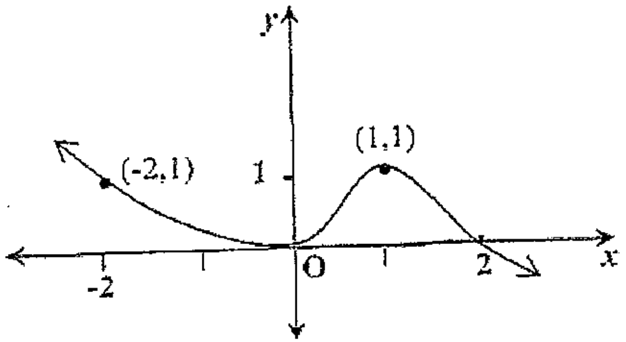
(ii) Prove that the greatest height H above the Earth's surface is given by

$$H = \frac{u^2 R}{2gR - u^2}$$ 2

(iii) Show that the body will escape from the Earth if $u \geq \sqrt{2gR}$ 1

(iv) If $u = \sqrt{2gR}$, prove that the time taken to reach a height $15R$ above the surface of the Earth is $42\sqrt{\dfrac{R}{2g}}$ 2

(b) The diagram shows the graph of $y = f(x)$. On separate diagrams sketch the graphs of the following. Clearly indicate any asymptotes and intercepts with the axes.



(i) $y = \ln[f(x)]$ 2

(ii) $y = \dfrac{1}{f(x)}$ 2

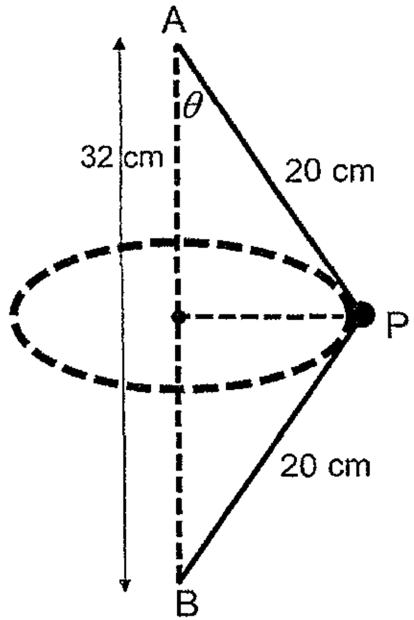
(iii) $y = -|f(x)|$ 2

(c) Sketch the curve showing vertical and slant asymptotes.

$$f(x) = \frac{x^2 - 3x - 4}{x + 3}$$ 2

Question 16 (15 marks)

(a) A particle P of mass m kg is tied to the midpoint of a light inextensible string of length 40 cm. One end of the string is fixed at point A, and the other end is fixed at point B which is 32 cm vertically below A. Particle P moves with constant speed v m/s in a horizontal circle around the midpoint of AB, while both sections of string AP and BP remain taut. The acceleration due to gravity is $g\ m/s^2$.

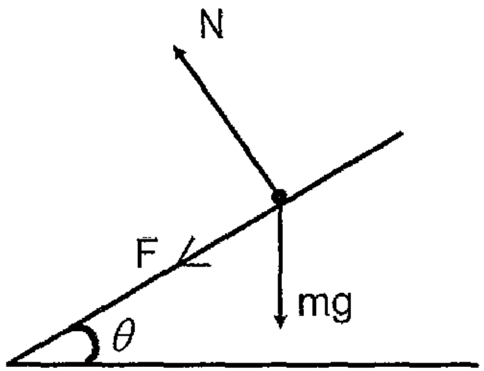


(i) Draw a diagram showing the forces acting on the particle P. 2

(ii) Find the tension in each part of the string in terms of m, v and g. 4

(iii) Show that $v \geq \frac{3}{10}\sqrt{g}$, for both strings to be taut. 1

(b) A car travels at a uniform speed of $v \ m/s$ around a banked circular track. The track is inclined at an angle θ to the horizontal and the car moves in a horizontal circle of radius r. The car experiences a normal reaction force, N, from the track, a vertical force of magnitude mg due to gravity and a sideways frictional force, F, acting down the slope. This information is shown in the diagram below.



(i) Resolve the forces along the slope and perpendicular to the slope, or otherwise. Hence, find expressions for F and N. 4

(ii) A track of radius 200 metres is banked at angle of 25° to the horizontal. Find the speed of cars around this track if there is no sideways friction force. Assume that $g = 9.8m/s$. 2

(iii) A motorist is riding around the track at 90 km/h. Find the frictional force experienced by the motorist and in what direction. The combined mass of the car and motorist is 1500 kg. 2

END OF TEST

Trial HSC Extension 2, 2012 – Solutions

Multiple choice (10 marks)

1 B 2 A 3 D 4 C 5 D 6 B 7 B

8 A 9 B 10 C

Question 11 (15 marks)

(a) $\displaystyle\int \frac{e^{x}-e^{-x}}{(e^{x}+e^{-x})^2}\,dx$

Let $u = e^{x}+e^{-x}$

$\dfrac{du}{dx} = e^{x}-e^{-x}$

$du = (e^{x}-e^{-x})\,dx$

$I = \displaystyle\int \frac{du}{u^2} = \int u^{-2}\,du$

$= \dfrac{u^{-1}}{-1} + C$

$= \dfrac{-1}{u} + C$

$= \dfrac{-1}{e^{x}+e^{-x}} + C$

(b) $\displaystyle\int x\tan^{-1}x\,dx$

$= \tan^{-1}x \cdot \dfrac{x^2}{2} - \displaystyle\int \frac{1}{1+x^2}\times\frac{x^2}{2}\,dx$

$= \dfrac{x^2}{2}\tan^{-1}x - \dfrac{1}{2}\displaystyle\int \frac{x^2}{1+x^2}\,dx$ —— (1)

$\displaystyle\int \frac{x^2}{x^2+1}\,dx = \int\left(\frac{x^2+1-1}{x^2+1}\right)dx$

$= \displaystyle\int \frac{x^2+1}{x^2+1}\,dx - \int\frac{1}{x^2+1}\,dx$

$= x - \tan^{-1}x + C$

(1) becomes

$I = \dfrac{x^2}{2}\tan^{-1}x - \dfrac{1}{2}\left(x - \tan^{-1}x + C\right)$

$= \dfrac{x^2}{2}\tan^{-1}x - \dfrac{x}{2} + \dfrac{1}{2}\tan^{-1}x + C$

(c) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$

Let $t = \tan\dfrac{x}{2}$ and $\cos x = \dfrac{1-t^2}{1+t^2}$

$\dfrac{dt}{dx} = \left(\sec^2\dfrac{x}{2}\right)\dfrac{1}{2}$

$= \dfrac{1}{2}\left(1+\tan^2\dfrac{x}{2}\right)$

$dt = \dfrac{1}{2}(1+t^2)\,dx$

$dx = \dfrac{2\,dt}{1+t^2}$

when $x=0$, $t=0$

when $x = \dfrac{\pi}{2}$, $t = \tan\dfrac{\pi}{4} = 1$

$2+\cos x = 2 + \dfrac{1-t^2}{1+t^2}$

$= \dfrac{2(1+t^2)+1-t^2}{1+t^2}$

$= \dfrac{2+2t^2+1-t^2}{1+t^2}$

$= \dfrac{3+t^2}{1+t^2}$

$I = \displaystyle\int_0^1 \frac{1+t^2}{3+t^2}\times\frac{2\,dt}{1+t^2}$

$= \displaystyle\int_0^1 \frac{2\,dt}{3+t^2}$

$= 2\displaystyle\int_0^1 \frac{dt}{t^2+(\sqrt{3})^2}$

$2\left[\dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{t}{\sqrt{3}}\right)\right]_0^1$

$= \dfrac{2}{\sqrt{3}}\left[\tan^{-1}\left(\dfrac{t}{\sqrt{3}}\right)\right]_0^1$

$= \dfrac{2}{\sqrt{3}}\left(\tan^{-1}\left(\dfrac{1}{\sqrt{3}}\right) - \tan^{-1}(0)\right)$

$= \dfrac{2}{\sqrt{3}}\left(\dfrac{\pi}{6} - 0\right)$

$= \dfrac{2\pi}{6\sqrt{3}} = \dfrac{\pi}{3\sqrt{3}}$

(d) Let $3x+1 \equiv A(x-2)(x+2)+B(x+2)+C(x-2)^2$

$x=2 \Longrightarrow 7=4B \qquad \underline{\underline{B=\frac{7}{4}}}$

$x=-2 \Longrightarrow -5=16C \qquad \underline{\underline{C=\frac{-5}{16}}}$

Comparing coefficients of x^2 on both sides of the identity ~~as~~ $A+C=0 \qquad \underline{\underline{A=\frac{5}{16}}}$

$$\frac{3x+1}{(x-2)^2(x+2)} = \frac{5}{16}\times\frac{1}{x-2} + \frac{7}{4}\times\frac{1}{(x-2)^2} - \frac{5}{16(x+2)}$$

$$\int\frac{(3x+1)\,dx}{(x-2)^2(x+2)} = \frac{5}{16}\int\frac{dx}{x-2} + \frac{7}{4}\int\frac{dx}{(x-2)^2} - \frac{5}{16}\int\frac{dx}{x+2}$$

$$= \underline{\underline{\frac{5}{16}\log(x-2) - \frac{7}{4(x-2)} - \frac{5}{16}\log(x+2) + C}}$$

(e) $\int_0^{\frac{\pi}{4}} \log(1+\tan x)\,dx$

$= \int_0^{\frac{\pi}{4}} \log\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx$

$= \int_0^{\frac{\pi}{4}} \log\left(1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\cdot\tan x}\right)dx$

$= \int_0^{\frac{\pi}{4}} \log\left(1+\frac{1-\tan x}{1+\tan x}\right)$

$= \int_0^{\frac{\pi}{4}} \log\left(\frac{1+\tan x+1-\tan x}{1+\tan x}\right)$

$= \int_0^{\frac{\pi}{4}} \log\frac{2}{1+\tan x}\,dx$

$= \int_0^{\frac{\pi}{4}} \left(\log 2 - \log(1+\tan x)\right)dx$

$= \int_0^{\frac{\pi}{4}} \log 2\,dx - \int_0^{\frac{\pi}{4}} \log(1+\tan x)\,dx$

$\int_0^{\frac{\pi}{4}} \log(1+\tan x)\,dx + \int_0^{\frac{\pi}{4}} \log(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} \log 2\,dx$

$2\int_0^{\frac{\pi}{4}} \log(1+\tan x)\,dx = \log 2\left[x\right]_0^{\frac{\pi}{4}}$

$= \log 2\left(\frac{\pi}{4}-0\right)$

$= \frac{\pi}{4}\log 2$

$\int_0^{\frac{\pi}{4}} \log(1+\tan x)\,dx = \underline{\underline{\frac{\pi}{8}\log 2}}$

Question 12 (15 marks)

(a)(i) Let $P(x) = (x-\alpha)^k \cdot Q(x)$ where $Q(\alpha) \neq 0$

$$\frac{dP}{dx} = k(x-\alpha)^{k-1}Q(x) + (x-\alpha)^k\frac{dQ}{dx}$$

$$= (x-\alpha)^{k-1}\left[kQ(x) + (x-\alpha)\frac{dQ}{dx}\right]$$

$$= (x-\alpha)^{k-1}S(x)$$

where $S(x) = kQ(x) + (x-\alpha)\frac{dQ}{dx}$ is a polynomial

and $S(\alpha) \neq 0$

$\therefore x=\alpha$ is a root of multiplicity $k-1$ of the equation $\frac{dP}{dx}=0$

(ii) Since $p(x)$ has a root of multiplicity 3, $p(x)$, $p'(x)$ and $p''(x)$ have a common zero.

$$p'(x) = 4x^3 - 11\times 3x^2 + 42\times 2x - 68$$
$$= 4x^3 - 33x^2 + 84x - 68$$

$$p''(x) = 4\times 3x^2 - 33\times 2x + 84$$
$$= 12x^2 - 66x + 84$$

$$p''(x) = 0 \Longrightarrow 12x^2 - 66x + 84 = 0$$
$$6(2x^2 - 11x + 14) = 0$$
$$2x^2 - 11x + 14 = 0$$

$-7, -4$ $\quad pq = 28$

$\quad p+q = -11$

$$2x^2 - 4x - 7x + 14 = 0$$
$$2x(x-2) - 7(x-2) = 0$$
$$(x-2)(2x-7) = 0$$
$$x = 2 \quad \text{or} \quad x = \frac{7}{2}$$

$$p'(2) = 4\times 8 - 33\times 4 + 84\times 2 - 68$$
$$= 200 - 200 = 0$$

$$p(2) = 16 - 11\times 8 + 42\times 4 - 68\times 2 + 40$$
$$= 16 - 88 + 168 - 136 + 40$$
$$= 0$$

Sum of the roots

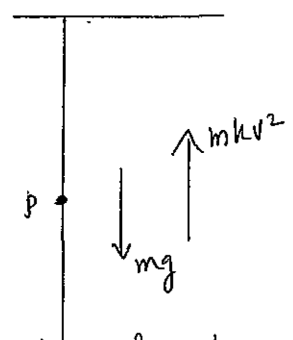
$$\alpha, 2, 2, 2$$
$$6 + \alpha = 11$$
$$\alpha = 5$$

Roots are $2, 2, 2, 5$

(b) $x^3 + px^2 + q = 0$

$$y = x^2 \qquad x = \sqrt{y}$$
$$(\sqrt{y})^3 + p(\sqrt{y})^2 + q = 0$$
$$y^{\frac{3}{2}} + py + q = 0$$
$$y^{\frac{3}{2}} = -(py + q)$$
$$y^3 = (py + q)^2$$
$$y^3 = p^2y^2 + 2pqy + q^2$$
$$y^3 - p^2y^2 - 2pqy - q^2 = 0$$
$$x^3 - p^2x^2 - 2pqx - q^2 = 0$$
$$x^3 + ax^2 + bx + c = 0$$
$$a = -p^2$$
$$b = -2pq \qquad c = -q^2$$

(c)



Equation of motion is

$$m\ddot{x} = mg - mkv^2$$
$$\ddot{x} = g - kv^2$$

Terminal velocity V_0 occurs when $\ddot{x} = 0$

$$g - kV_0^2 = 0$$
$$kV_0^2 = g\ ; \ V_0^2 = \frac{g}{k}$$
$$V_0 = \sqrt{\frac{g}{k}}$$

(ii) $v\dfrac{dv}{dx} = g - kv^2$

$$\frac{dv}{dx} = \frac{g - kv^2}{v}$$
$$\frac{dx}{dv} = \frac{v}{g - kv^2}$$
$$dx = \frac{v\,dv}{g - kv^2}$$
$$[x]_0^S = \int_0^W \frac{v\,dv}{g - kv^2}$$
$$S = \int_0^W \frac{-2kv\,dv}{-2k(g - kv^2)}$$
$$= \frac{-1}{2k}\left[\log(g - kv^2)\right]_0^W$$
$$= \frac{-1}{2k}\left(\log(g - kW^2) - \log g\right)$$
$$= \frac{1}{2k}\log\left(\frac{g}{g - kW^2}\right)$$

(iii) $S = H$

$$\frac{1}{2k}\log\left(\frac{g}{g-kW^2}\right) = \frac{1}{2k}\log\left(\frac{g+kU^2}{g}\right)$$

$$\frac{g}{g-kW^2} = \frac{g+kU^2}{g}$$

$$(g-kW^2)(g+kU^2) = g^2$$

$$g^2 + gkU^2 - gkW^2 - k^2U^2W^2 = g^2$$

$$gkU^2 - gkW^2 - k^2U^2W^2 = 0$$

divide by U^2W^2gk

$$\frac{gkU^2}{U^2W^2gk} - \frac{gkW^2}{U^2W^2gk} - \frac{k^2U^2W^2}{U^2W^2gk} = 0$$

$$\frac{1}{W^2} - \frac{1}{U^2} - \frac{k}{g} = 0$$

$$\frac{1}{W^2} - \frac{1}{U^2} = \frac{k}{g}$$

$$\frac{1}{W^2} = \frac{1}{U^2} + \frac{1}{V_0^2}$$

(d) $x = 4\cos\theta$, $y = 3\sin\theta$

$$\frac{dx}{d\theta} = 4\times -\sin\theta = -4\sin\theta$$

$$\frac{dy}{d\theta} = 3\cos\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= \frac{3\cos\theta}{-4\sin\theta}$$

Equation of tangent

$$y - 3\sin\theta = \frac{-3\cos\theta}{4\sin\theta}(x - 4\cos\theta)$$

$$4\sin\theta\, y - 12\sin^2\theta = -3x\cos\theta + 12\cos^2\theta$$

$$4y\sin\theta + 3x\cos\theta = 12$$

$$\frac{4y\sin\theta}{12} + \frac{3x\cos\theta}{12} = 1$$

$$\frac{x}{4}\cos\theta + \frac{y}{3}\sin\theta = 1$$

(ii) $x = 0 \Rightarrow \dfrac{y\sin\theta}{3} = 1$

$$y = \frac{3}{\sin\theta}$$

$y = 0 \Rightarrow \dfrac{x}{4}\cos\theta = 1$

$$x = \frac{4}{\cos\theta}$$

Area of ΔOQR

$$= \frac{1}{2}\times OQ \times OR$$

$$= \frac{1}{2}\times\frac{4}{\cos\theta}\times\frac{3}{\sin\theta} = \frac{6}{\sin\theta\cos\theta}$$

$$= \frac{12}{2\sin\theta\cos\theta}$$

$$= \frac{12}{\sin 2\theta}$$

(iii) $\dfrac{12}{\sin 2\theta}$ is minimum when $\sin 2\theta$ is maximum

ie $\sin 2\theta = 1$

$$2\theta = \frac{\pi}{2} \qquad \theta = \frac{\pi}{4}$$

The coordinates of P are

$$\left(4\cos\frac{\pi}{4}, 3\sin\frac{\pi}{4}\right)$$

$$= \left(4\times\frac{1}{\sqrt{2}}, 3\times\frac{1}{\sqrt{2}}\right)$$

$$= \left(\frac{4}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$$

Question 13 (15 marks)

(a) $z = 4\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

$= 4\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

$= -2\sqrt{3} + 2i$

(b)(i) $z_1 = 1 + i\sqrt{3}$

$r = \sqrt{1^2 + (\sqrt{3})^2} = 2$

$\tan\alpha = \frac{\sqrt{3}}{1} = \sqrt{3}$

$\alpha = \frac{\pi}{3}$

argument $\theta = \alpha = \frac{\pi}{3}$

$1 + i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$z_2 = 1 - i$

$r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\tan\alpha = 1$

$\alpha = \frac{\pi}{4}$

$\theta = -\alpha = -\frac{\pi}{4}$

$1 - i = \sqrt{2}\left(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right)$

$|z_1 z_2| = |z_1||z_2| = 2\sqrt{2}$

$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

$= \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$

$z_1^n z_2^n = (z_1 z_2)^n$

$= (2\sqrt{2})^n\left[\cos\frac{n\pi}{12} + i\sin\frac{n\pi}{12}\right]$

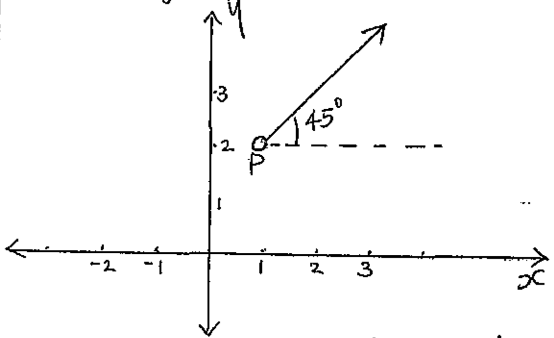
$z_1^n z_2^n$ is purely imaginary means $\arg(z_1^n z_2^n)$ is a multiple of $\frac{\pi}{2}$

when $n = 6$, $\arg(z_1^n z_2^n) = \frac{\pi}{2}$

$z_1^6 z_2^6 = (2\sqrt{2})^6\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]$

$= 512i$

(c)(i) $\arg(z - (1 + 2i)) = \frac{\pi}{4}$



The locus is the half ray at P(1,2), making 45° to the x axis, the point P is not included.

(ii) $z\bar{z} - 3(z + \bar{z}) \le 0$

If $z = x + iy$ then $\bar{z} = x - iy$

$z\bar{z} = (x + iy)(x - iy)$

$= x^2 + y^2$

$z + \bar{z} = x + iy + x - iy$

$= 2x$

$z\bar{z} - 3(z + \bar{z})$

$= x^2 + y^2 - 6x$

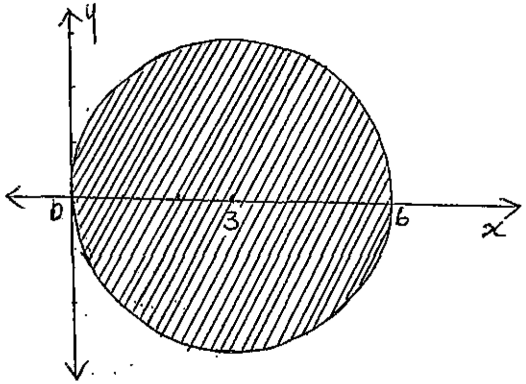
$= x^2 - 6x + y^2$

$= x^2 - 6x + 9 - 9 + y^2$

$(x-3)^2 + y^2 - 9 \le 0$

$(x-3)^2 + y^2 \le 9$

The locus is all points that are on and inside the circle of radius 3 units and centre at (3,0)



(d)(i) $z^7 = -1 = \cos\pi + i\sin\pi$

$= \cos(2k\pi + \pi) + i\sin(2k\pi + \pi)$

$k = 0, 1, 2 \ldots$

The seven seventh roots of -1 are given by

$z = \left[\cos(2k\pi + \pi) + i\sin(2k\pi + \pi)\right]^{\frac{1}{7}}$

$= \cos\frac{(2k+1)\pi}{7} + i\sin\frac{(2k+1)\pi}{7}$

where $k = 0, 1, 2, 3, 4, 5, 6$

by De Moivre's theorem.

$k = 0$

$z_1 = \cos\frac{\pi}{7} + i\sin\frac{\pi}{7}$

$k = 1$

$z_2 = \cos\frac{3\pi}{7} + i\sin\frac{3\pi}{7}$

$k = 2$

$z_3 = \cos\frac{5\pi}{7} + i\sin\frac{5\pi}{7}$

$k = 3$

$z_4 = \cos\frac{7\pi}{7} + i\sin\frac{7\pi}{7} = -1$

$k = 4$

$z_5 = \cos\frac{9\pi}{7} + i\sin\frac{9\pi}{7}$

$k = 5$

$z_6 = \cos\frac{11\pi}{7} + i\sin\frac{11\pi}{7}$

$k = 6$

$z_7 = \cos\frac{13\pi}{7} + i\sin\frac{13\pi}{7}$

(ii) $Z_7 = \overline{Z_1}$ $\quad Z_1 + Z_7 = 2\cos\frac{\pi}{7}$

$Z_6 = \overline{Z_2}$ $\quad Z_2 + Z_6 = 2\cos\frac{3\pi}{7}$

$Z_5 = \overline{Z_3}$ $\quad Z_3 + Z_5 = 2\cos\frac{5\pi}{7}$

$$Z^7+1 = (Z-Z_1)(Z-Z_2)(Z-Z_3)(Z-Z_4)(Z-Z_5)(Z-Z_6)(Z-Z_7)$$

$$=(Z+1)(Z-Z_1)(Z-Z_7)(Z-Z_2)(Z-Z_6)(Z-Z_3)(Z-Z_5)$$

$$=(Z+1)\left(Z^2-Z(Z_1+Z_7)+1\right)\left(Z^2-Z(Z_2+Z_6)+1\right)\left(Z^2-Z(Z_3+Z_5)+1\right)$$

$$=(Z+1)\left(Z^2-Z\times 2\cos\frac{\pi}{7}+1\right)\left(Z^2-Z\times 2\cos\frac{3\pi}{7}+1\right)\left(Z^2-Z\times 2\cos\frac{5\pi}{7}+1\right) \quad \text{—— (1)}$$

Also

$$Z^7+1=(Z+1)(Z^6-Z^5+Z^4-Z^3+Z^2-Z+1) \quad \text{—— (2)}$$

From (1) and (2) we get

$$\left(Z^2-2\cos\frac{\pi}{7}Z+1\right)\left(Z^2-2\cos\frac{3\pi}{7}Z+1\right)\left(Z^2-2\cos\frac{5\pi}{7}Z+1\right)$$
$$=Z^6-Z^5+Z^4-Z^3+Z^2-Z+1$$

$$Z^4-2\cos\frac{3\pi}{7}Z^3+Z^2-2\cos\frac{\pi}{7}Z^3+4\cos\frac{\pi}{7}\cos\frac{3\pi}{7}\,Z^2$$
$$-2\cos\frac{\pi}{7}Z+Z^2-2\cos\frac{3\pi}{7}Z+1\Big)\left(Z^2-2\cos\frac{5\pi}{7}Z+1\right)$$
$$=Z^6-Z^5+Z^4-Z^3+Z^2-Z+1$$

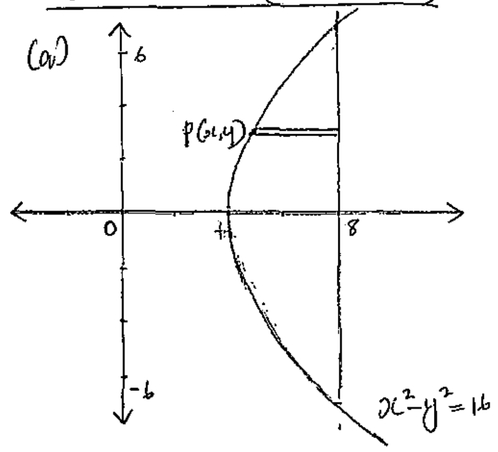
Equating coefficients of Z^5 on both sides

$$-2\cos\frac{5\pi}{7}Z^5-2\cos\frac{3\pi}{7}Z^5-2\cos\frac{\pi}{7}Z^5=-Z^5$$

$$\underline{\underline{2\cos\frac{\pi}{7}+2\cos\frac{3\pi}{7}+2\cos\frac{5\pi}{7}=1}}$$

Question 14 (15 marks)

(a)



Substitute $x=8$ in

$x^2-y^2=16$

$64-y^2=16$

$y^2=48$

$y=\pm\sqrt{16\times 3}$

$=\pm 4\sqrt{3}$

Area of annulus

$=\pi(8^2-x^2)$

Volume of annulus

$\Delta V=\pi(8^2-x^2)\Delta y$

Volume of solid

$$V=\lim_{\Delta y\to 0}\sum_{-4\sqrt{3}}^{4\sqrt{3}}\pi(64-x^2)\Delta y$$

$$=\int_{-4\sqrt{3}}^{4\sqrt{3}}\pi(64-x^2)\,dy$$

$$=\int_{-4\sqrt{3}}^{4\sqrt{3}}\pi(64-16-y^2)\,dy$$

$$=2\pi\int_0^{4\sqrt{3}}(48-y^2)\,dy$$

$$=2\pi\left[48y-\frac{y^3}{3}\right]_0^{4\sqrt{3}}$$

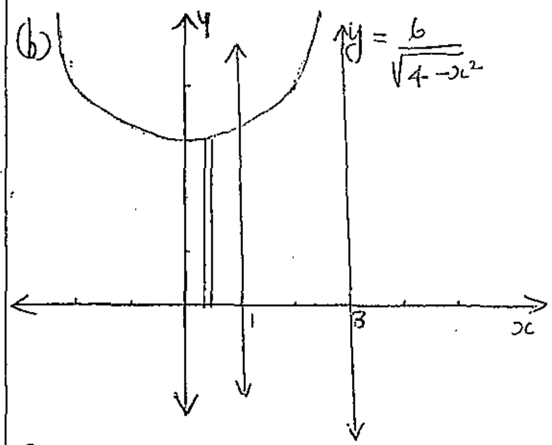
$$=2\pi\left\{\left(48\times 4\sqrt{3}-\frac{64\times 3\sqrt{3}}{3}\right)-(\right.$$

$$=2\pi\left(192\sqrt{3}-64\sqrt{3}\right)$$

$$=2\pi\times 128\sqrt{3}$$

$$=\underline{\underline{256\sqrt{3}\,\pi \text{ cubic units.}}}$$

(b)



Radius $=3-x$

height $=y$

Volume of cylindrical shell

$\Delta V=2\pi rh\,\Delta x$

$=2\pi(3-x)y\,\Delta x$

$$\therefore V = \lim_{\Delta x \to 0} \sum_{x=0}^{1} 2\pi(3-x) \times \frac{6}{\sqrt{4-x^2}} \Delta x$$

$$= \int_0^1 \frac{2\pi(3-x)\times 6}{\sqrt{4-x^2}}\, dx$$

$$= 12\pi \int_0^1 \frac{3-x}{\sqrt{4-x^2}}\, dx$$

(ii) $V = 12\pi \int_0^1 \frac{3}{\sqrt{4-x^2}}\, dx - 12\pi \int_0^1 \frac{x}{\sqrt{4-x^2}}\, dx$

$$= \underbrace{36\pi \int_0^1 \frac{dx}{\sqrt{4-x^2}}}_{I_1} - \underbrace{12\pi \int_0^1 \frac{x}{\sqrt{4-x^2}}\, dx}_{I_2}$$

$I_1 = 36\pi \left[\sin^{-1}\frac{x}{2}\right]_0^1$

$= 36\pi\left(\sin^{-1}\frac{1}{2} - \sin^{-1} 0\right)$

$= 36\pi \times \frac{\pi}{6} = 6\pi^2$

$I_2 = \int_0^1 \frac{x}{\sqrt{4-x^2}}\, dx$

Let $u = 4 - x^2$

$\frac{du}{dx} = -2x$

$-2x\,dx = du$

$x\,dx = -\frac{du}{2}$

when $x=0$, $u=4$

when $x=1$, $u=3$

$= \int_4^3 \frac{-\frac{du}{2}}{\sqrt{u}} = -\frac{1}{2}\int_4^3 \frac{du}{\sqrt{u}}$

$= \frac{1}{2}\int_3^4 u^{-\frac{1}{2}}\, du$

$= \frac{1}{2}\left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right]_3^4 = \frac{1}{2}\times 2\left[u^{\frac{1}{2}}\right]_3^4$

$= 4^{\frac{1}{2}} - 3^{\frac{1}{2}} = 2 - \sqrt{3}$

$I_2 = 12\pi(2-\sqrt{3})$

$= 24\pi - 12\sqrt{3}\pi$

$V = 6\pi^2 - 24\pi + 12\sqrt{3}\pi$

$= 49\, m^3$

(c) $P(p, \frac{1}{p})$ $Q(q, \frac{1}{q})$

$m_{PQ} = \frac{\frac{1}{p} - \frac{1}{q}}{p - q}$

$= \frac{q-p}{pq} \times \frac{1}{p-q} = \frac{-1}{pq}$

Equation of PQ

$y - \frac{1}{p} = \frac{-1}{pq}(x - p)$

$pqy - q = -x + p$

$x + pqy - q - p = 0$

$x + pqy - (p+q) = 0$

(ii) $\frac{|0 + 0 - (p+q)|}{\sqrt{1^2 + (pq)^2}} = \sqrt{2}$

$\frac{|p+q|}{\sqrt{1+p^2q^2}} = \sqrt{2}$

$\frac{(p+q)^2}{1+p^2q^2} = 2$

$(p+q)^2 = 2(1+p^2q^2)$

(iii) $M = \left(\frac{p+q}{2}, \frac{\frac{1}{p}+\frac{1}{q}}{2}\right)$

$= \left(\frac{p+q}{2}, \frac{p+q}{2pq}\right)$

$x = \frac{p+q}{2}$ $\quad y = \frac{p+q}{2pq}$

$x^2 = \frac{(p+q)^2}{4}$ $\quad y^2 = \frac{(p+q)^2}{4p^2q^2}$

$\frac{x^2}{y^2} = \frac{(p+q)^2}{4} \times \frac{4p^2q^2}{(p+q)^2}$

$\frac{x^2}{y^2} = p^2q^2$ —— (1)

From (ii) $(p+q)^2 = 2(1+p^2q^2)$

$\frac{(p+q)^2}{2} = 1 + p^2q^2$

$\frac{(2x)^2}{2} = 1 + p^2q^2$

$\frac{4x^2}{2} = 1 + p^2q^2$

$2x^2 - 1 = p^2q^2$

(1) becomes $\frac{x^2}{y^2} = 2x^2 - 1$

$x^2 = 2x^2y^2 - y^2 = y^2(2x^2-1)$

$y^2 = \frac{x^2}{2x^2-1}$ which is the locus of M.

Domain of $y^2 = \frac{x^2}{2x^2-1}$

$2x^2 - 1 > 0$

$2x^2 > 1$

$x^2 > \frac{1}{2}$

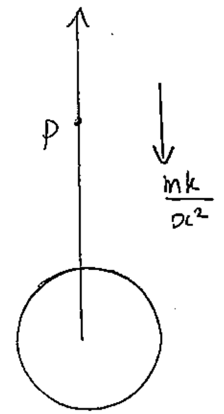
$x > \frac{1}{\sqrt{2}}$ or $x < -\frac{1}{\sqrt{2}}$

$D : \{x : |x| > \frac{1}{\sqrt{2}}\}$

$y^2 = \frac{x^2}{2x^2-1}$ is symmetric in x and y. Hence the range $\{y : |y| > \frac{1}{\sqrt{2}}\}$

Question 15 (15 marks)

)



Equation of motion is

$m\ddot{x} = -\frac{mk}{x^2}$ —— (1)

Substitute $\ddot{x} = v\frac{dv}{dx}$

(1) becomes

$mv\frac{dv}{dx} = -\frac{mk}{x^2}$

$v\frac{dv}{dx} = -\frac{k}{x^2}$ —— (2)

At the Earth's surface $x = R$ and

$g = \frac{k}{R^2}$ $\therefore k = gR^2$

(2) becomes

$v\frac{dv}{dx} = -\frac{gR^2}{x^2}$

$v\,dv = -\frac{gR^2}{x^2}dx$

$\int v\,dv = -gR^2\int\frac{dx}{x^2}$

$\frac{v^2}{2} = -gR^2 \times \frac{x^{-1}}{-1} + c$

$\frac{v^2}{2} = \frac{gR^2}{x} + c$ —— (3)

when $x = R$, $v = u$

$\frac{u^2}{2} = gR + c$

$c = \frac{u^2}{2} - gR$

(3) becomes

$\frac{v^2}{2} = \frac{gR^2}{x} + \frac{u^2}{2} - gR$

$v^2 = \frac{2gR^2}{x} + u^2 - \frac{2gR^2}{R}$

$v^2 = u^2 + 2gR^2\left(\frac{1}{x} - \frac{1}{R}\right)$

(ii) At the greatest height $v = 0$

$0 = u^2 + 2gR^2\left(\frac{1}{x} - \frac{1}{R}\right)$

$2gR^2\left(\frac{1}{x} - \frac{1}{R}\right) = -u^2$

$\frac{1}{x} - \frac{1}{R} = \frac{-u^2}{2gR^2}$

$\frac{1}{x} = \frac{1}{R} - \frac{u^2}{2gR^2}$

$\frac{1}{x} = \frac{2gR - u^2}{2gR^2}$

$x = \frac{2gR^2}{2gR - u^2}$

The greatest height H above the Earth is

$H = \frac{2gR^2}{2gR - u^2} - R$

$= \frac{2gR^2 - R(2gR - u^2)}{2gR - u^2}$

$= \frac{2gR^2 - 2gR^2 + u^2R}{2gR - u^2}$

$= \frac{u^2R}{2gR - u^2}$

(iii) If the particle escapes from the Earth, there is no maximum height, since the particle never turns downward again. This is equivalent to saying $H \to \infty$

$H \to \infty \Rightarrow 2gR - u^2 = 0$

$2gR = u^2$

$u^2 = \sqrt{2gR}$ (since $u > 0$)

The body will escape the Earth if $u \geq \sqrt{2gR}$

(iv) $g = 10\text{ m/s}^2$ $R = 6400\text{ km}$

$= \frac{10}{1000}\text{ km/s}^2$

$u = \sqrt{2 \times \frac{1}{100} \times 6400}$ km/s

$= \sqrt{128}$ km/s

$= 11.31$ km/s

(v) Let t_1 be the time taken by the body to rise to a height of $15R$ above the Earth's surface. During this time x changes from R to $16R$

$$V^2 = u^2 + 2gR^2\left(\frac{1}{x} - \frac{1}{R}\right)$$

Substitute $u^2 = 2gR$

$$V^2 = 2gR + \frac{2gR^2}{x} - 2gR$$

$$= \frac{2gR^2}{x}$$

$$V = \frac{\sqrt{2g}\,R}{x^{\frac{1}{2}}}$$

$$\frac{dx}{dt} = \frac{\sqrt{2g}\,R}{x^{\frac{1}{2}}}$$

$$\frac{dt}{dx} = \frac{x^{\frac{1}{2}}}{\sqrt{2g}\,R}$$

$$dt = \frac{x^{\frac{1}{2}}}{\sqrt{2g}\,R}\,dx$$

$$\int_0^{t_1} dt = \int_R^{16R} \frac{x^{\frac{1}{2}}}{\sqrt{2g}\,R}\,dx$$

$$= \frac{1}{\sqrt{2g}\,R}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_R^{16R}$$

$$= \frac{1}{\sqrt{2g}\,R} \times \frac{2}{3}\left[x^{\frac{3}{2}}\right]_R^{16R}$$

$$= \frac{1}{\sqrt{2g}\,R} \times \frac{2}{3}\left((16R)^{\frac{3}{2}} - R^{\frac{3}{2}}\right)$$

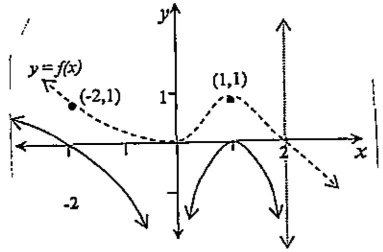
$$= \frac{1}{\sqrt{2g}\,R} \times \frac{2}{3}\left(64R^{\frac{3}{2}} - R^{\frac{3}{2}}\right)$$

$$= \frac{1}{\sqrt{2g}\,R} \times \frac{2}{3} \times 63R^{\frac{3}{2}}$$

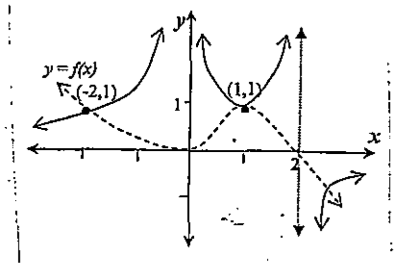
$$= \frac{1}{\sqrt{2g}} \times \frac{2}{3} \times 63R^{\frac{1}{2}}$$

$$= 42\sqrt{\frac{R}{2g}}$$

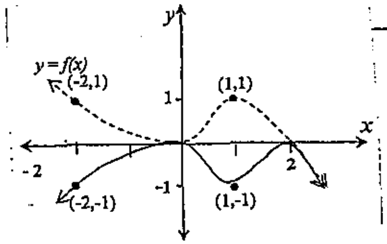
(b) (i) $y = \ln[f(x)]$



(ii) $y = \frac{1}{f(x)}$



(iii) $y = -|f(x)|$



(c) $f(x) = \frac{x^2 - 3x - 4}{x+3}$

Vertical asymptote: $x = -3$

$f(x) = 0 \Longrightarrow x^2 - 3x - 4 = 0$

$(x-4)(x+1) = 0$

$x = 4$ or -1

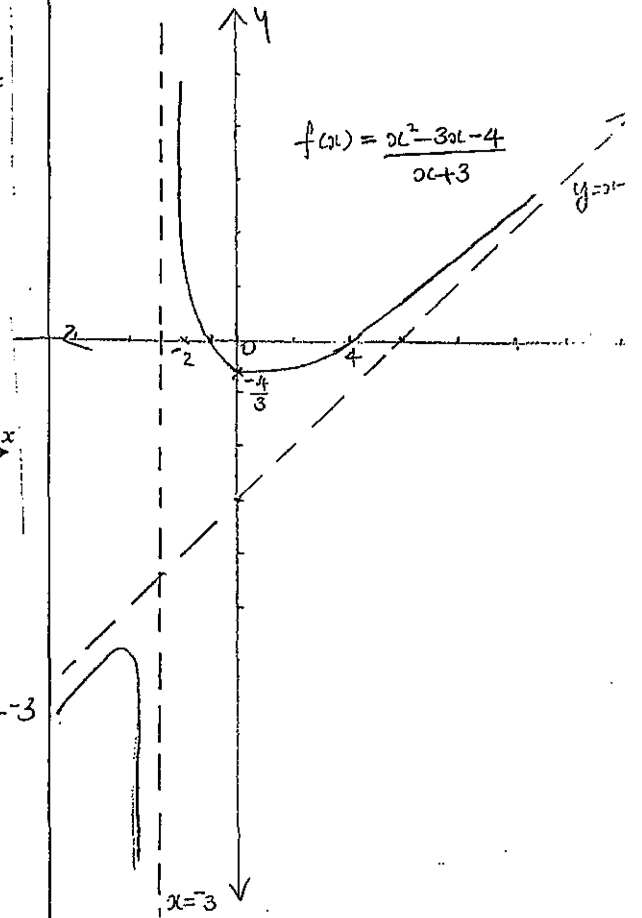
x intercepts are $(4,0)$ $(-1,0)$

$x = 0 \Longrightarrow f(x) = \frac{-4}{3}$

Y intercept $\left(0, -\frac{4}{3}\right)$

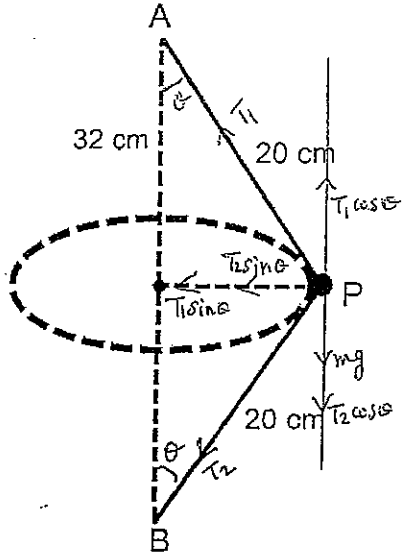
$$\begin{array}{r} x - 6 \\ x+3\overline{\big)\,x^2 - 3x - 4} \\ x^2 + 3x \\ \hline -6x - 4 \\ -6x - 18 \\ \hline 14 \end{array}$$

Slant asymptote is $y = x - 6$



Question 16 (15 marks)

a) (i)



ii) Resolving the forces at P

Vertically

$T_1\cos\theta = T_2\cos\theta + mg$

$T_1\cos\theta - T_2\cos\theta = mg$

$(T_1 - T_2)\cos\theta = mg$

$T_1 - T_2 = \frac{mg}{\cos\theta}$

$\cos\theta = \frac{16}{20} = \frac{4}{5}$

$T_1 - T_2 = \frac{5}{4}mg$

Horizontally

$T_1\sin\theta + T_2\sin\theta = \frac{mv^2}{r}$

$(T_1 + T_2)\sin\theta = \frac{mv^2}{0.12}$

$T_1 + T_2 = \frac{mv^2}{0.12} \times \frac{1}{\sin\theta}$

$\sin\theta = \frac{12}{20} = \frac{3}{5}$

$T_1 + T_2 = \frac{mv^2}{0.12} \times \_ = \frac{5}{0.36}mv^2$

$T_1 - T_2 = \frac{5}{4}mg$ —— (1)

$T_1 + T_2 = \frac{5}{0.36}mv^2$ —— (2)

adding (1) and (2)

$2T_1 = \frac{5}{4}mg + \frac{5}{0.36}mv^2$

$= m\left(\frac{5}{4}g + \frac{5}{0.36}v^2\right)$

(2) − (1) gives

$2T_2 = \frac{5}{0.36}mv^2 - \frac{5}{4}mg$

$= m\left(\frac{5v^2}{0.36} - \frac{5}{4}g\right)$

$T_1 = \frac{m}{2}\left(\frac{5}{4}g + \frac{5}{0.36}v^2\right)$

$T_2 = \frac{m}{2}\left(\frac{5v^2}{0.36} - \frac{5}{4}g\right)$

$T_2 \ge 0$

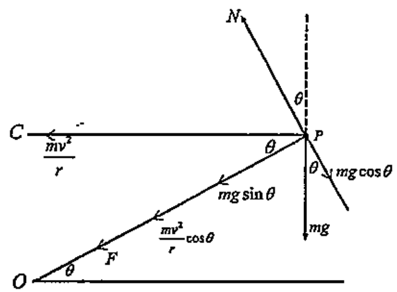
$\frac{5v^2}{0.36} - \frac{5}{4}g \ge 0$

$v^2 \ge \frac{g}{4} \times 0.36$

$v \ge 0.3\sqrt{g}$

$v \ge \frac{3}{10}\sqrt{g}$

(b) (i)



Resolving along the slope

$\frac{mv^2}{r}\cos\theta = F + mg\sin\theta$

$F = \frac{mv^2}{r}\cos\theta - mg\sin\theta$

Resolving perpendicular to the slope

$\frac{mv^2}{r}\sin\theta = N - mg\cos\theta$

$N = \frac{mv^2}{r}\sin\theta + mg\cos\theta$

(ii) $F = 0$

$\therefore \frac{mv^2}{r}\cos\theta = mg\sin\theta$

$\frac{v^2}{r}\cos\theta = g\sin\alpha$

$\frac{v^2}{rg} = \tan\alpha$; $v^2 = rg\tan\alpha$

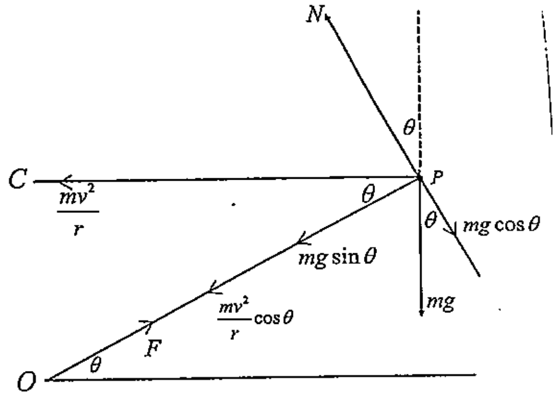
$v = \sqrt{rg\tan\alpha}$

$= \sqrt{200 \times 9.8 \times \tan 25°}$

$= 30$ m/s $= 108$ km/h

(iii) 90 km/h = 25 m/s

The car is travelling at a speed less than the design speed of the track. The car tends to slip down the track. So friction will react to this by pushing up the slope.

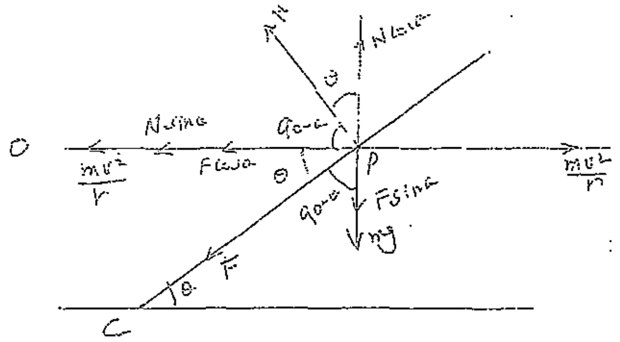


$F = mg\sin\theta - \frac{mv^2}{r}\cos\theta$

$= 1500 \times 9.8 \times \sin 25 - \frac{1500 \times 625}{200} \times \cos 25°$

= 1964 Newtons up the track

Alternative solution - Resolving forces along horizontal and vertical direction.



horizontally :

$$N\sin\theta + F\cos\theta = \frac{mv^2}{r} \quad (1)$$

Vertically :

$$N\cos\theta - F\sin\theta = mg \quad (2)$$

(1) × sin θ $\quad N\sin^2\theta + F\sin\theta\cos\theta = \frac{mv^2}{r}\sin\theta \quad (3)$

(2) × cos θ $\quad N\cos^2\theta - F\sin\theta\cos\theta = mg\cos\theta \quad (4)$

(3) + (4) $\quad N(\sin^2\theta + \cos^2\theta) = \frac{mv^2}{r}\sin\theta + mg\cos\theta$

$$N = \frac{mv^2}{r}\sin\theta + mg\cos\theta$$

(1) × cos θ $\quad N\sin\theta\cos\theta + F\cos^2\theta = \frac{mv^2}{r}\cos\theta \quad (5)$

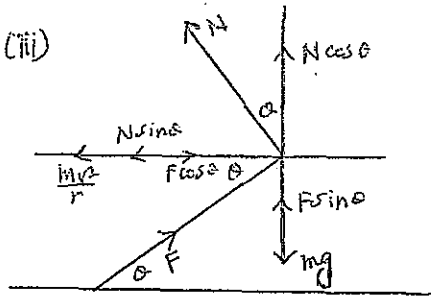
(2) × sin θ $\quad N\sin\theta\cos\theta - F\sin^2\theta = mg\sin\theta \quad (6)$

(5) − (6) $\quad F\cos^2\theta + F\sin^2\theta = \frac{mv^2}{r}\cos\theta - mg\sin\theta$

$$F = \frac{mv^2}{r}\cos\theta - mg\sin\theta$$

(ii) Same as the previous method

(iii)



horizontally:

$$\frac{mv^2}{r} = N\sin\theta - F\cos\theta \quad (1)$$

Vertically :

$$mg = F\sin\theta + N\cos\theta \quad (2)$$

(1) × cos θ $\quad N\sin\theta\cos\theta - F\cos^2\theta = \frac{mv^2}{r}\cos\theta \quad (3)$

(2) × sin θ $\quad F\sin^2\theta + N\sin\theta\cos\theta = mg\sin\theta \quad (4)$

(4) − (3) $\quad F\sin^2\theta + F\cos^2\theta = mg\sin\theta - \frac{mv^2}{r}\cos\theta$

$$F = mg\sin\theta - \frac{mv^2}{r}\cos\theta$$

$$= 1500 \times 9.8 \times \sin 25 - \frac{1500 \times 625}{200}\cos 25^\circ$$

= 1964 Newtons up the track.