GIRRAWEEN HIGH SCHOOL

MATHEMATICS EXTENSION 2

TASK 4 2013 – TRIAL EXAMINATION

ANSWERS COVER SHEET

FINAL MARK

Name:____________________________ **Teacher:**____________________________

QUESTION	MARK	E2	E3	E4	E5	E6	E7	E8	E9
Multiple Choice 1-3	/3	√	√						√
Multiple Choice 4	/1	√		√					√
Multiple Choice 5,6	/2	√	√	√					√
Multiple Choice 7	/1	√						√	√
Multiple Choice 8	/1	√					√		√
Multiple Choice 9,10	/2	√			√				√
TOTAL	/10								
11ab	/11	√						√	√
c	/4	√	√						√
	/15								
12ab	/9	√	√						√
c	/6			√					√
	/15								
13	/15	√	√	√					√
	/15								
14a-c	/11	√					√		√
d	/4	√						√	√
	/15								
15ab	/10	√			√				√
c	/5		√						
	/15								
16a	/10	√			√				√
b	/5							√	
	/15								
TOTAL	/100	/100	/38	/24	/22		/12	/21	/100

HSC Outcomes **Mathematics Extension 2**

E1 appreciates the creativity, power and usefulness of mathematics to solve a broad range of problems.

E2 chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings.

E3 uses the relationship between algebraic and geometric representations of complex numbers and of conic sections.

E4 uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials.

E5 uses ideas and techniques from calculus to solve problems in mechanics involving resolution of forces, resisted motion and circular motion

E6 combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions.

E7 uses the techniques of slicing and cylindrical shells to determine volumes.

E8 applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems.

E9 communicates abstract ideas and relationships using appropriate notation and logical argument.

GIRRAWEEN HIGH SCHOOL

TRIAL EXAMINATION

2013

MATHEMATICS

EXTENSION 2

Time allowed - Three hours

(Plus 5 minutes' reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- For Multiple choice questions 1 – 10: Circle the correct answer on your examination paper.
- For Questions 11 – 16: All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board-approved calculators may be used.
- Each question attempted is to be returned on a *separate* piece of paper clearly marked Question 1 , Question 2 , etc. Each piece of paper must show your name.
- You may ask for extra pieces of paper if you need them.
- A list of board approved integrals is provided.

Multiple choice: Questions 1 – 10: Circle the correct answer on this question paper.

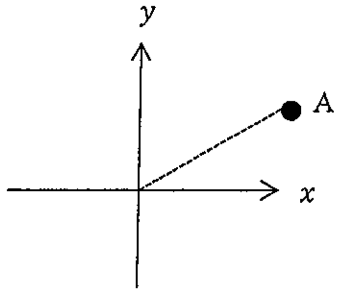
Question 1

If $z = 3 - i$ and $w = 2i - 1$ then $\bar{z} - w =$

(A) $4 - 3i$ (B) $4 - i$ (C) $2 - 3i$ (D) $4 + i$

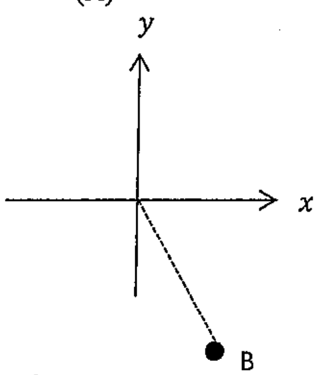
Question 2

If $\overrightarrow{OA} = z$ on the diagram below:

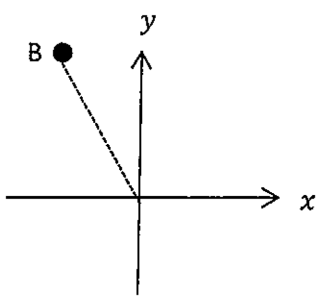


In which of the following diagrams does $\overrightarrow{OB}$ represent iz?

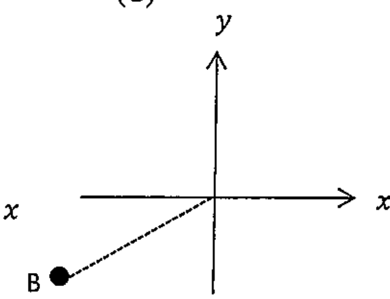
(A)



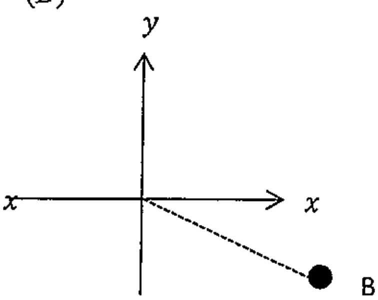
(B)



(C)



(D)



Question 3

The modulus and argument of $\sqrt{6} - i\sqrt{2}$ are

(A) $2\sqrt{2}\ and\ \frac{\pi}{6}$ (B) $2\sqrt{2}\ and\ -\frac{\pi}{6}$ (C) $2\sqrt{2}\ and\ \frac{\pi}{3}$ (D) $2\sqrt{2}\ and\ -\frac{\pi}{3}$

Question 4

If $\alpha, \beta, and\ \gamma$ are the roots of the polynomial equation $2x^3 - x^2 + 6x - 3 = 0$ then $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} =$

(A) $\frac{1}{3}$ (B) 3 (C) $\frac{1}{2}$ (D) 2

Question 5

The ellipse $\frac{x^2}{9} + \frac{y^2}{25} = 1$ has foci at

(A) $(\pm 4, 0)$ (B) $(\pm 3, 0)$ (C) $(0, \pm 4)$ (D) $(0, \pm 3)$

Question 6

The hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$ has directrices at

(A) $x = \pm\frac{9}{5}$ (B) $y = \pm\frac{9}{5}$ (C) $x = \pm\frac{16}{5}$ (D) $y = \pm\frac{16}{5}$

Question 7

$$\int \frac{e^x}{\sqrt{1-e^{2x}}}.dx =$$

(A) $tan^{-1}(e^x) + C$ (B) $sin^{-1}(e^x) + C$ (C) $\frac{1}{2}\ln(1-e^{2x}) + C$ (D) $-\frac{1}{2}\ln(1-e^{2x}) + C$

Question 8

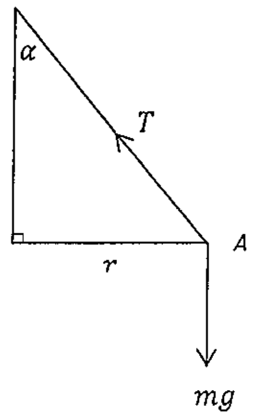
The area enclosed by the parabola $y = 4 - x^2$, the x axis and the y axis in the first quadrant is rotated about the x axis. An expression for finding the resulting volume is

(A) $V = \pi\int_0^2 (4-x^2)^2.dx$ (B) $V = \pi\int_0^4 (4-x^2)^2.dx$ (C) $V = 2\pi\int_0^2 y\sqrt{4-y}.dy$

(D) $V = 2\pi y\int_0^4 \sqrt{4-y}.dy$

Question 9

When resolving forces on this conical pendulum in the horizontal and vertical directions, if T is the tension in the string, mg is the force on the particle at A due to gravity, r is the radius of the horizontal circle the particle is rotating around, α is the angle the string makes with the vertical and w is the angular velocity of the particle, $\tan\alpha =$



(A) $\frac{w^2}{rg}$ (B) $\frac{g}{rw^2}$ (C) $\frac{rw^2}{g}$ (D) $\frac{gr}{w^2}$

Question 10

The force of Earth's gravity on an object is inversely proportional to the square of the distance of that object from the centre of Earth *(see diagram).*

$$F = ma = \frac{mk}{x^2}$$

If the radius of the Earth is R and the force due to gravity on the object at the Earth's surface is mg then $k =$

(A) $\frac{g}{R^2}$ (B) gR^2 (C) gR (D) Can't be determined at the Earth's surface because $x = 0$.

Question 11 ***(15 marks) Show all necessary working on a separate page*** **Marks**

(a) Evaluate the following integrals:

(i) $\int x^8 lnx.dx$ **2**

(ii) $\int e^x sinx.dx$ **3**

(iii) $\int \frac{1}{cosx-1}.dx$ **2**

(b) Express $\frac{-9}{(x+2)^2(x-1)}$ in the form $\frac{A}{(x+2)^2}+\frac{B}{(x+2)}+\frac{C}{(x-1)}$. Hence find $\int \frac{-9}{(x+2)^2(x-1)}.dx$ **4**

(c) (i) If $(x+iy)^2 = 5-12i, x, y, real$ find the values of x and y. **3**

(ii) Hence solve the quadratic equation $z^2+(1-2i)z+(2i-2)=0$ **1**

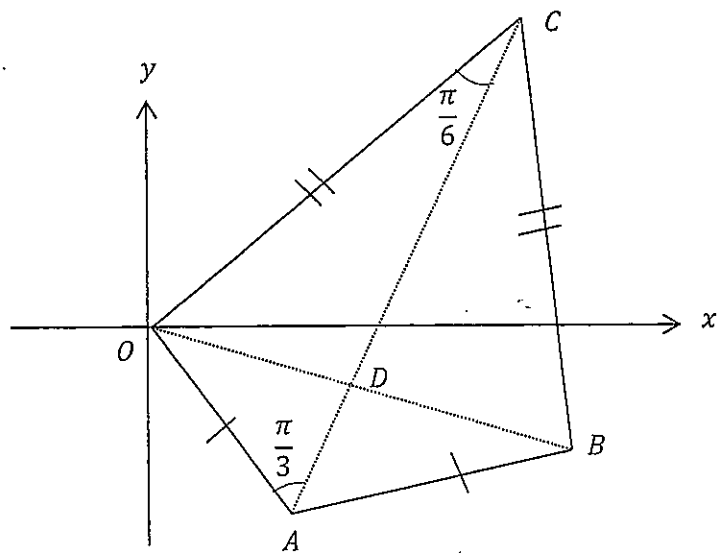
Question 12 ***(15 marks) Show all necessary working on a separate page***

(a) (i) Find $(1+i\sqrt{3})(1+i)$ in Cartesian form. **1**

(ii) Express $1+i\sqrt{3}$ and $1+i$ in modulus/argument form. Hence find $(1+i\sqrt{3})(1+i)$ in modulus/argument form. **2**

(iii)Hence find the exact value of $tan\frac{7\pi}{12}$. **2**

(b) In the Argand diagram below, $OABC$ is a kite. $OA = AB, OC = CB$, $\overrightarrow{OA} = w$, $\angle OAC = \frac{\pi}{3}$ and $\angle OCA = \frac{\pi}{6}$. Find the complex numbers $\overrightarrow{OB}$ and $\overrightarrow{OC}$ in terms of w. **4**



(c) If the roots of the polynomial equation $3x^3-11x^2+17x+7=0$ are α, β and γ

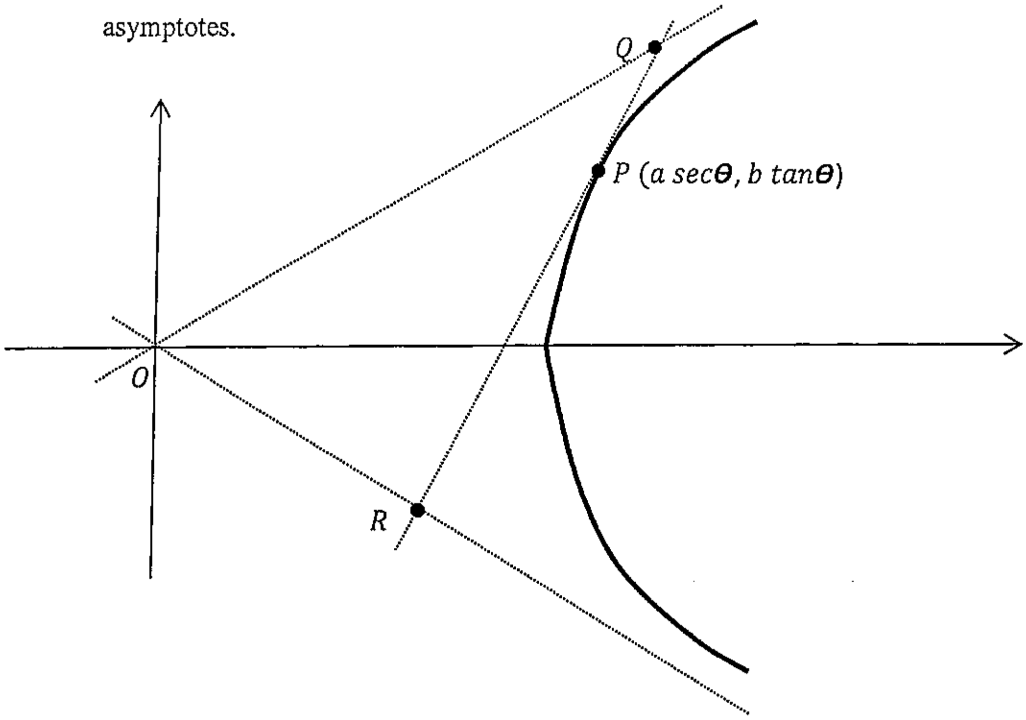
(i) Find the value of $\alpha^3+\beta^3+\gamma^3$ **3**

(ii) Form the polynomial equation with roots α^2, β^2 and γ^2 **3**

Question 13 *(15 marks) Show all necessary working on a separate page* Marks

(a) $P\,(a\sec\theta, b\tan\theta)$ is an arbritrary point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Q and R are the points where the tangent to the hyperbola at P intersects with the asymptotes.



(i) Show that the equation of the tangent to the hyperbola at P is **2**

$$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$$

(ii) Show that the Q is the point $\left(\frac{a}{\sec\theta - \tan\theta}, \frac{b}{\sec\theta - \tan\theta}\right)$ **2**

(iii) The coordinates of R are $\left(\frac{a}{\sec\theta + \tan\theta}, \frac{-b}{\sec\theta + \tan\theta}\right)$ **DO NOT PROVE THIS!**

Show that the distance OR is $\frac{ae}{\sec\theta + \tan\theta}$. **2**

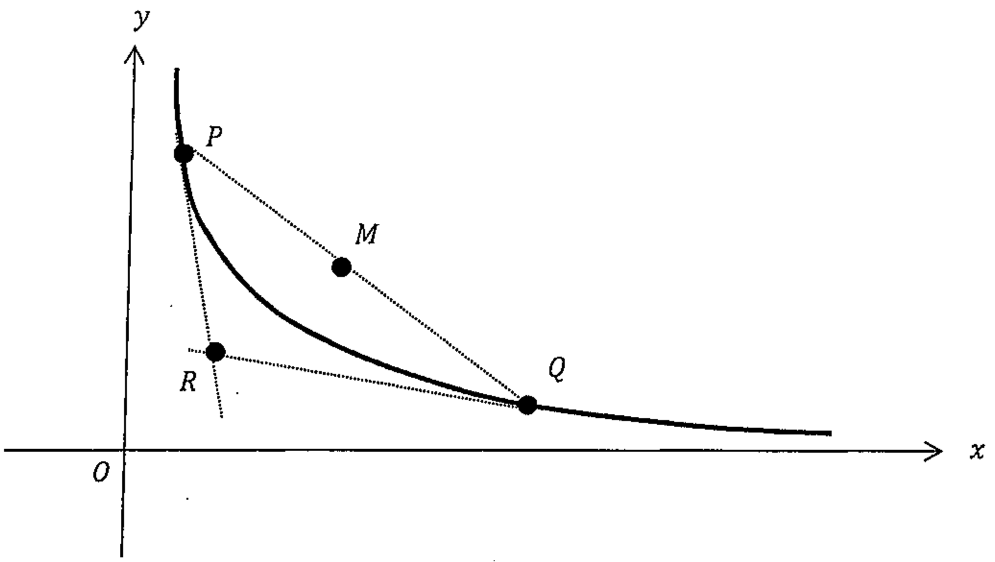
(iv) Show that the distance from the line OR to Q is $\frac{2ab}{ae(\sec\theta - \tan\theta)}$. **2**

(v) Show that the area of triangle OQR is a constant. **1**

Question 13 continues on the next page

(b) $P\ (cp, \frac{c}{p})$ and $Q\ (cq, \frac{c}{q})$ are two points on the rectangular hyperbola $xy = c^2$.

R is the point where the tangents at P and Q meet and M is the midpoint of the chord PQ.



(i)The tangent to the hyperbola at P is $x + p^2y = 2cp$. **DO NOT PROVE THIS!**

State the equation of the tangent at Q and show that R is the point $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$ and **3**

M is the point $\left(\frac{c(p+q)}{2}, \frac{c(p+q)}{2pq}\right)$.

(ii)If R is the midpoint of OM show that $(p + q)^2 = 8pq$. **1**

(iii)Hence find the locus of M if R is the midpoint of OM. **2**

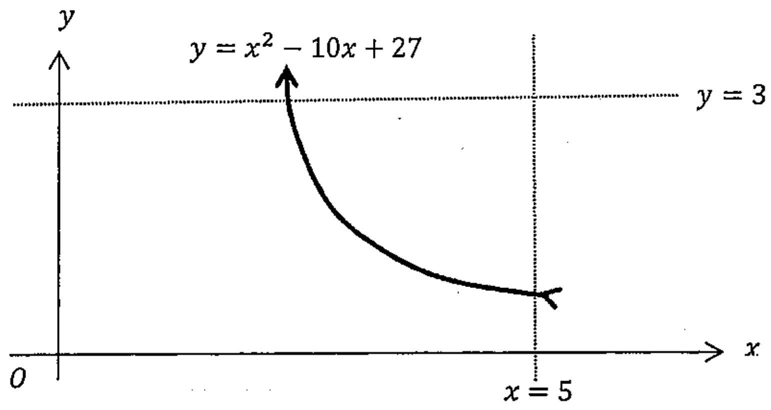
Examination continues on the next page

Question 14 *(15 Marks)* **Marks**

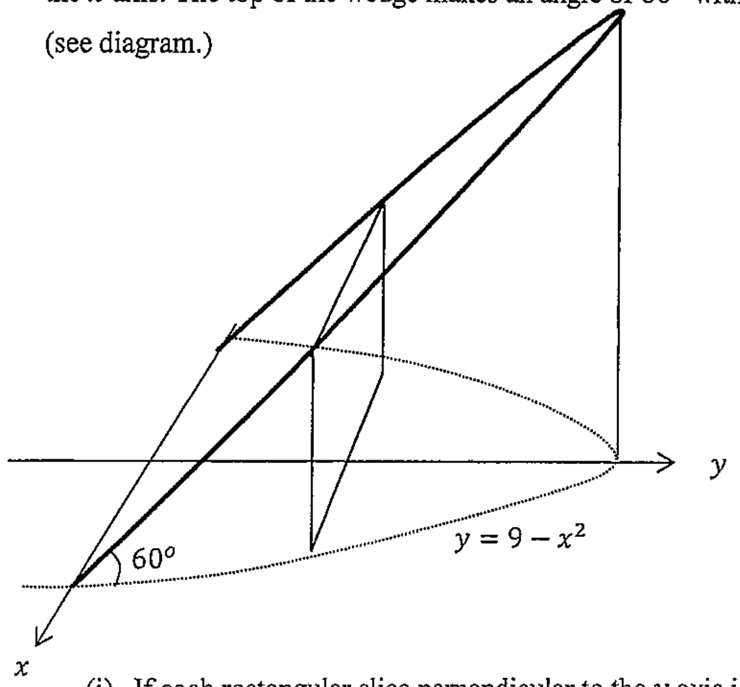
(a) (i) Prove that $\int_0^a f(a-x).dx = \int_0^a f(x).dx$ **1**

(ii)Hence find $\int_0^3 x^2(3-x)^{20}.dx$ **2**

(b) The area enclosed by the curve $y = x^2 - 10x + 27$ and the lines $y = 3$ and $x = 5$ is rotated about the line $x = 5$ (see below.) Find the volume of the solid formed using the method of cylindrical shells. **4**



(c) The base of a wedge is the area enclosed by the curve $y = 9 - x^2$ and the x axis. The top of the wedge makes an angle of 60^o with the xy plane (see diagram.)



(i) If each rectangular slice perpendicular to the y axis is δy thick, show that the volume of a slice is $2y\sqrt{3}\sqrt{(9-y)}.\delta y$. **2**

(ii)Find the volume of the wedge. **2**

Question 14 continues on the next page

Question 14 (continued) **Marks**

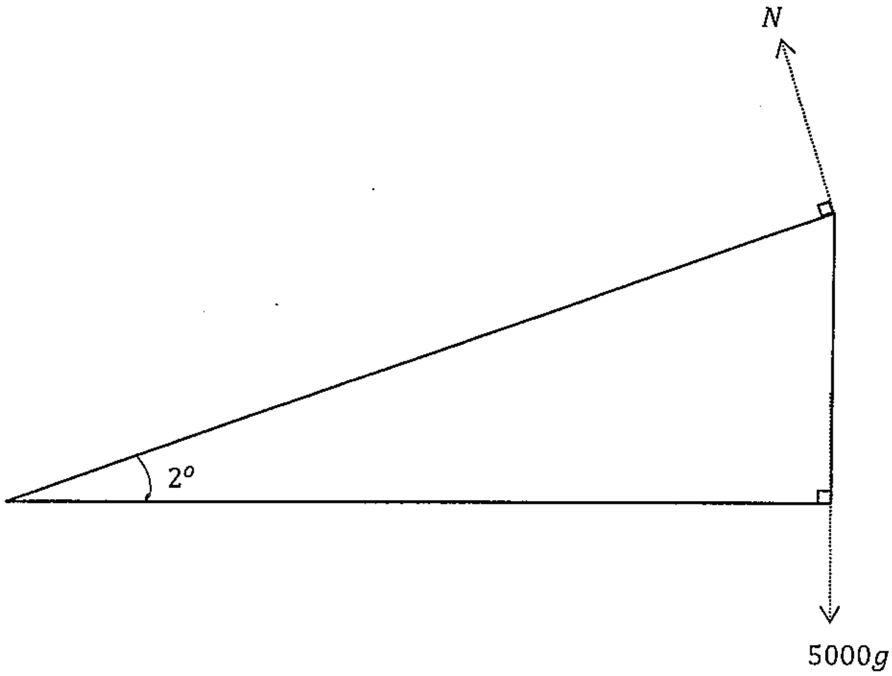
(d) Let $I_n = \int_0^{\frac{\pi}{2}} \cos^n x.\, dx$

(i) Show that $I_n = \frac{n-1}{n} I_{n-2}$ 3

(ii) Hence or otherwise find $\int_0^{\frac{\pi}{2}} \cos^8 x.\, dx$ 1

Question 15 *(15 Marks)*

(a) A 5 ton truck is rounding a curve with a radius of 500 metres which is banked at an angle of 2^o to the horizontal (see diagram.)



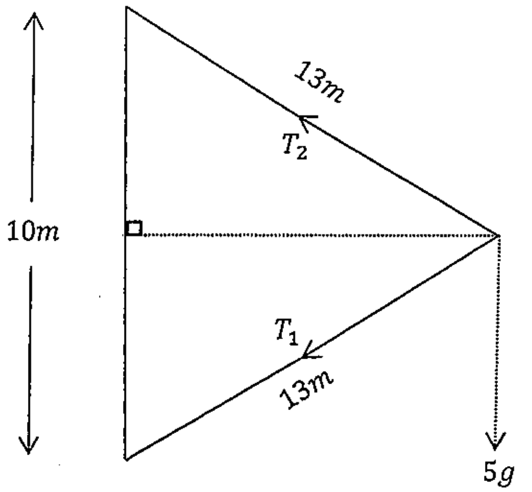
(i) By resolving forces (either vertically and horizontally or parallel and perpendicular to the plane), determine the optimum speed that the truck can take the bend at so that there is no lateral friction on the tyres. (Use $g = 9.8m/s^2$) 2

(ii) The truck rounds the curve at $72km/h$ (which is faster than the optimum speed). How much friction (in Newtons) is exerted on the tyres? 3

Question 15 continues on the next page

Question 15 (continued) **Marks**

(b) A particle weighing $5kg$ is attached by two strings each 13 metres long to the top and base of a vertical pole 10 metres long. It is rotating around the pole in a horizontal circle at $20m/s$. The tensions in each string are T_1 and T_2 respectively (see diagram.)



(i) By resolving forces in the vertical and horizontal directions, show that **2**

$$T_1 \times \frac{12}{13} + T_2 \times \frac{12}{13} = \frac{500}{3}$$

and $T_2 \times \frac{5}{13} - T_1 \times \frac{5}{13} = 5g$

(ii) Find the tension in each string (use $g = 9.8m/s^2$) **3**

(c) Let $w = cos\frac{2\pi}{9} + isin\frac{2\pi}{9}$.

(i) Show that w^n is a root of $z^9 - 1 = 0$, n an integer. **1**

(ii) Show that $w + w^8 = 2cos\frac{2\pi}{9}$ **1**

(iii) Show that $(w^3 + w^6)(w^2 + w^7) = w + w^8 + w^4 + w^5$ **1**

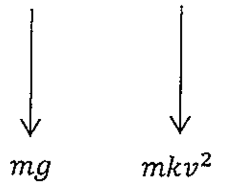
(iv) Hence show that $cos\frac{2\pi}{9} + cos\frac{4\pi}{9} = cos\frac{\pi}{9}$. (You may assume **2**

that $cos\frac{2\pi}{3} = -\frac{1}{2}$)

Examination continues on the next page

Question 16 *(15 marks)* **Marks**

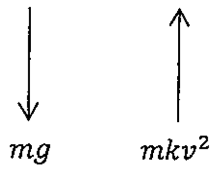
(a) A particle is launched vertically upwards from the ground ($x = 0$) at $U\ m/s$. It experiences gravity (mg) and air resistance proportional to the square of its velocity (mkv^2).(se diagram)



(i) If x is the distance upwards from the ground, show that $x = -\frac{1}{2k}\ln\left(\frac{kv^2+g}{kU^2+g}\right)$. **3**

(ii) If H is the maximum height that the particle reaches, show that $H = -\frac{1}{2k}\ln\left(\frac{g}{kU^2+g}\right)$. **1**

(iii) The particle starts to fall from its maximum height (see diagram).



Show that the terminal velocity T (the velocity that the particle can never exceed as it falls) is given by $T = \sqrt{\frac{g}{k}}$. **1**

(iv) Taking $x = 0$ as the TOP position (when the particle starts to fall), show that $x = -\frac{1}{2k}\ln\left(\frac{g-kv^2}{g}\right)$ when the particle is falling. **3**

(v) The particle hits the ground ($x = H$) with velocity W. Show that $H = -\frac{1}{2k}\ln\left(\frac{g-kW^2}{g}\right)$. **1**

(vi) Using your answers to (ii), (iii) and (v), show that $\frac{1}{U^2} + \frac{1}{T^2} = \frac{1}{W^2}$. **1**

Question 16 continues on the next page

(b) Let $I_n = \int_0^1 x^n e^{-x}.\,dx$

(i) Show that $I_n = nI_{n-1} - \frac{1}{e}$ **1**

(ii) Find I_3 **1**

(iii) Prove by induction that $I_n = n! - \left(\dfrac{\frac{n!}{0!}+\frac{n!}{1!}+\frac{n!}{2!}+\cdots+\frac{n!}{n!}}{e}\right)$ **2**

(iv) Using the fact that $\lim_{n\to\infty} \int_0^1 x^n e^{-x}.\,dx = 0$, show that **1**

$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$

Here endeth the examination!!!

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Solutions:

Multiple Choice: (1) B (2) B (3) B (4) D (5) C (6) A (7) B (8) A (9) C (10) B

(11)(a)(i) $\int x^8 \ln x \, dx$ $\quad u = \ln x \quad v = \frac{1}{9}x^9$

$u' = \frac{1}{x} \quad v' = x^8$

By $\int u \cdot \frac{dv}{dx} \, dx = uv - \int v \cdot \frac{du}{dx} \, dx$

$$\int x^8 \ln x \, dx = \frac{x^9 \ln x}{9} - \frac{1}{9}\int x^9 \times \frac{1}{x} \, dx \quad 1$$

$$= \frac{x^9 \ln x}{9} - \frac{1}{9}\int x^8 \, dx \quad \underline{2}$$

$$= \frac{x^9 \ln x}{9} - \frac{x^9}{81} + C. \quad 1$$

(ii) $\int e^x \sin x \, dx$ $\quad u = e^x \quad v = -\cos x$

$u' = e^x \quad v' = \sin x$

By $\int u \cdot \frac{dv}{dx} \, dx = uv - \int v \cdot \frac{du}{dx} \, dx$

$$\int e^x \sin x \, dx = -\cos x e^x + \int \cos x e^x \, dx \quad (1) \quad 1$$

Taking $\int \cos x e^x \, dx$ out of (1) and evaluating

$u = e^x \quad v = \sin x$

$u' = e^x \quad v' = \cos x$ $\quad \underline{3}$

$$\int u \cdot \frac{dv}{dx} \, dx = uv - \int v \cdot \frac{du}{dx} \, dx$$

$$= e^x \sin x - \int e^x \sin x \, dx \quad (2)$$

Sub. (2) in (1): $\int e^x \sin x \, dx = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$ $\quad 1$

$$\int e^x \sin x \, dx = \frac{1}{2}e^x(\sin x - \cos x) + C. \quad 1$$

Q. (11)(a)(iii) $\displaystyle\int \frac{1}{\cos x - 1}\,dx$

Using $t = \tan\left(\frac{x}{2}\right)$

$$dx = \frac{2}{t^2+1}\,dt$$

$$= \int \frac{1}{\frac{1-t^2}{1+t^2} - 1} \cdot \frac{2}{1+t^2}\,dt \qquad 1$$

$$= \int \frac{2}{-2t^2}\,dt$$

$$= \int -\frac{1}{t^2}\,dt \qquad \underline{2}$$

$$= \frac{1}{t} + C \qquad 1$$

$$= \frac{1}{\tan\left(\frac{x}{2}\right)} + C \text{ or } = \cot\left(\tfrac{x}{2}\right) + C$$

(b) Let $\displaystyle\frac{-9}{(x+2)^2(x-1)} \equiv \frac{A}{(x+2)^2} + \frac{B}{(x+2)} + \frac{C}{(x-1)}$

$\therefore A(x-1) + B(x+2)(x-1) + C(x+2)^2 \equiv 0x^2 + 0x - 9$

Sub. in $x = -2$:

$-3A = -9 \Rightarrow A = 3.$ 1

Sub. in $x = 1$

$3^2 \times C = -9 \Rightarrow 9C = -9 \Rightarrow C = -1.$ 1

Sub. in $x = 0$ $\underline{4}$

$-A - 2B + 4C = -9$

$-3 - 2B - 4 = -9 \Rightarrow B = 1.$ 1

$$\therefore \int \frac{-9}{(x+2)^2(x-1)}\,dx = \int \frac{3}{(x+2)^2} + \frac{1}{(x+2)} - \frac{1}{(x-1)}\,dx$$

$$= -\frac{3}{(x+2)} + \ln(x+2) - \ln(x-1) + C.$$

$$= \underline{-\frac{3}{(x+2)} + \ln\left(\frac{x+2}{x-1}\right) + C.} \qquad 1$$

Solutions p. 3

Q.(11)(c)(i) $(x+iy)^2 = 5-12i$

$(x^2-y^2)+2ixy = 5-12i$

Equating real parts, $x^2-y^2 = 5$ (1)

Equating imaginary parts, $2xy = -12$ (2)] 1

$y = \frac{-6}{x}$

Sub. (2) in (1): $x^2 - \frac{36}{x^2} = 5$ 1

$x^4 - 5x^2 - 36 = 0$

$(x^2-9)(x^2+4) = 0$ $\underline{3}$

$x = \pm 3$ $x \neq \pm 2i$ as x is real.

If $\underline{x = \pm 3, y = \mp 2}$ (as $y = \frac{-6}{x}$) 1

(ii) $z^2 + (1-2i)z + (2i-2) = 0$

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-1+2i \pm \sqrt{(1-2i)^2 - 4\times 1\times(2i-2)}}{2\times 1}$$

$$= \frac{-1+2i \pm \sqrt{5-12i}}{2}$$

$$= \frac{-1+2i \pm (3-2i)}{2}$$

$\underline{z = 1 \quad \text{or} \quad z = -2+2i}$ $\underline{1}$

Solutions p. 4

Q. (12) (a) (i) $(1+i\sqrt{3})(1+i)$

$= 1 + i + i\sqrt{3} - \sqrt{3}$

$= (1-\sqrt{3}) + i(1+\sqrt{3})$ 1

(ii) $1+i\sqrt{3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ 1

$\therefore (1+i\sqrt{3})(1+i)$

$= 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \times \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ 2

$= 2\sqrt{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$ 1

(iii) Using (1) and (2): and equating imaginaries:

$2\sqrt{2}\sin\frac{7\pi}{12} = 1+\sqrt{3} \Rightarrow \sin\frac{7\pi}{12} = \frac{1+\sqrt{3}}{2\sqrt{2}}$

Equating reals: 1

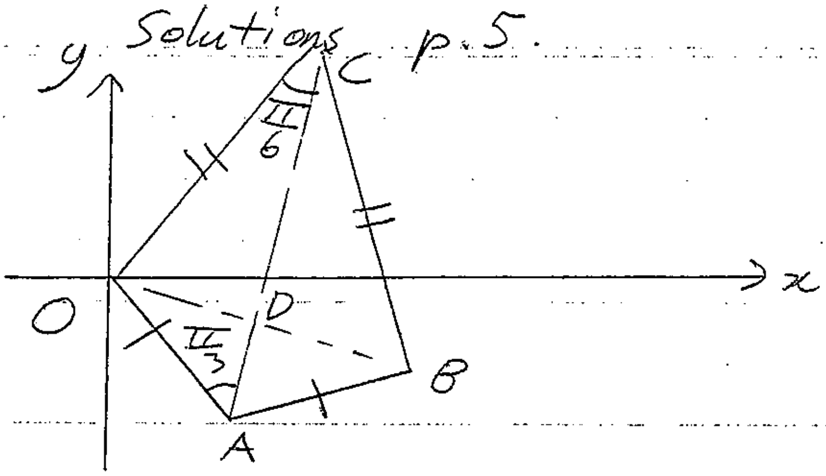
$2\sqrt{2}\cos\frac{7\pi}{12} = 1-\sqrt{3} \Rightarrow \cos\frac{7\pi}{12} = \frac{1-\sqrt{3}}{2\sqrt{2}}.$

$\therefore \tan\frac{7\pi}{12} = \dfrac{\sin\frac{7\pi}{12}}{\cos\frac{7\pi}{12}}$ 2

$= \frac{1+\sqrt{3}}{2\sqrt{2}} \div \frac{1-\sqrt{3}}{2\sqrt{2}}$ 1

$\tan\frac{7\pi}{12} = \frac{1+\sqrt{3}}{1-\sqrt{3}}$

Solutions p.5.



$\angle ODA = \frac{\pi}{2}$ [Diagonals of kite perpendicular].

$\therefore \angle AOB = \frac{\pi}{6}$ [$\angle$ sum $\triangle AOD$].

$\angle COD = \frac{\pi}{3}$ [$\angle$ sum $\triangle COD$]. 1

$\therefore \angle AOC = \frac{\pi}{6} + \frac{\pi}{3}$

$= \frac{\pi}{2}$ [adjacent $\angle$'s].

Using $\triangle OAD$, $\sin\frac{\pi}{3} = \frac{OD}{OA}$

$\therefore OA\sin\frac{\pi}{3} = OD.$

$\frac{OA\sqrt{3}}{2} = OD$

$OA\sqrt{3} = OB$ [as AC bisects OB, one diagonal bisects other in kite].

$\therefore \vec{OB} = w \times \sqrt{3} \times (\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$

$= w\left(\frac{3}{2} + i\frac{\sqrt{3}}{2}\right)$ 1

Using $\triangle ODC$: $\sin\frac{\pi}{6} = \frac{OD}{OC}$

$OC = \frac{OD}{\sin\frac{\pi}{6}}$

$= 2OD.$

$= 2 \times OA\frac{\sqrt{3}}{2}$

$= OA\sqrt{3}.$ 1

4

$\therefore \vec{OC} = w \times \sqrt{3}(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$

$= iw\sqrt{3}.$ 1

Solutions p. 6

Q.12 (a)(i) Firstly, $\alpha+\beta+\gamma = -\frac{b}{a} = \frac{11}{3}$

$$\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$$
$$= \left(\frac{11}{3}\right)^2 - 2\times\frac{17}{3}$$
$$= \frac{19}{9}. \qquad 1$$

Next, as α is a root of $3x^3-11x^2+17x+7=0$,

$3\alpha^3-11\alpha^2+17\alpha+7=0$ (1)

As β is a root, $3\beta^3-11\beta^2+17\beta+7=0$ (2) +

As γ is a root, $3\gamma^3-11\gamma^2+17\gamma+7=0$ (3) $\qquad 1$

$$3(\alpha^3+\beta^3+\gamma^3)-11(\alpha^2+\beta^2+\gamma^2)+17(\alpha+\beta+\gamma)+21=0$$
$$3(\alpha^3+\beta^3+\gamma^3)-11\times\frac{19}{9}+17\times\frac{11}{3}+21=0 \qquad \underline{3}$$
$$\alpha^3+\beta^3+\gamma^3 = -\frac{541}{27} \text{ or } -20\tfrac{1}{27} \qquad 1$$

(ii) Let $x=\sqrt{y}$

$\therefore$ $3x^3-11x^2+17x+7=0$ becomes

$$3y\sqrt{y}-11y+17\sqrt{y}+7=0 \qquad 1$$
$$3y\sqrt{y}+17\sqrt{y} = 11y-7$$
$$\sqrt{y}(3y+17) = 11y-7$$

Squaring BS:

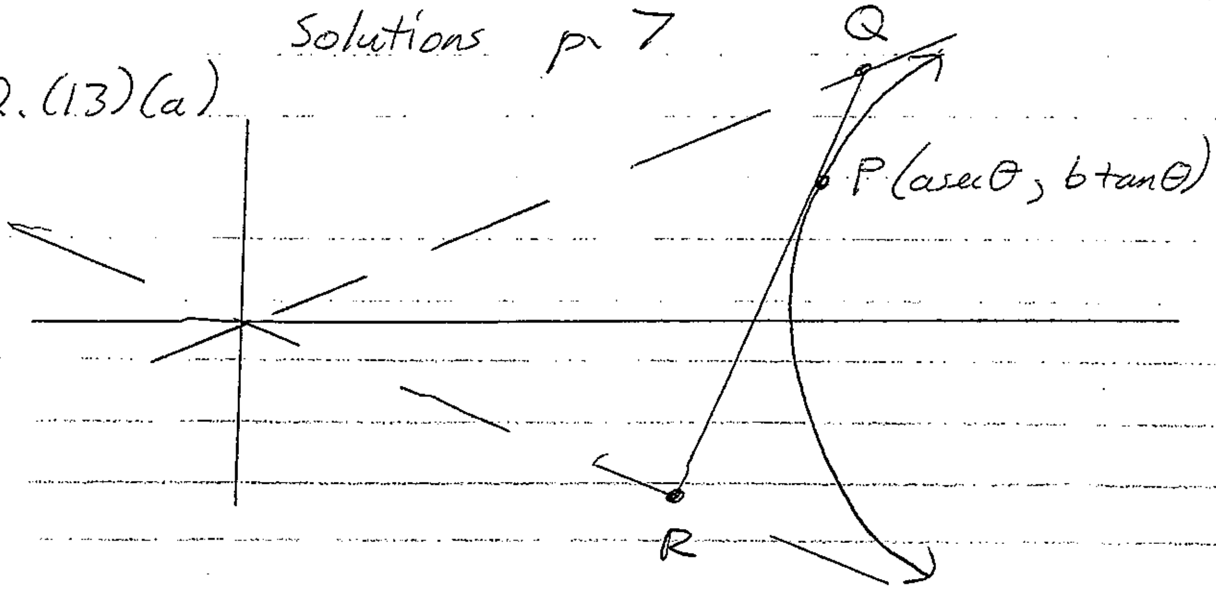
$$y(9y^2+102y+289) = 121y^2-154y+49 \qquad 1$$
$$9y^3-19y^2+443y-49=0$$

Equation with roots $\alpha^2, \beta^2, \gamma^2$ is

$$9x^3-19x^2+443x-49=0 \qquad 1 \qquad \underline{3}$$

Solutions p. 7

Q. (13)(a)



(i) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

$\frac{2x}{a^2} - \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$

$\frac{2x}{a^2} = \frac{2y}{b^2} \cdot \frac{dy}{dx}$

$\frac{b^2 x}{a^2 y} = \frac{dy}{dx}$

or could do:

$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$, $x = a\sec\theta$, $y = b\tan\theta$

$= b\sec^2\theta \times \frac{1}{a\sec\theta\tan\theta}$

$= \frac{b\sec\theta}{a\tan\theta}$. 1

or

At P $(a\sec\theta, b\tan\theta)$, $\frac{dy}{dx} = \frac{b^2 a\sec\theta}{a^2 b\tan\theta}$.

$= \frac{b\sec\theta}{a\tan\theta}$ 1

By $y - y_1 = m(x - x_1)$

$y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$ 1

2

$ay\tan\theta - ab\tan^2\theta = bx\sec\theta - ab\sec^2\theta$

$ab(\sec^2\theta - \tan^2\theta) = bx\sec\theta - ay\tan\theta$ $\div ab$ & as $\sec^2\theta - \tan^2\theta = 1$,

$1 = \frac{x\sec\theta}{a} - \frac{y\tan\theta}{b}$

or $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$ as required.

Solutions: p. 8

Q. (13)(a)(ii) Q is where

$$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1 \text{ intersects with } y = \frac{b}{a}x$$

$$\therefore \quad \frac{x\sec\theta}{a} - \frac{b}{a}\frac{x\tan\theta}{b} = 1.$$

$$x\left(\frac{\sec\theta - \tan\theta}{a}\right) = 1.$$

2

$$x = \frac{a}{\sec\theta - \tan\theta}$$

As $y = \frac{b}{a}x$, $y = \dfrac{b}{\sec\theta - \tan\theta}$

1

Co-ordinates of Q are $\left(\dfrac{a}{\sec\theta - \tan\theta}, \dfrac{b}{\sec\theta - \tan\theta}\right)$ as required.

(iii) Distance OR $= \sqrt{\dfrac{a^2 + b^2}{(\sec\theta + \tan\theta)^2}}$ 1

$$= \frac{\sqrt{a^2+b^2}}{\sec\theta + \tan\theta}.$$

As $e^2 = 1 + \dfrac{b^2}{a^2}$, $a^2e^2 = a^2 + b^2$

$$\therefore \quad OR = \frac{\sqrt{a^2e^2}}{\sec\theta + \tan\theta}$$

2

$$= \frac{ae}{\sec\theta + \tan\theta}.$$

1

PTO →

Solutions: p. 9

Q. (13)(a)(iv)

Using perpendicular distance from $y = -\frac{b}{a}x$

i.e. $bx + ay = 0$

to $Q\left(\frac{a}{\sec\theta - \tan\theta}, \frac{b}{\sec\theta - \tan\theta}\right)$

$$\text{Distance} = \left|\frac{Ax_1 + By_1 + C}{\sqrt{A^2+B^2}}\right|$$

$A = b$
$B = a$

$$= \left|\frac{\frac{ab}{\sec\theta - \tan\theta} + \frac{ab}{\sec\theta - \tan\theta} + 0}{\sqrt{b^2+a^2}}\right| \quad 1$$

$$= \frac{2ab}{(\sec\theta - \tan\theta)\sqrt{a^2+b^2}}$$

Noting that $ae = \sqrt{a^2+b^2}$ [proven in (iii)].

$$\text{Distance from } OR \text{ to } Q = \frac{2ab}{ae(\sec\theta - \tan\theta)} \quad 1$$

$\underline{2}$

(v) Area $\triangle OQR = \frac{1}{2} \times B \times h$

$$= \frac{1}{2} \times OR \times \text{Distance } OR \text{ to } Q$$

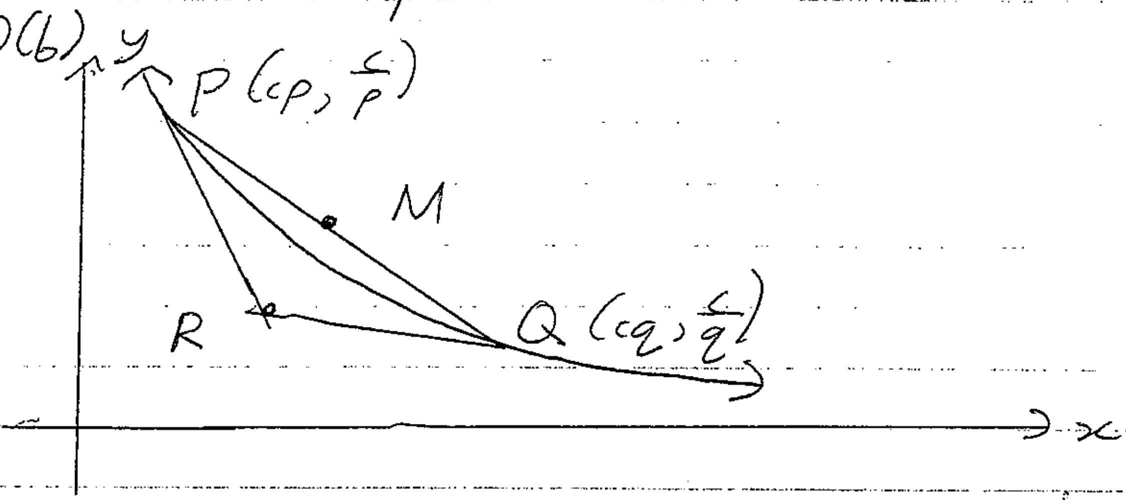
$$= \frac{1}{\cancel{2}} \times \frac{\cancel{ae}}{(\sec\theta + \tan\theta)} \times \frac{\cancel{2}ab}{\cancel{ae}(\sec\theta - \tan\theta)}$$

$$= \frac{ab}{\sec^2\theta - \tan^2\theta}$$

$\underline{1}$

$= ab$ [which is constant as required, as $\sec^2\theta - \tan^2\theta = 1$].

Q.(13)(b)



(i) Tangent at Q is $x + q^2 y = 2cq$ (1) 1

at P $x + p^2 y = 2cp$ (2)

co-ordinates of R.

$$(q^2 - p^2)y = 2c(q-p)$$
$$(q-p)(p+q)y = 2c(q-p)$$
$$y = \frac{2c}{p+q}.$$

As R is on $x + p^2 y = 2cp$

$$x + \frac{2cp^2}{p+q} = 2cp$$ 3

$$(p+q)x + 2cp^2 = 2cp^2 + 2cpq$$
$$(p+q)x = 2cpq$$
$$x = \frac{2cpq}{(p+q)}$$

$\therefore$ R is $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$. 1

M is midpoint PQ

$$= \left(\frac{c(p+q)}{2}, \left(\frac{\frac{c}{p} + \frac{c}{q}}{2}\right)\right)$$

$$= \left(\frac{c(p+q)}{2}, \frac{c(p+q)}{2pq}\right).$$ 1

Solutions p. 11

If R is midpoint OM

~~then~~ $\left[\frac{c(p+q)}{4}, \frac{c(p+q)}{4pq}\right] = \left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$

Using x co-ordinates:

$\frac{c(p+q)}{4} = \frac{2cpq}{(p+q)}$

$\times$ BS by $\frac{4(p+q)}{c}$

$\underline{(p+q)^2 = 8pq.}$ 1

OR using y co-ordinates:

$\frac{c(p+q)}{4pq} = \frac{2c}{p+q}$

$\times$ BS by $\frac{4pq(p+q)}{c}$

$\underline{(p+q)^2 = 8pq.}$

(Either will do).

(iii) Locus of M: $x = \frac{c(p+q)}{2}$

$\therefore \frac{2x}{c} = p+q.$ (1)

$y = \frac{c(p+q)}{2pq}.$

$y^2 = \frac{c^2(p+q)^2}{4(pq)^2}$

$= \frac{c^2(p+q)^2}{4\times\left[\frac{1}{8}(p+q)^2\right]^2}$ 1

$= \frac{16c^2}{(p+q)^2}.$ $\underline{2}$

$\therefore y = \frac{4c}{p+q}$ (2)

Sub. (1) in (2): $y = 4c \div \frac{2x}{c}$

$\underline{y = \frac{2c^2}{x}.}$ 1

R can't be midpoint OM if P is in Q_1 & Q in Q_3.

NO MARKS FOR THIS.

Note:

$y = \frac{2c^2}{x}$ if $p>q$ in Q_1

If $p>q$ in Q_3 then $y = \frac{2c^2}{x}$ as well.

Solutions: p.12

Q.(14)(a)(i) $\int_0^a f(a-x).dx$

Let $u = a - x$, $du = -1.dx$

$$= -\int_0^a f(a-x).-1.dx$$

$$= -\int_{u=a}^{u=0} f(u).du \text{ changing the variable}$$

1

$$= \int_0^a f(u).du \quad \left[\text{as } \int_a^b f(x).dx = -\int_b^a f(x).dx\right].$$

$$= \underline{\int_0^a f(x).dx} \quad \left[\text{as } \int_a^b f(u).du = \int_a^b f(x).dx\right].$$

(ii) Hence $\int_0^3 x^2(3-x)^{20}.dx$

$$= \int_0^3 (3-x)^2.x^{20}.dx \quad \left[\text{as } \int_0^a f(a-x).dx = \int_0^a f(x).dx\right]$$

1

$$= \int_0^3 9x^{20} - 6x^{21} + x^{22}.dx$$

$$= \left[\frac{3}{7}x^{21} - \frac{3}{11}x^{22} + \frac{1}{23}x^{23}\right]_0^3$$

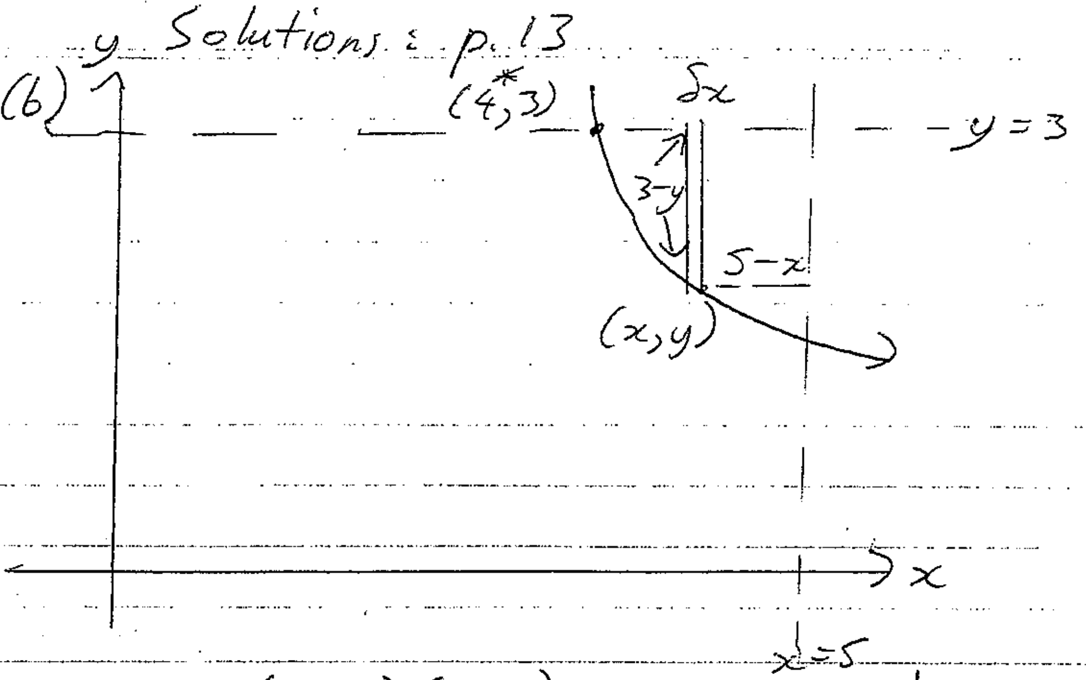
2

$$= \underline{17\ 719\ 401{\cdot}25}$$

1

Solutions : p. 13

Q. (14)(b)



$\delta V = 2\pi(5-x)(3-y).$ 1

$= 2\pi\,(5-x)[3-(x^2-10x+27)].$

$= 2\pi\,(5-x)[-x^2+10x-24].$

$= 2\pi\,(x^3-15x^2+74x-120).\delta x$ 1

Note: When $y=3$,

$x^2-10x+27=3$

$x^2-10x+24=0$ $\underline{4}$

$(x-\overset{*}{4})(x-6)=0.$

$x=4$ [$x=6$ is outside our volume].

$V = \lim_{\delta x\to 0} 2\pi \sum_{x=4}^{x=5} (x^3-15x^2+74x-120).\delta x$

Letting $\delta x \to 0$

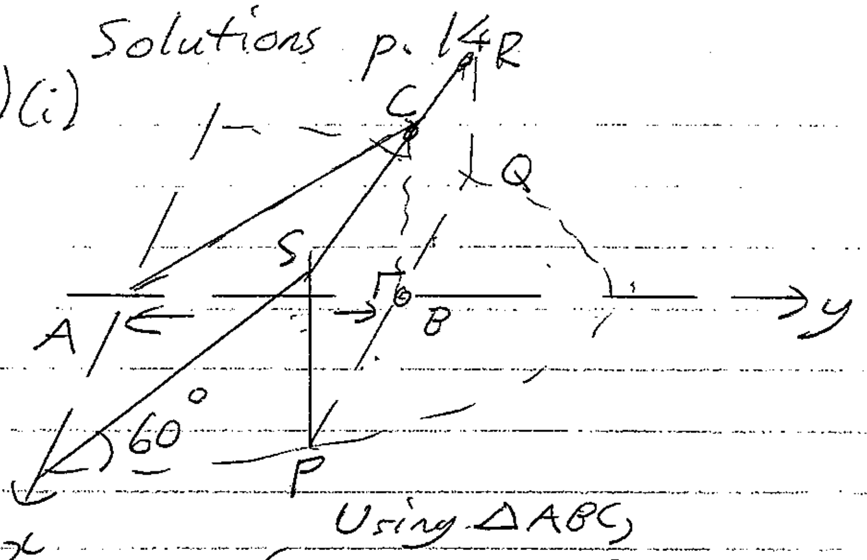
$V = 2\pi \int_4^5 (x^3-15x^2+74x-120).dx$ 1

$= 2\pi \left[\frac{1}{4}x^4-5x^3+37x^2-120x\right]_4^5$

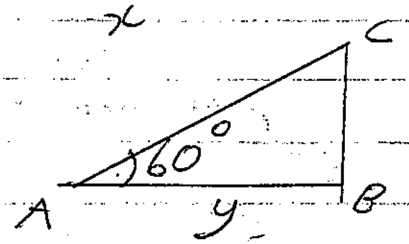
$= \underline{\frac{\pi}{2} \text{ cubic units.}}$ 1

Solutions p. 14R

Q. (14)(c)(i)



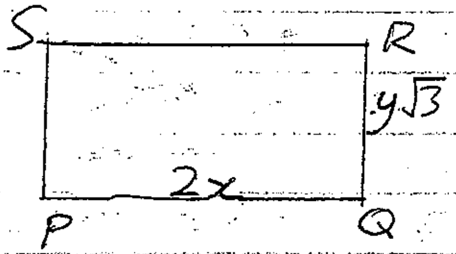
Using ΔABC,



$$\tan 60^\circ = \frac{CB}{y}$$

$$y \tan 60^\circ = CB$$

$$y\sqrt{3} = CB. \qquad 1$$



Area of rectangle PQRS

$$= 2x \times y\sqrt{3} \qquad \underline{2}$$

As $y = 9 - x^2$

$$x^2 = 9 - y$$

$$x = \sqrt{9-y}$$ [we need a LENGTH → take POSITIVE root].

$$\therefore 2x = 2\sqrt{9-y}. \qquad 1$$

$$\therefore \text{Area} = 2x \times y\sqrt{3}$$

$$= 2\sqrt{3}\, y\sqrt{9-y}.$$

$\therefore$ $\underline{\delta V = 2y\sqrt{9-y}\,.\,\sqrt{3}\,.\,\delta y}$ as required.

(ii) $V = \lim_{\delta y \to 0} \sum_{y=0}^{y=9} 2y\sqrt{9-y}\,.\,\sqrt{3}\,.\,\delta y$ [Note: Max. y value = 9].

Letting $\delta y \to 0$

$$V = 2\sqrt{3}\int_0^9 y\sqrt{9-y}\; dy \qquad 1$$

PTO →

Q. (14)(c)(ii) (continued).

$$V = -2\sqrt{3}\int_0^9 y\sqrt{9-y}\,.-1.dy$$

Letting $u = 9-y$, $du = -1.dy$

$$V = -2\sqrt{3}\int_9^0 (9-u)\sqrt{u}\,.du$$

$$= 2\sqrt{3}\int_0^9 9\sqrt{u} - u\sqrt{u}\,.du$$

$$= 2\sqrt{3}\left[6u\sqrt{u} - \frac{2u^2\sqrt{u}}{5}\right]_0^9 \qquad \underline{2}$$

$$= 2\sqrt{3} \times \frac{324}{5}$$

$$= \underline{\frac{648\sqrt{3}}{5} \text{ cubic units.}} \qquad 1$$

(d)(i) $I_n = \int_0^{\frac{\pi}{2}} \cos^n x.dx$ $\quad u = \cos^{n-1}x \quad v = \sin x$

$u' = (n-1)\cos^{n-2}x.\sin x \quad v' = \cos x$

$$= \left[\cos^{n-1}x.\sin x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}x.\sin x.\sin x.d$$

$$= 0^* + (n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}x(1-\cos^2 x).dx \qquad 1$$

$$I_n = 0 + (n-1)\int_0^{\frac{\pi}{2}}\cos^{n-2}x.dx - (n-1)\int_0^{\frac{\pi}{2}}\cos^n x.dx$$

$$I_n = (n-1)I_{n-2} - (n-1)I_n$$

$$nI_n = (n-1)I_{n-2} \qquad \underline{3}$$

$$\underline{I_n = \frac{(n-1)}{n}I_{n-2}.} \qquad 1$$

*Note: As $\cos\frac{\pi}{2} = 0$, $\sin 0 = 0$

$\left[\cos^{n-1}x\sin x\right]_0^{\frac{\pi}{2}} = 0 - 0 = 0.$

Q. (14)(d)(ii) [continued]:

Hence $I_8 = \frac{7}{8} \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} I_0$ $\quad \underline{1}$

and as $I_0 = \int_0^{\frac{\pi}{2}} \cos^0 x \, dx$

$= \int_0^{\frac{\pi}{2}} 1 \, dx$

$= [x]_0^{\frac{\pi}{2}}$

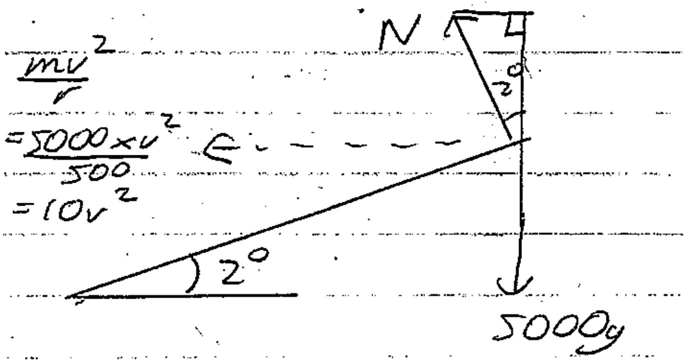
$= \frac{\pi}{2}$

$I_8 = \frac{7}{8} \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$

$\underline{\int_0^{\frac{\pi}{2}} \cos^8 x \, dx = \frac{35\pi}{256}.}$

Q. (15)(a)(i) $\quad$ OR

Horizontal & vertical:



Horizontal: $N \sin 2^\circ = 10v^2$ (1)

Vertical: $N \cos 2^\circ = 5000g$ (2) $\quad 1$

(1) ÷ (2)

$\tan 2^\circ = \frac{10v^2}{5000g}$

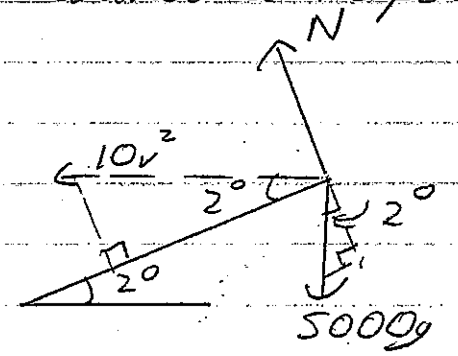
$\frac{5000g \tan 2^\circ}{10} = v^2$ $\quad \underline{2}$

$\underline{13 \cdot 0809\ldots = v}$ $\quad 1$

Ideal velocity $= \underline{13 \cdot 08\ldots \text{ m/s}}$

$= \underline{47 \cdot 1 \text{ km/h}}$

Parallel & Perpendicular:



Parallel: $5000g \sin 2^\circ = 10v^2 \cos 2^\circ$ $\quad 1$

$\therefore \frac{5000g \sin 2^\circ}{10 \cos 2^\circ} = v^2$ $\quad 1$ or $\underline{2}$

$13 \cdot 08\ldots = v$

$\underline{\text{Ideal velocity} = 13 \cdot 08 \text{ m/s}}$

$\underline{= 47 \cdot 1 \text{ km/h}}$

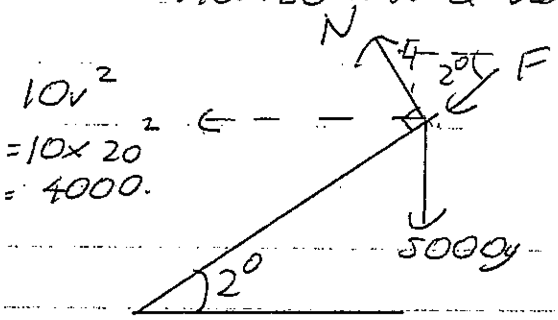
[Note: Perpendicular forces are

$N - 5000g \cos 2^\circ = 10v^2 \sin 2^\circ$.]

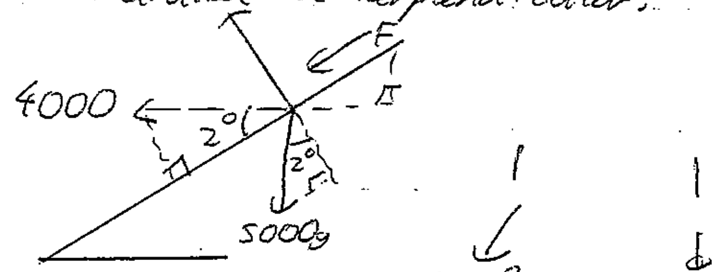
Solutions: p.17

Q.(15)(a)(ii)

Horizontal & Vertical:



Parallel & Perpendiculars:



Parallel: $F + 5000g\sin 2° = 4000\cos 2°$

$F = 4000\cos 2° - 5000g\sin 2°$

$= 2287$ Newtons.

or $\underline{3}$

Horizontal: $N\sin 2° + F\cos 2° = \dfrac{mv^2}{r}$

$N\sin 2° + F\cos 2° = 4000 \quad (1)$ | 1

Vertical: $N\cos 2° - F\sin 2° = 5000g \quad (2)$

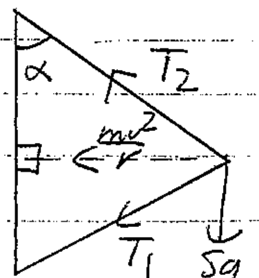
$(1)\times\cos 2° = (3)$ $N\sin 2°\cos 2° + F\cos^2 2° = 4000\cos 2°$ | 1

$(2)\times\sin 2° = (4)$ $N\sin 2°\cos 2° - F\sin^2 2° = 5000g\sin 2°$

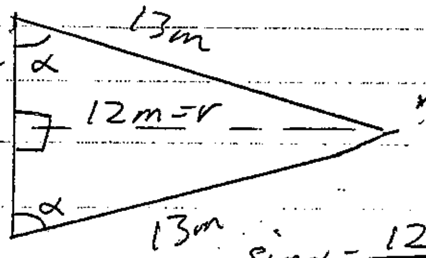
$F(\cos^2 2° + \sin^2 2°) = 4000\cos 2° - 5000g\sin 2°$ | 1 $\quad \underline{3}$

$\underline{F \qquad = 2287 \text{ Newtons [nearest Newton]}}$.

Q.(15)(b)



Distances:



$r = 12m$ by Pythagoras.

$\sin\alpha = \frac{12}{13}$ $\quad \cos\alpha = \frac{5}{13}$.

Resolving horizontally: $T_2\sin\alpha + T_1\sin\alpha = \dfrac{mv^2}{r}$ | 1

or $T_2 \times \frac{12}{13} + T_1 \times \frac{12}{13} = \frac{5\times 20^2}{12} = \frac{500}{3}$ (1)

Resolving vertically, $T_2\cos\alpha - T_1\cos\alpha = mg$ | 1

$T_2 \times \frac{5}{13} - T_1 \times \frac{5}{13} = 5g$. (2)

PTO →

$\underline{2}$

Solutions: p. 18

Q. (15)(b)(ii) $T_2 \times \frac{12}{13} + T_1 \times \frac{12}{13} = \frac{500}{3}$ (1) $\times 65 = (3)$

$T_2 \times \frac{5}{13} - T_1 \times \frac{5}{13} = 5g$ (2) $\times 156 = (4)$

$$\left.\begin{aligned} 60T_2 + 60T_1 &= \frac{32500}{3} \quad (3) \\ 60T_2 - 60T_1 &= 780g \quad (4) \end{aligned}\right] 1$$

$(3)+(4)$: $120T_2 = \frac{32500}{3} + 780g$

$T_2 = \quad$ Newtons $153.9\dot{7}$

$\underline{T_2 \doteqdot 154 \text{ Newtons. [nearest Newton].}}$ 1

$(3)-(4)$: $120T_1 = \frac{32500}{3} - 780g$ $\quad$ $\underline{3}$

$T_1 \doteqdot 26.5\dot{7}$ Newtons.

$\underline{T_1 \doteqdot 27 \text{ Newtons.}}$ [nearest [Newton]. 1

(c) If $w = \cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}$,

$w^9 = \cos 2\pi + i\sin 2\pi$

$= 1$ [by De Moivre].

$\therefore (w^n)^9 = w^{9n}$ $\quad$ $\underline{1}$

$= (w^9)^n$

$= 1^n$

$= 1$ if n an integer.

(ii) $w^8 = \left(\cos\frac{16\pi}{9} + i\sin\frac{16\pi}{9}\right)$ [by De Moivre]

$= \cos\frac{2\pi}{9} - i\sin\frac{2\pi}{9}$

$\therefore w + w^8 = \left(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}\right) + \left(\cos\frac{2\pi}{9} - i\sin\frac{2\pi}{9}\right)$ $\quad$ $\underline{1}$

$\underline{= 2\cos\frac{2\pi}{9}}$

Solutions p. 19

Q.(15)(c)(iii) $(\omega^3+\omega^6)(\omega^2+\omega^7)$

$= \omega^5+\omega^{10}+\omega^8+\omega^{13}$ 1

$= \omega^5+\omega+\omega^8+\omega^4$ [as $\omega^9=1$].

$= \omega+\omega^8+\omega^4+\omega^5$.

(iv) Hence as $\omega^3+\omega^6 = (\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3})+(\cos\frac{2\pi}{3}-i\sin\frac{2\pi}{3})$

$= 2\cos\frac{2\pi}{3}$ (A)

$= -1$.

$\omega^2+\omega^7 = (\cos\frac{4\pi}{9}+i\sin\frac{4\pi}{9})+(\cos\frac{4\pi}{9}-i\sin\frac{4\pi}{9})$

1 $= 2\cos\frac{4\pi}{9}$ (B)

and $\omega^4+\omega^5 = (\cos\frac{8\pi}{9}+i\sin\frac{8\pi}{9})+(\cos\frac{8\pi}{9}-i\sin\frac{8\pi}{9})$

$= 2\cos\frac{8\pi}{9}$

$= -2\cos\frac{\pi}{9}$, (C)

and $(\omega^3+\omega^6)(\omega^2+\omega^7) = \omega+\omega^8+\omega^4+\omega^5$ [from Part (iii)

Substituting (ii), (A), (B) and (C) in (iii)

$-1 \times 2\cos\frac{4\pi}{9} = 2\cos\frac{2\pi}{9} - 2\cos\frac{\pi}{9}$ 1 2

$2\cos\frac{\pi}{9} = 2\cos\frac{2\pi}{9} + 2\cos\frac{4\pi}{9}$

$\cos\frac{\pi}{9} = \cos\frac{2\pi}{9} + \cos\frac{4\pi}{9}$.

Q.(16)(a)(i) $F = ma = -mg - mkv^2$

$\therefore a = v \cdot \frac{dv}{dx} = -g - kv^2$ 1

$$\frac{dv}{dx} = \frac{-g-kv^2}{v}$$

$$\frac{dx}{dv} = \frac{-v}{g+kv^2}$$

$$\therefore x = \int \frac{-v}{g+kv^2} \cdot dv$$

$$= -\frac{1}{2k}\int \frac{-2kv}{g+kv^2} \cdot dv$$

$$x = -\frac{1}{2k}\ln(g+kv^2) + c$$ 1

As $v = U$ when $x = 0$ $\underline{3}$

$$0 = -\frac{1}{2k}\ln(g+kU^2) + c$$

$$c = \frac{1}{2k}\ln(g+kU^2)$$

$$x = -\frac{1}{2k}\ln(g+kv^2) + \frac{1}{2k}\ln(g+kU^2)$$

$$\therefore \underline{x = -\frac{1}{2k}\ln\left(\frac{kv^2+g}{kU^2+g}\right)}$$ 1

(ii) Max. height H is x when $v = 0$

$$= -\frac{1}{2k}\ln\left(\frac{g}{g+kU^2}\right)$$ $\underline{1}$

(iii) When falling, $F = ma = mg - mkv^2$

$$\therefore a = g - kv^2.$$

Terminal velocity is when $a = 0$ i.e. $g - kv^2 = 0$

$$g = kv^2$$

$$\underline{\sqrt{\frac{g}{k}} = v}$$ $\underline{1}$

$$\therefore \underline{T = \sqrt{\frac{g}{k}}}$$

Q.(16)(a)(iv) $v \cdot \dfrac{dv}{dx} = g - kv^2$ 1

$$\frac{dv}{dx} = \frac{g-kv^2}{v}$$

$$\frac{dx}{dv} = \frac{v}{g-kv^2}$$

$$x = -\frac{1}{2k}\int \frac{-2kv}{g-kv^2} \cdot dv$$

$$x = -\frac{1}{2k}\ln(g-kv^2)+C$$ 1

As $x=0$ when $v=0$ [top of particle's arc]

$$0 = -\frac{1}{2k}\ln g + C$$

3

$$\therefore C = \frac{1}{2k}\ln g$$

$$x = -\frac{1}{2k}\ln(g-kv^2)+\frac{1}{2k}\ln g$$

$$\underline{x = -\frac{1}{2k}\ln\left(\frac{g-kv^2}{g}\right)}$$ 1

(v) Particle has fallen back H [$x=H$] when $v=W$

$$\therefore \underline{H = -\frac{1}{2k}\ln\left(\frac{g-kW^2}{g}\right)}$$

(vi) From (ii) and (v) $H = -\dfrac{1}{2k}\ln\left(\dfrac{g}{g+kU^2}\right) = -\dfrac{1}{2k}\ln\left(\dfrac{g-kW^2}{g}\right)$

$$\therefore \frac{g}{g+kU^2} = \frac{g-kW^2}{g}$$

$\times g(g+kU^2)$

$$g^2 = (g+kU^2)(g-kW^2)$$ 1

$$g^2 = g^2 - gkW^2 + gkU^2 - k^2U^2W^2$$

$$0 = gkU^2 - gkW^2 - k^2U^2W^2 \quad \div k^2$$

$$0 = \frac{g}{k}U^2 - \frac{g}{k}W^2 - U^2W^2$$

As $T=\sqrt{\frac{g}{k}}$, $T^2 = \frac{g}{k}$ from (iii)

$$0 = T^2U^2 - T^2W^2 - U^2W^2 \quad \div T^2U^2W^2$$

$$0 = \frac{1}{W^2} - \frac{1}{U^2} - \frac{1}{T^2} \Rightarrow \underline{\frac{1}{U^2} + \frac{1}{T^2} = \frac{1}{W^2}}$$

p.22

Q. (16)(b)(i) $I_n = \int_0^1 x^n e^{-x}\,dx \qquad u = x^n \qquad v = -e^{-x}$

$u' = nx^{n-1} \qquad v' = e^{-x}$

$I_n = uv - \int v \cdot \frac{du}{dx} \cdot dx$

$= \left[-x^n e^{-x}\right]_0^1 + n\int_0^1 x^{n-1} e^{-x}\,dx$

$= \left[-1^n \times e^{-1} - 0^n \times e^0\right] + nI_{n-1} \qquad \underline{1}$

$= -\frac{1}{e} + nI_{n-1}$

$\underline{= nI_{n-1} - \frac{1}{e}}$

(ii) $I_3 = 3I_2 - \frac{1}{e}$

$I_2 = 2I_1 - \frac{1}{e}$

$I_1 = 1I_0 - \frac{1}{e}$

$I_0 = \int_0^1 e^{-x}\,dx$

$= \left[-e^{-x}\right]_0^1$

$= -e^{-1} - -e^0$

$= 1 - \frac{1}{e} \qquad \underline{1}$

$\therefore I_1 = 1\left(1 - \frac{1}{e}\right) - \frac{1}{e}$

$= 1 - \frac{1}{e} - \frac{1}{e}$

$= 1 - \frac{2}{e}$

$I_2 = 2\left(1 - \frac{2}{e}\right) - \frac{1}{e}$

$= 2 - \frac{4}{e} - \frac{1}{e}$

$= 2 - \frac{5}{e}$

$I_3 = 3\left(2 - \frac{5}{e}\right) - \frac{1}{e}$

$= 6 - \frac{15}{e} - \frac{1}{e}$

$\underline{I_3 = 6 - \frac{16}{e}}$

p. 23

Q. C(16)(b)(iii) Step 1: Show true for $n=0$.

$$LHS = 0! - \frac{\left(\frac{0!}{0!}\right)}{e} = 1 - \frac{1}{e}$$

RHS: $I_0 = 1 - \frac{1}{e}$ [From Part (ii)].

True for $n=0$.

Step 2: Assume true for $n=k$

i.e. $I_k = k! - \frac{\left(\frac{k!}{0!} + \frac{k!}{1!} + \frac{k!}{2!} + \dots + \frac{k!}{k!}\right)}{e}$

Step 3: Prove true for $n=k+1$

i.e. $I_{k+1} = (k+1)! - \frac{\left(\frac{(k+1)!}{0!} + \frac{(k+1)!}{1!} + \frac{(k+1)!}{2!} + \dots + \frac{(k+1)!}{(k+1)!}\right)}{e}$

1

LHS: I_{k+1}

$= (k+1) \quad I_k \quad - \frac{1}{e}$ 2

$= (k+1)\left[k! - \frac{\left(\frac{k!}{0!} + \frac{k!}{1!} + \frac{k!}{2!} + \dots + \frac{k!}{k!}\right)}{e}\right] - \frac{1}{e}$ 1

$= (k+1)! - \left[\frac{\frac{(k+1)!}{0!} + \frac{(k+1)!}{1!} + \frac{(k+1)!}{2!} + \dots + \frac{(k+1)!}{k!}}{e}\right] - \frac{1}{e}$

$= (k+1)! - \left[\frac{\frac{(k+1)!}{0!} + \frac{(k+1)!}{1!} + \frac{(k+1)!}{2!} + \dots + \frac{(k+1)!}{k!}}{e}\right] - \frac{\left[\frac{(k+1)!}{(k+1)!}\right]}{e}$

$= (k+1)! - \frac{\frac{(k+1)!}{0!} + \frac{(k+1)!}{1!} + \dots + \frac{(k+1)!}{k!} + \frac{(k+1)!}{(k+1)!}}{e}$

= RHS QED If it true for $n=k$ then it will be true for $n=k+1$. Hence as it is true for $n=0$ it will be true for $n=0+1=1$ and so on for all positive integers n [and 0].

p. 24

Q. (16)(b)(iv) As $\lim\limits_{n\to\infty} \int_0^1 x^n e^{-x}\,dx = 0$

$$\lim_{n\to\infty}\left(n! - \frac{\frac{n!}{0!} + \frac{n!}{1!} + \frac{n!}{2!} + \cdots + \frac{n!}{n!}}{e}\right) = 0$$

$\times \dfrac{e}{n!}$

$$\lim_{n\to\infty}\left(e - \left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right)\right) = 0 \qquad \underline{1}$$

$$\therefore \lim_{n\to\infty}\left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right) = e.$$