Girraween High School

2017 Year 12 Trial Higher School Certificate

Mathematics Extension 2

Instructions

- Attempt all questions.
- For Questions 1 -10, shade the circle for the letter corresponding to the correct answer on your answer sheet.
- For Questions 11 – 16, start each question on a new page. Each question should be clearly labelled.
- All necessary working must be shown for Questions 11– 16.
- Marks may be deducted for careless or badly arranged work.
- Board approved calculators may be used.
- A Mathematics reference sheet is provided.

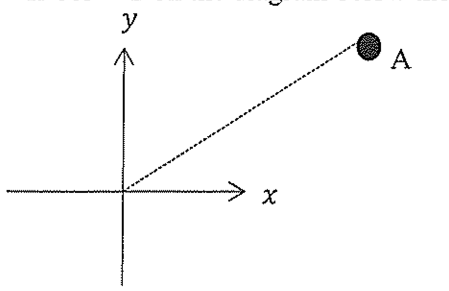
Question 1

If $z = 5 - 5i$ and $w = 2 + i$ then $\frac{z}{w} =$

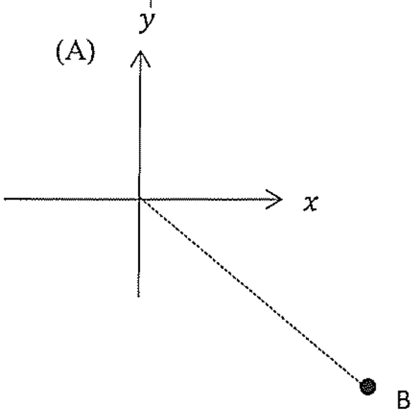
(A) $1 + 3i$ (B) $1 - 3i$ (C) $3 + i$ (D) $3 - i$

Question 2

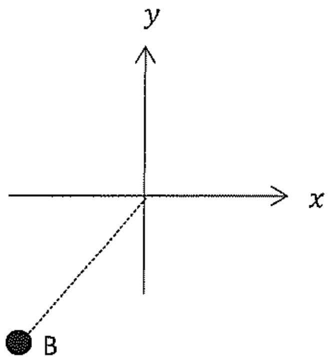
If $\overrightarrow{OA} = z$ on the diagram below then $\overrightarrow{OB} = z\, cis\, \frac{3\pi}{2}$ could be



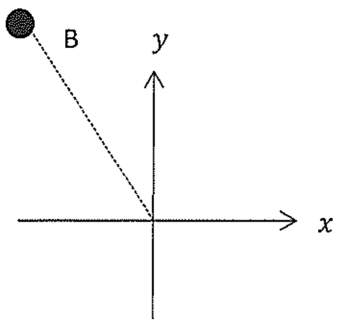
(A)



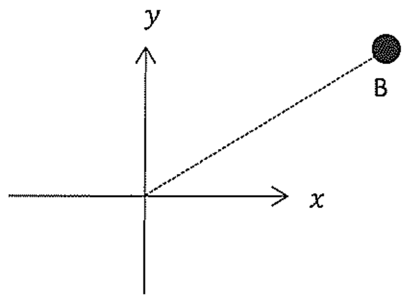
(B)



(C)



(D)



Question 3 If α, β and γ are the roots of the polynomial equation $3x^3 + 5x - 1 = 0$ then

$\alpha^3 + \beta^3 + \gamma^3 =$

(A) 0 (B) -12 (C) $\frac{34}{9}$ (D) 1

Examination continues on the next page

Question 4

The sum of the eccentricities of two conics is $\frac{3}{2}$. The two conics could *not* be

(A) An ellipse and a hyperbola (B) An ellipse and a parabola

(C) A circle and a hyperbola (D) A hyperbola and a parabola

Question 5

The foci of the hyperbola $\frac{x^2}{25} - \frac{y^2}{144} = 1$ are

(A) $(\pm 13,0)$ (B) $(0, \pm 13)$ (C) $(\pm \frac{25}{13}, 0)$ (D) $(0, \pm \frac{25}{13})$

Question 6

$$\int \frac{1}{xln(x)}.dx =$$

(A) $\frac{(\ln(x))^2}{2} + C$ (B) $-\ln(x) + C$ (C) $\frac{-1}{\ln(x)} + C$ (D) $\ln(\ln(x)) + C$

Question 7

$$\int sin^3x\, dx =$$

(A) $\frac{sin^4x}{4}$ +C (B) $\frac{cos^4x}{4} + C$ (C) $cosx - \frac{cos^3x}{3} + C$ (D) $\frac{cos^3x}{3} - \cos x + C$

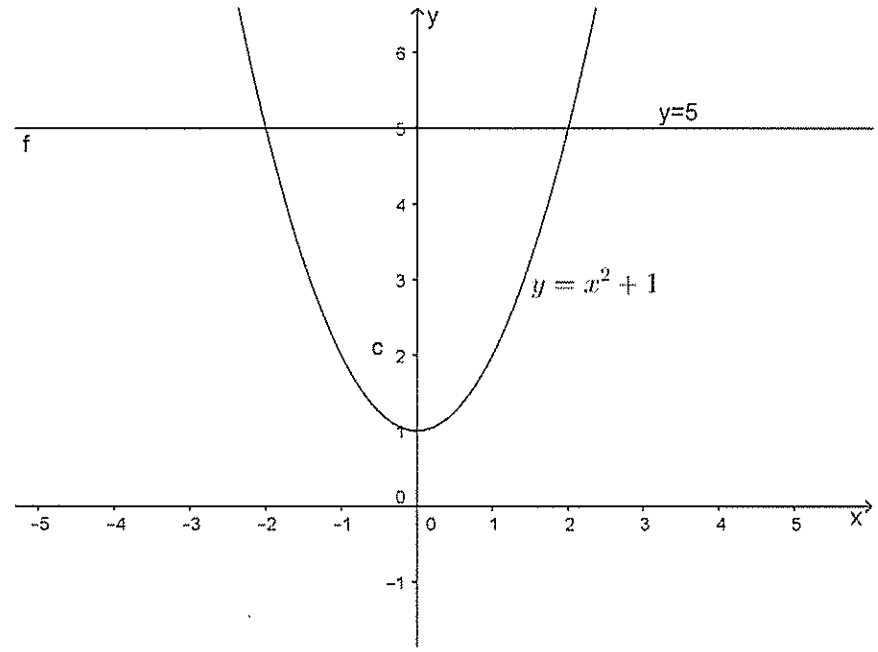
Examination continues on the next page

Question 8

The volume obtained by rotating the area enclosed by $y = x^2 + 1$ and the line $y = 5$ about the x axis can be found using the expression

(A) $\pi \int_{-2}^{2} (24 - 2x^2 - x^4).dx$ (B) $\pi \int_{-2}^{2} (676 - 52x^2 + x^4).dx$

(C) $2\pi \int_{1}^{5} y\sqrt{y-1}.dy$ (D) $4\pi \int_{1}^{5} (5-y)\sqrt{y-1}.dy$



Question 9

A particle is launched vertically upwards from the surface of Earth. As it ascends it experiences gravity downwards which is inversely proportional to the square of its distance from the centre of Earth (i.e. $F = \frac{mk}{x^2}$ where k is a constant and x is the distance from the centre of Earth). Given that the radius of Earth is R and the acceleration due to gravity at the Earth's surface is $g\ m/s^2$, the value of k is:

(A) $\frac{g}{R^2}$ (B) $\sqrt{\frac{g}{R}}$ (C) $\sqrt{gR}$ (D) gR^2

Question 10

An astronaut in a centrifuge with a radius of $8m$ is experiencing a centripetal force of $3g$ Newtons per kilogram of mass. Given that $g = 9.8m/s^2$, the speed at which the centrifuge is rotating is

(A) $1.91m/s$ (B) $3.68m/s$ (C) $15.34m/s$ (D) $235.2m/s$

Examination continues on the next page

Question 11 ***(15 marks) Show all necessary working on a separate page*** **Marks**

(a) If $z = -\sqrt{3} - i$ and $w = 1 - i$

(i) Find zw in Cartesian form. **2**

(ii) Convert both z and w to modulus/argument form. **3**

(iii) Use your answers to (i) and (ii) to find the exact value of $sin\frac{11\pi}{12}$. **1**

(b) Sketch the region in the complex plane where $|z-1| \leq 1$ and $0 \leq \arg z < \frac{\pi}{4}$ **3**

(c) (i) Prove that if a polynomial equation $P(x) = 0$ has a double root then $P'(x) = 0$ will also have the same root. **2**

(ii) Solve $18x^3 + 3x^2 - 4x - 1 = 0$ given that it has a double root. **4**

Question 12 ***(15 marks) Show all necessary working on a separate page***

(a) Find

(i) $\int x^8 \ln(x) . dx$ **2**

(ii) $\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2+\cos x} . dx$ **2**

(iii) $\int \frac{\sqrt{x^2-1}}{x} . dx$ **3**

(b) (i) Express $\frac{-2x^2-8x-2}{(x+1)^2(x+3)}$ in the form $\frac{A}{(x+1)^2} + \frac{B}{x+1} + \frac{C}{x+3}$ **3**

(ii) Hence find $\int \frac{-2x^2-8x-2}{(x+1)^2(x+3)} . dx$ **2**

(c) For the ellipse $25x^2 + 9y^2 = 225$

(i) Find the eccentricity **1**

(ii) Find the co-ordinates of the foci and equations of the directrices **1**

(iii) Sketch the graph of $25x^2 + 9y^2 = 225$ showing all of these features. **1**

Examination continues on the next page

Question 13 ***(15 marks) Show all necessary working on a separate page*** **Marks**

(a) If α, β and γ are the roots of the polynomial equation **2**

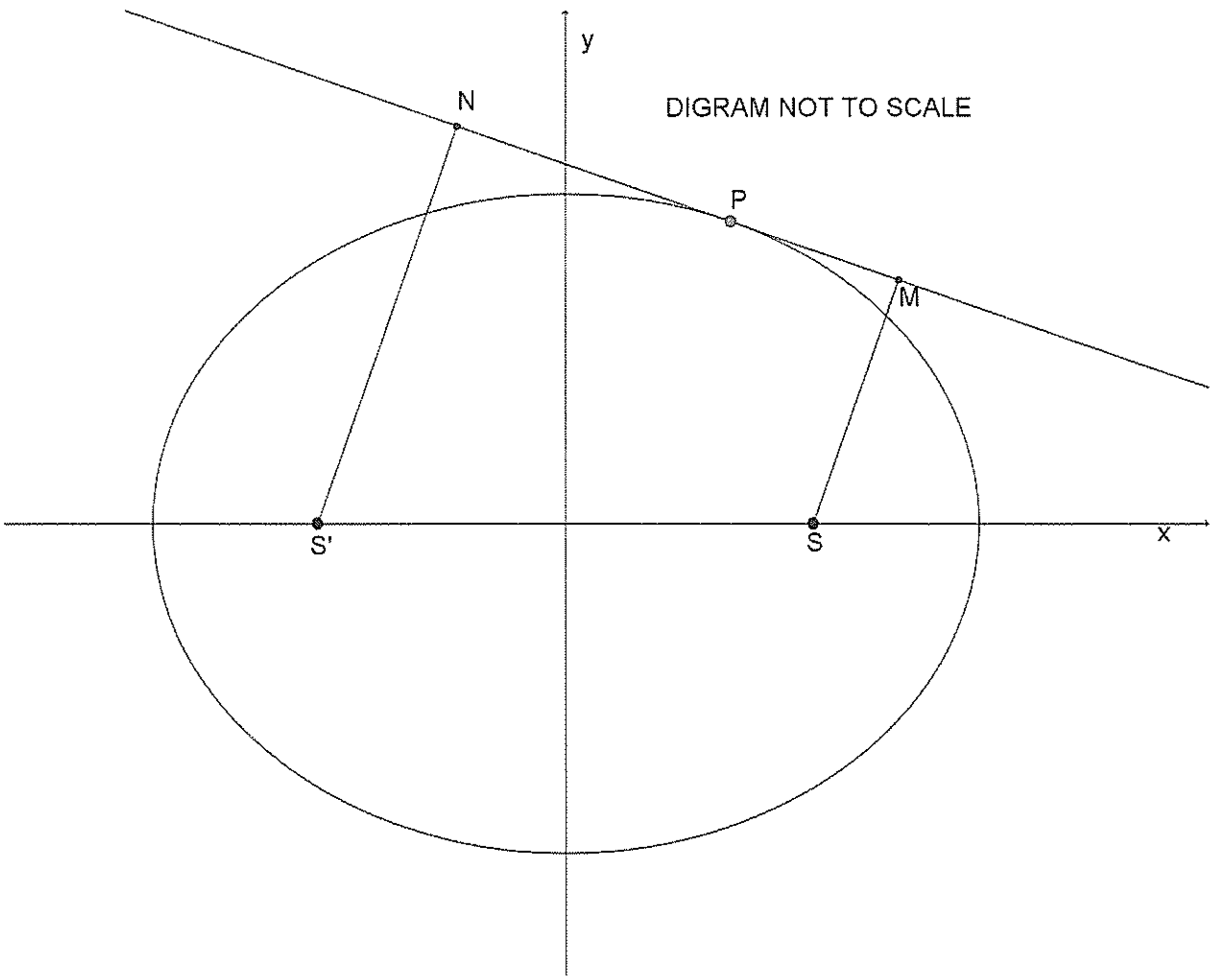
$2x^3 + 7x^2 + 10x - 7 = 0$, form the polynomial equation with roots

$\alpha + 2, \beta + 2$ and $\gamma+2$.

(b) The point $P(acos\Theta, bsin\Theta)$ is on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $a > b$. M and N

are points on the tangent to the ellipse at P so that the perpendiculars to the tangent

at M and N intersect with the x axis at the foci S and S' respectively *(see diagram).*



Question 13(b) continues on the following page

Question 13(b) (continued) **Marks**

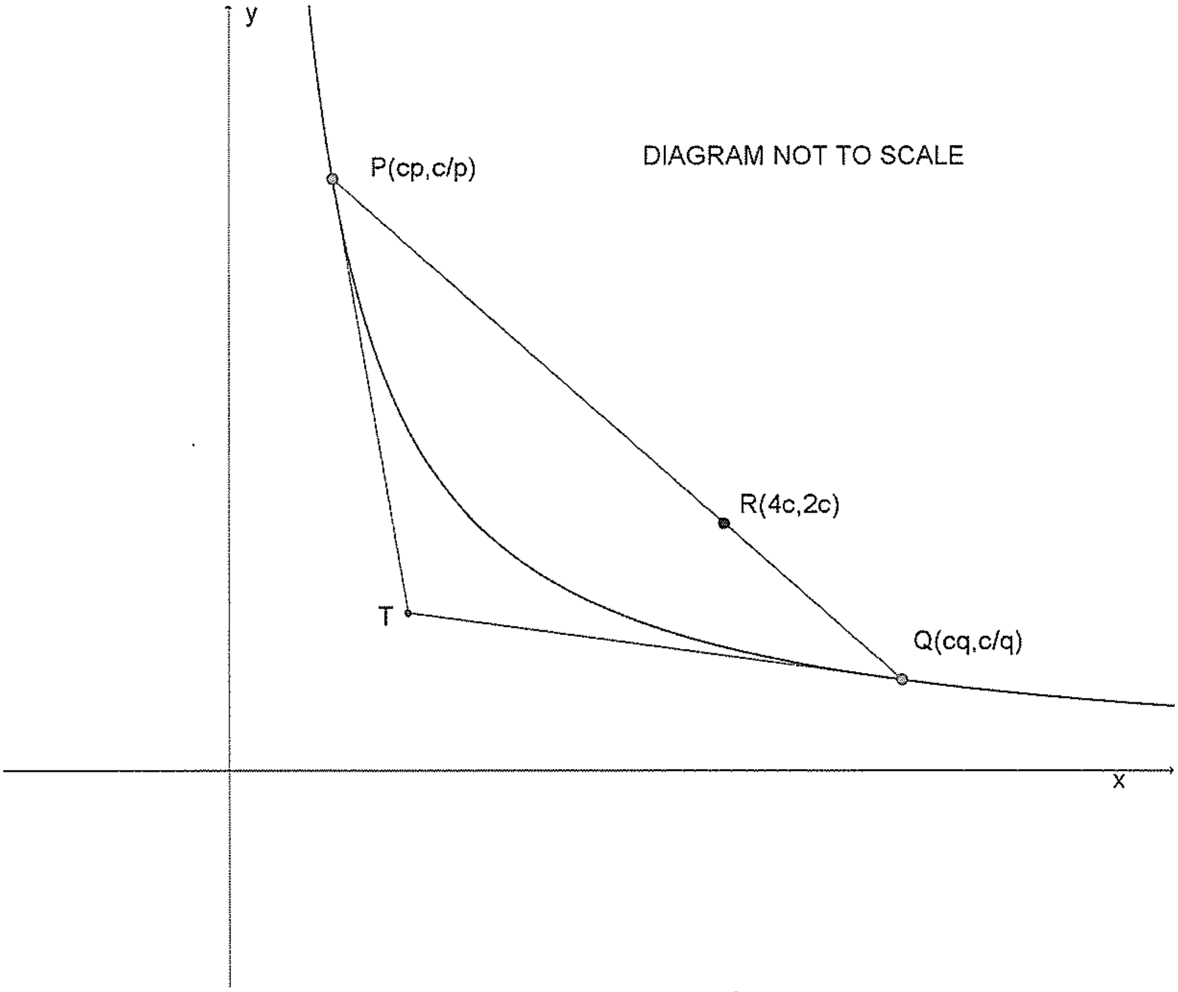
(i) Show that the equation of the tangent at P is $bx\cos\theta + ay\sin\theta = ab$. **2**

(ii) Show that the distance $MS = \frac{ab - abe\cos\theta}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$ **2**

(iii) Show that $MS \times NS' = b^2$. **3**

(c) $P\ (cp, \frac{c}{p})$ and $Q\ (cq, \frac{c}{q})$ are points on the rectangular hyperbola $xy = c^2$.

T is the point of intersection of the tangents at P and Q (*see diagram*).



(i) Show that the equation of the tangent at P is $x + p^2y = 2cp$ and state the equation of the tangent at Q. **2**

(ii) Show that T is the point $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$. **2**

(iii) Show that the chord PQ is $x + pqy = c(p+q)$ **1**

(iv) Find the locus of T if the chord PQ passes through the point $R(4c, 2c)$ **1**

Examination continues on the next page

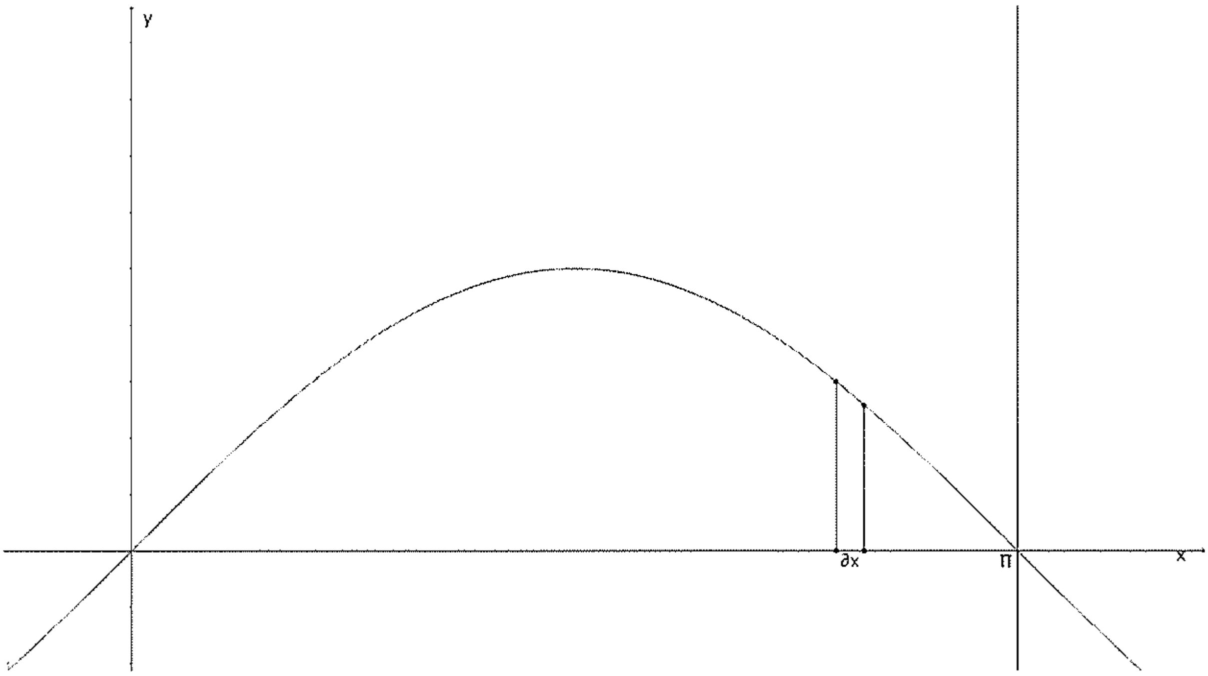
Question 14 *(15 Marks)* **Marks**

(a) (i) Prove that $\int_0^a f(a-x).dx = \int_0^a f(x).dx$ **1**

(ii) Hence or otherwise find $\int_0^{\pi} xsinx.dx$ **2**

(b) A solid is formed by rotating the area between $y = sinx$, the x axis

and the lines $x = 0$ and $x = \pi$ around the line $x = \pi$ *(see diagram).*



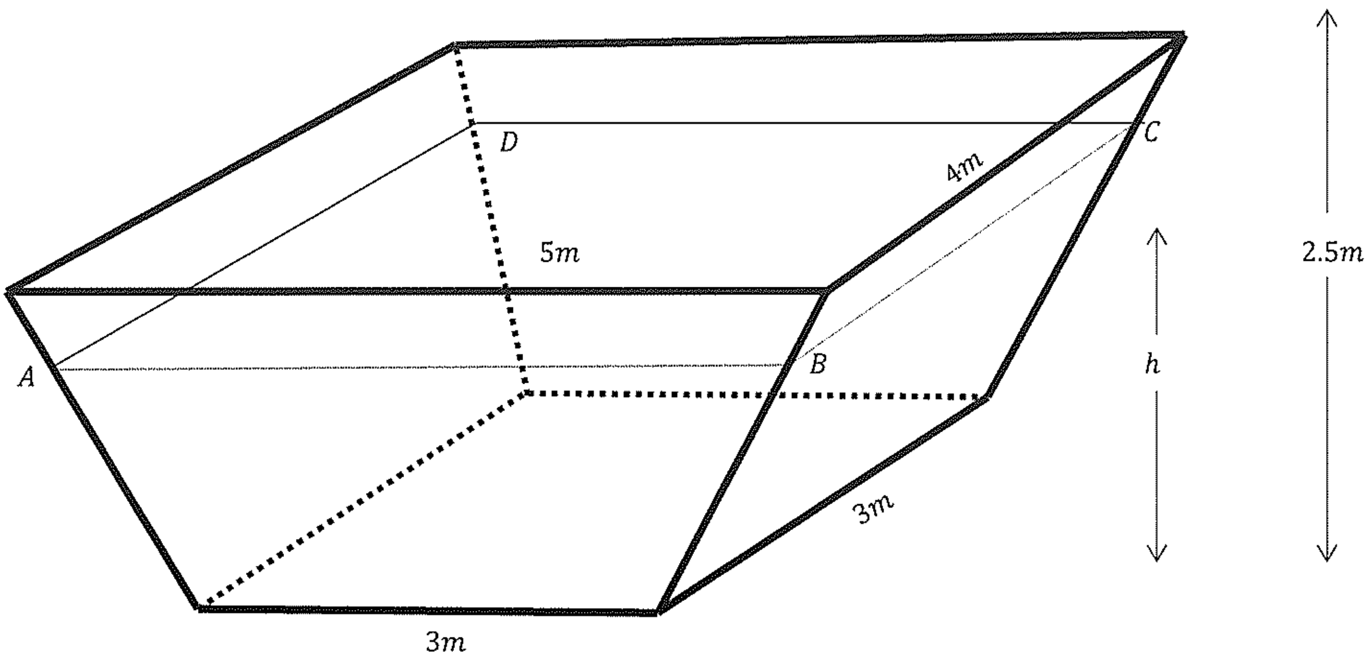
Show that the volume of this solid using cylindrical shells is given by **3**

$V = 2\pi \int_0^{\pi} (\pi - x)sinx.dx$ and find this volume *(you may use your answer to Part (a)*

Question 14 continues on the following page

Question 14 (continued) **Marks**

(c) A rectangular skip bin with a perpendicular height of 2.5 metres has a square base 3 metres by 3 metres and a rectangular top 5 metres by 4 metres. h is the vertical distance up from the base of the skip bin to rectangle $ABCD$ *(see diagram).*



(i) Show that the area of rectangle $ABCD = \frac{8h^2}{25} + \frac{18h}{5} + 9$ **3**

(ii) Find the volume of the skip bin. **2**

(d) Let $I_n = \int_0^{\frac{\pi}{4}} sin^n x.dx$

(i) Show that $I_n = \frac{-\left(\frac{1}{\sqrt{2}}\right)^n}{n} + \frac{n-1}{n} I_{n-2}$ **2**

(ii) Hence find the exact value of $\int_0^{\frac{\pi}{4}} sin^6 x.dx$ **2**

Examination continues on the next page

Question 15 ***(15 marks) Show all necessary working on a separate page*** **Marks**

(a) A cyclist and bicycle with a total mass of $80kg$ is riding around a velodrome with a radius of $50m$ which is banked at an angle of 20^o to the horizontal.

If riding at optimum speed (when she experiences no lateral friction) she is subject to a normal force from the track and gravity $(g = 9.8m\ per\ s^2)$

(i) By resolving forces in the horizontal and vertical directions, find the optimum speed the cyclist can ride at (the speed where no lateral friction is experienced). **3**

(ii) The cyclist decreases her speed to $11m/s$ (which as stated is BELOW optimum speed). How much lateral friction does she now experience? **2**

(iii) The maximum friction that the track can provide before the cyclist slips outward is $0.2 \times N$, where N is the normal force provided by the track. What is the maximum speed that the cyclist can attain before slipping outwards? **2**

(b) A $10kg$ weight is attached by a taut rope to the top of a solid cone with semi vertical angle 30^o. The radius of the cone where the weight is rotating is $6m$.

While rotating the weight is subject to tension from the rope, gravity $(g = 9.8m\ per\ s^2)$ and a normal force while it is touching the cone *(see diagram).*

Question 15(b) continues on the following page

Question 15(b) (continued) **Marks**

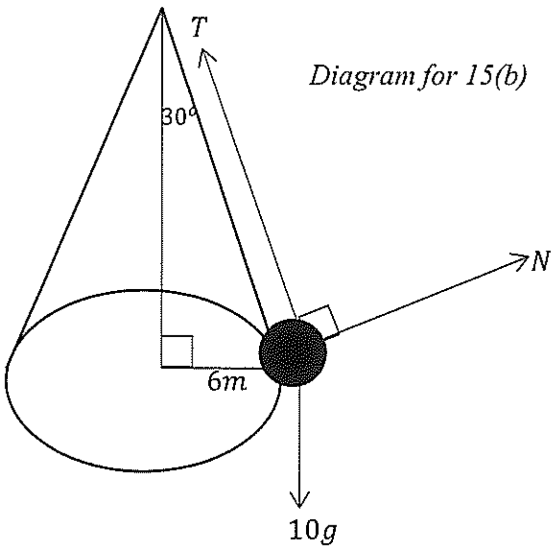


Diagram for 15(b)

(i) By resolving forces vertically and horizontally, find the tension in the rope and the normal force if the particle is rotating at $4m/s$. **4**

(ii) The particle now speeds up until it loses contact with the surface of the cone. At what speed does this happen? **4**

Question 16 ***(15 marks) Show all necessary working on a separate page***

(a) A skydiver jumps out of a stationary balloon and starts falling freely to Earth.

She experiences gravity of mg downwards and air resistance of $\frac{mv^2}{360}$ upwards.

Given that down is *positive*, $x = t = 0$ at the balloon and that $g = 9.8m \; per \; s^2$,

(i) Show that $\ddot{x} = g - \frac{v^2}{360}$ and find her terminal velocity. **2**

(ii) Show that $x = 180ln\left(\frac{360g}{360g - v^2}\right)$ and find the distance fallen when the skydiver reaches $50m/s$. **3**

(iii) Find the time taken for the skydiver to reach this speed. **3**

(b) (i) Use DeMoivre's theorem and the expansion of $(cos\Theta + isin\Theta)^4$ to show that $cos4\Theta = 8cos^4\Theta - 8cos^2\Theta + 1$. **2**

(ii) Hence or otherwise show that $cos\frac{\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2}$. **2**

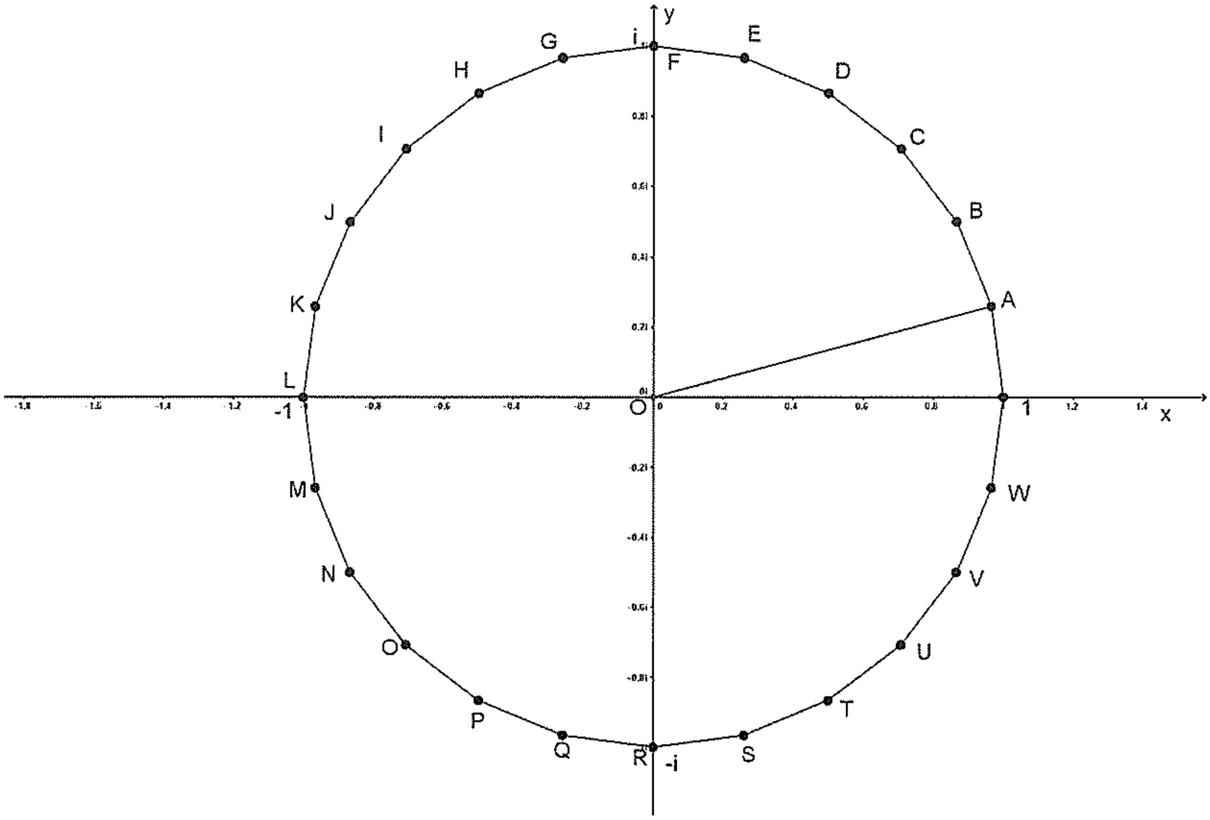
Question 16(b) continues on the following page

Question 16(b) (continued) **Marks**

(iii) $A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W\ AND\ X$

below represent all of the roots of $w^{24} = 1$ on the complex plane.

(see diagram).



If $\overrightarrow{OA} = w$ is the root of $w^{24} = 1$ with the smallest positive argument, show that **1**

$|w - 1| = \sqrt{2}\sqrt{1 - \cos\frac{\pi}{12}}$

(iv) Find the exact value of the perimeter of polygon $ABCDEFGHIJKLMNOPQRSTUVWX$. **1**

(v) Find $\lim\limits_{n \to \infty} n\sqrt{2}\sqrt{1 - \cos\frac{2\pi}{n}}$ **1**

END OF EXAMINATION

Multiple Choice:

(1) B (2) A (3) D (4) D (5) A (6) D (7) D (8) A (9) D (10) C

Q(1) $\frac{z}{w}$

$= \frac{5-5i}{2+i} \times \frac{(2-i)}{(2-i)}$

$= 1-3i$ (B)

(3) $3\alpha^3+5\alpha-1=0$

$3\beta^3+5\beta-1=0$

$3\gamma^3+5\gamma-1=0$

$3(\alpha^3+\beta^3+\gamma^3)+5(\alpha+\beta+\gamma)-3=0$

As $\alpha+\beta+\gamma=0$ (D)

$\alpha^3+\beta^3+\gamma^3=1.$

(5) $e^2 = 1+\frac{144}{25}$

$\therefore e = \frac{13}{5}.$

Foci $= (\pm ae, 0)$

$= (\pm\frac{13}{5}\times 5, 0)$

$= (\pm 13, 0)$ (A)

(6) $u = \ln x.\ du = \frac{1}{x}.dx$

$\int \frac{1}{x\ln x}.dx$

$= \int \frac{1}{u}.du$ (D)

$= \ln u + C$

$= \ln(\ln x) + C$

(7) $\int \sin^3 x.dx$

$= \int \sin^2 x\cdot\sin x.dx$

$= \int (1-\cos^2 x)\sin x.dx$

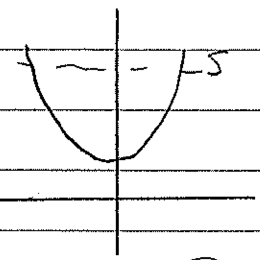
$= \int (\cos^2 x-1).-\sin x.dx$ $u=\cos x$. $du=-\sin x.dx$

$= \int u^2-1.du$ (D)

$= \frac{1}{3}u^3-u+C$

$= \frac{1}{3}\cos^3 x-\cos x+C$

(8) By slices:



$\delta V = \pi[5^2-y^2].\delta x$

$= \pi[5^2-(x^2+1)^2.\delta x]$

$= \pi[24-x^4-2x^2].\delta x$

$\therefore V = \pi\int_{-2}^{2} 24-2x^2-x^4.dx$ (A)

Note: By shells:

$V = 4\pi\int_1^5 y\sqrt{y-1}.dy$ [Ans. $= \frac{1088\pi}{15}$ c.u.]

(9) $\downarrow$

$\frac{mk}{x^2} = mg$ when $x=R$.

$\frac{k}{R^2}=g$ (D)

$k=gR^2$

(10) $\frac{mv^2}{r} = m\times 3g$

$\frac{v^2}{8} = 3g$ (C)

$v = 15.34\,m/s$

Q.(11) (a)(i) $zw = (-\sqrt{3} - i)(1 - i)$

$= -\sqrt{3} - 1 + i(\sqrt{3} - 1)$ 2

(ii) $z = 2\left(\cos\frac{-5\pi}{6} + i\sin\frac{-5\pi}{6}\right)$ 3

$w = \sqrt{2}\left(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right)$

(iii) $zw = 2\sqrt{2}\left(\cos\frac{-13\pi}{12} + i\sin\frac{-13\pi}{12}\right)$

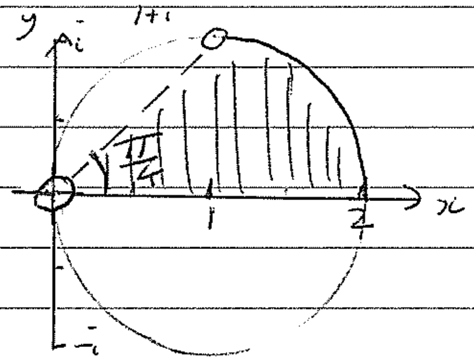
$= 2\sqrt{2}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$ 1

Equating imaginary parts in (i) & (iii)

$\sqrt{3} - 1 = 2\sqrt{2}\sin\frac{11\pi}{12}$

$\therefore \sin\frac{11\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$

(b) 3



(c) (i) If $P(x) = (x-\alpha)^2 Q(x)$ where $Q(x)$ is a polynomial

$P'(x) = 2(x-\alpha)Q(x) + (x-\alpha)^2 Q'(x)$ 2

$= (x-\alpha)[2Q(x) + (x-\alpha)Q'(x)]$

$\therefore$ If $P'(x) = 0$, $x - \alpha = 0$

$x = \alpha$ is a root of $P'(x) = 0$.

(ii) $P(x) = 18x^3 + 3x^2 - 4x - 1 = 0$ has double root. 4

$P'(x) = 54x^2 + 6x - 4 = 0$

or $27x^2 + 3x - 2 = 0$ has same root.

Root must be a factor of $\frac{1}{18}$ & $\frac{2}{27}$ $= \pm 1, \pm\frac{1}{3}, \pm\frac{1}{9}$.

~~$P(\frac{-1}{3}) = 18(\frac{-1}{3})^3 + 3(-\frac{1}{3})^2 - 4(-\frac{1}{3}) - 1 = 0$~~

~~$P'(-\frac{1}{3})$ or $27(-\frac{1}{3})^2 + 3(-\frac{1}{3}) - 2 = 0$~~

Double root $= -\frac{1}{3}$.

Other root:

By $\alpha + \beta + \gamma = \frac{-b}{a}$

$-\frac{2}{3} + \gamma = \frac{-3}{18}$

$\gamma = \frac{1}{2}.$

Roots are

$-\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}.$

Q.(12)(a)(i) $\int x^8 \ln x \, dx$ $\quad u = \ln x \quad v = \frac{1}{9}x^9$

$u' = \frac{1}{x} \quad v' = x^8$

By $\int uv' \, dx = uv - \int v \, u' \, dx$

$$\int x^8 \ln x \, dx = \frac{1}{9}x^9 \ln x - \int \frac{1}{9}x^8 \, dx$$

$$= \frac{1}{9}x^9 \ln x - \frac{1}{81}x^9 + C.$$

2

(ii) $\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2+\cos x} \, dx$ $\quad$ Let $t = \tan\left(\frac{x}{2}\right)$ $\frac{dt}{dx} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right) = \frac{t^2+1}{2}$

$dx = \frac{2}{t^2+1} \, dt$

$$= \int_{\tan\frac{\pi}{3}}^{\tan\frac{2\pi}{3}} \frac{1}{2 + \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} \, dt$$

$$= \int_{\sqrt{3}}^{-\sqrt{3}} \frac{2}{t^2+3} \, dt$$

$$= \frac{2}{\sqrt{3}} \left[\tan^{-1}\left(\frac{t}{\sqrt{3}}\right) \right]_{\sqrt{3}}^{-\sqrt{3}}$$

2

$$= \frac{2}{\sqrt{3}} \left[\tan^{-1}(-1) - \tan^{-1}(1) \right]$$

$$= \frac{2}{\sqrt{3}} \times -\frac{\pi}{2}$$

$$= -\frac{\pi}{\sqrt{3}}.$$

(iii) $\int \frac{\sqrt{x^2-1}}{x} \, dx$ $\quad$ Let $x = \sec\theta$

$dx = \sec\theta \tan\theta \, d\theta$

$$= \int \frac{\tan\theta}{\sec\theta} \cdot \sec\theta \tan\theta \, d\theta$$

$$= \int \tan^2\theta \, d\theta$$

$$= \int (\sec^2\theta - 1) \, d\theta$$

$$= \tan\theta - \theta + C$$

$$= \sqrt{x^2-1} - \sec^{-1}(x) + C$$

3

Q.(B)(b)(i) $\frac{-2x^2-8x-2}{(x+1)^2(x+3)} = \frac{A}{(x+1)^2} + \frac{B}{x+1} + \frac{C}{x+3}$

$x(x+1)^2(x+3)$

$-2x^2-8x-2 = A(x+3)+B(x+1)(x+3)+C(x+1)^2 \quad (1)$

Either: Sub. in $x=-1$, in (1)

$4 = A(-1+3)$

$A = 2.$

Sub. in $x=-3$ in (1):

$4 = C(-3+1)^2$

$C = 1.$

Sub. in $x=0$, $A=2$, $C=1$ in (1)

$-2 = 3\times 2 + 3B + 1\times 1$ 3

10:30 $B = -3$

OR Expanding (1): & equating co-efficients:

$(B+C)x^2+(A+4B+2C)x+(3A+3B+C) = -2x^2-8x-2.$

Equating x^2 co-efficients: $B+C=-2 \Rightarrow C=-2-B$ (2)

x co-efficients. $A+4B+2C=-8$

Sub. $C=-2-B$ in: $A+4B+2(-2-B)=-8 \Rightarrow A+2B=-4$ (3)

Constants: $3A+3B+C=-2.$

Sub. $C=-2-B$ in: $3A+3B-2-B=-2 \Rightarrow 3A+2B=0$ (4)

(4) − (3): $2A=4 \Rightarrow A=2.$

Sub. $A=2$ in (3): $2+2B=-4 \Rightarrow B=-3.$

Sub. $B=-3$ in (2): $C=-2--3 \Rightarrow C=1.$

$\therefore A=2,\ B=-3,\ C=1.$

(ii) $\int \frac{-2x^2-8x-2}{(x+1)^2(x+3)}\,dx$ 2

$= \int \frac{2}{(x+1)^2} - \frac{3}{x+1} + \frac{1}{x+3}\,dx$

$= -\frac{2}{(x+1)} - 3\ln(x+1) + \ln(x+3) + C.$

Q.(12)(c)(i) $25x^2 + 9y^2 = 225$
$\div 225.$

$$\frac{x^2}{9} + \frac{y^2}{25} = 1.$$

$e^2 = 1 - \frac{a^2}{b^2}$ as $b > a$. 1

$= 1 - \frac{9}{25}$

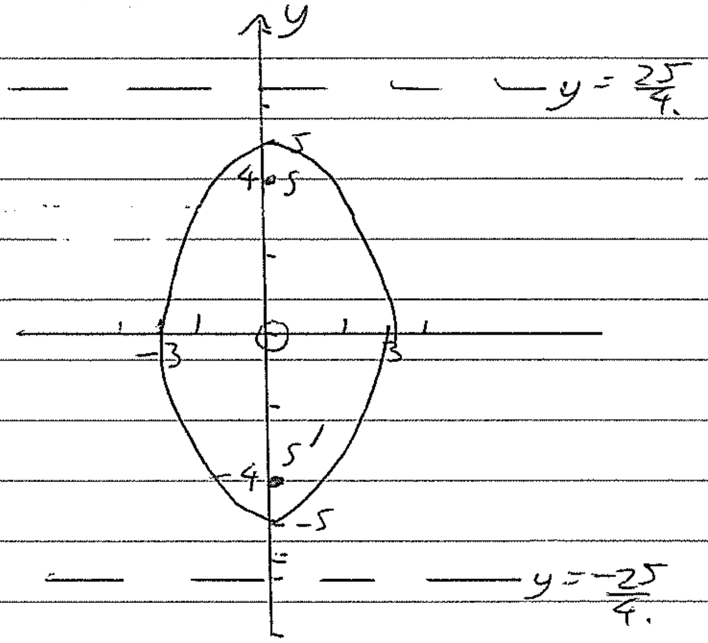
$e = \frac{4}{5}.$

(ii) Foci $= (0, \pm be)$ Directrices: $y = \pm \frac{b}{e}$

$= (0, \pm 5 \times \frac{4}{5})$ $y = \pm 5 \div \frac{4}{5}$

$= (0, \pm 4)$ $y = \pm \frac{25}{4}.$ 1

(iii)



1

(13)(a) Let $y = x + 2$

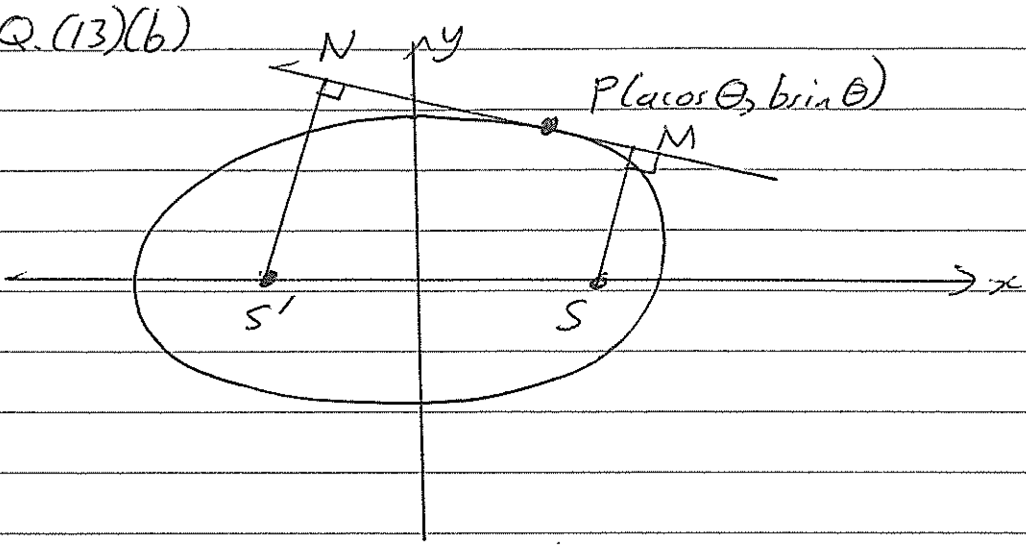
$\therefore x = y - 2.$

Equation is $2(y-2)^3 + 7(y-2)^2 + 10(y-2) - 7 = 0$ 2

$2y^3 - 5y^2 + 6y - 15 = 0.$

or $2x^3 - 5x^2 + 6x - 15 = 0.$

Q.(13)(b)



Tangent at P: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

$$\frac{2x}{a^2} + \frac{2y}{b^2}\cdot\frac{dy}{dx} = 0$$

$$\frac{2y}{b^2}\frac{dy}{dx} = \frac{-2x}{a^2}$$

$$\frac{dy}{dx} = \frac{-b^2x}{a^2y}$$

At $P(a\cos\theta, b\sin\theta)$ $\frac{dy}{dx} = \frac{-b^2 a\cos\theta}{a^2 b\sin\theta}$ 2

$$= \frac{-b\cos\theta}{a\sin\theta}$$

By $y - y_1 = m(x - x_1)$

$$y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$$

$$ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta.$$

$$bx\cos\theta + ay\sin\theta = ab(\sin^2\theta + \cos^2\theta)$$

$$bx\cos\theta + ay\sin\theta = ab.$$

(ii) Distance MS is perpendicular distance from tangent at P to $S(ae, 0)$

PTO →

By perpendicular distance $= \dfrac{|Ax_1 + By_1 + C|}{\sqrt{a^2+b^2}}$

$$MS = \frac{|b\cos\theta \times ae + a\sin\theta \times 0 - ab|}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

$$= \frac{|abe\cos\theta - ab|}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

2

As $|e|<1$ & $|\cos\theta|<1$, $abe\cos\theta < ab$

$\therefore |abe\cos\theta - ab| = ab - abe\cos\theta$

$$\therefore MS = \frac{ab - abe\cos\theta}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

(iii) $$NS' = \frac{|b\cos\theta \times -ae - ab|}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

$$= \frac{ab + abe\cos\theta}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

3

$$\therefore MS \times NS' = \frac{(ab - abe\cos\theta)}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} \times \frac{(ab + abe\cos\theta)}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

$$= \frac{a^2b^2 - a^2b^2e^2\cos^2\theta}{b^2\cos^2\theta + a^2\sin^2\theta}$$

$$= \frac{b^2(a^2 - a^2e^2\cos^2\theta)}{b^2\cos^2\theta + a^2\sin^2\theta}$$

PTO →

As $e^2 = 1 - \frac{b^2}{a^2}$ $\therefore a^2e^2 = a^2 - b^2$

$$= \frac{b^2[a^2 - (a^2-b^2)\cos^2\theta]}{b^2\cos^2\theta + a^2\sin^2\theta}$$

$$= \frac{b^2[a^2(1-\cos^2\theta) + b^2\cos^2\theta]}{b^2\cos^2\theta + a^2\sin^2\theta}$$

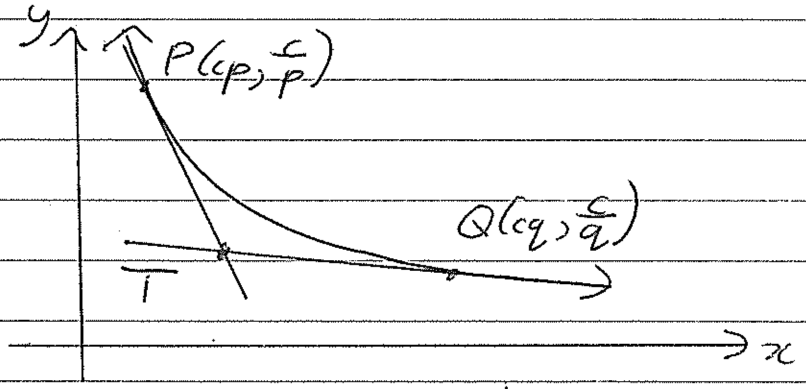
As $\sin^2\theta = 1 - \cos^2\theta$

$$= \frac{b^2[a^2\sin^2\theta + b^2\cos^2\theta]}{b^2\cos^2\theta + a^2\sin^2\theta}$$

$= b^2$

$= RHS$ QED.

(c)



(i) $xy = c^2$

$y = \frac{c^2}{x}$

$y' = -\frac{c^2}{x^2}$

At $P(cp, \frac{c}{p})$

$y' = -\frac{c^2}{c^2p^2}$

$= -\frac{1}{p^2}$

$\therefore$ By $y - y_1 = m(x - x_1)$

$y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$

$p^2y - cp = -x + cp$

$x + p^2y = 2cp$.

2

(i) (continued):

Tangent at $P = x + p^2y = 2cp$. (1)

Tangent at Q is $x + q^2y = 2cq$ (2)

(ii) (1) − (2) above.

$(p^2 - q^2)y = 2c(p-q)$

$(p-q)(p+q)y = 2c(p-q)$

$y = \frac{2c}{p+q}$ (3)

Sub. (3) in (1):

$x = 2cp - \frac{2cp^2}{p+q}$

$= \frac{2cp(p+q) - 2cp^2}{p+q}$

$= \frac{2cp^2 + 2cpq - 2cp^2}{p+q}$

$x = \frac{2cpq}{p+q}$

(13)(c)(ii)(cont). p. 9

$$\therefore T = \left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$$

(iii) m PQ $= \dfrac{\frac{c}{q} - \frac{c}{p}}{cq - cp} \quad \begin{matrix} \times pq \\ \times pq \end{matrix}$

$$= \frac{c(p-q)}{cpq(q-p)}$$

$$= \frac{-1}{pq}$$

$$y - \frac{c}{q} = -\frac{1}{pq}(x - cq)$$

$$pqy - cp = -x + cq$$

$$+x \quad +cp$$

$$\underline{x + pqy = c(p+q)}$$

1

(iv) Chord PQ passes through $R(4c, 2c)$:

$$4c + 2cpq = c(p+q)$$

$$2pq + 4 = p+q$$

$$2pq = p+q-4 \quad (1)$$

At T, $y = \dfrac{2c}{p+q}$

$$\therefore p+q = \frac{2c}{y} \quad (2)$$

At T $x = \dfrac{2cpq}{p+q} \quad (3)$

Sub. (1) in (3):

$$x = \frac{c(p+q-4)}{p+q}$$

$$(p+q)x = c[(p+q)-4]. \quad (4)$$

Sub. (2) in (4):

$$\frac{2cx}{y} = c\left[\frac{2c}{y} - 4\right] \quad \times \frac{y}{c}$$

$$2x = 2c - 4y$$

$$\underline{x + 2y - c = 0}$$

Locus of T is 1

$\underline{x + 2y - c = 0.}$

11:35

→ 90 mins so far.

Q.(14)(a)(i) $\int_0^a f(a-x).dx$ $\quad u = a-x$, $du = -dx$

$= -\int_{u=a}^{a-0} f(u).du$

$= \int_0^a f(u).du$ as $\int_b^a f(x).dx = -\int_a^b f(x).dx$ $\qquad \underline{1}$

$= \int_0^a f(x).dx$ by $\int_a^b f(x).dx = \int_a^b f(y).dy$

= RHS QED.

(ii) $\int_0^{\pi} x\sin x.dx$

$= \int_0^{\pi} (\pi - x)\sin(\pi - x).dx$ $\qquad \underline{2}$

$= \int_0^{\pi} (\pi - x)\sin x.dx$ as $\sin(\pi - x) = \sin x$.

$\int_0^{\pi} x\sin x.dx = \pi\int_0^{\pi}\sin x.dx - \int_0^{\pi} x\sin x.dx$

$2\int_0^{\pi} x\sin x.dx = \pi\int_0^{\pi}\sin x.dx$

$\int_0^{\pi} x\sin x.dx = \frac{\pi}{2}\int_0^{\pi}\sin x.dx$ (1)

NOTE:

$\int_0^{\pi}\sin x.dx$

$= \left[-\cos x\right]_0^{\pi}$

$= [-(-1) - -1]$

$= 2.$ (2)

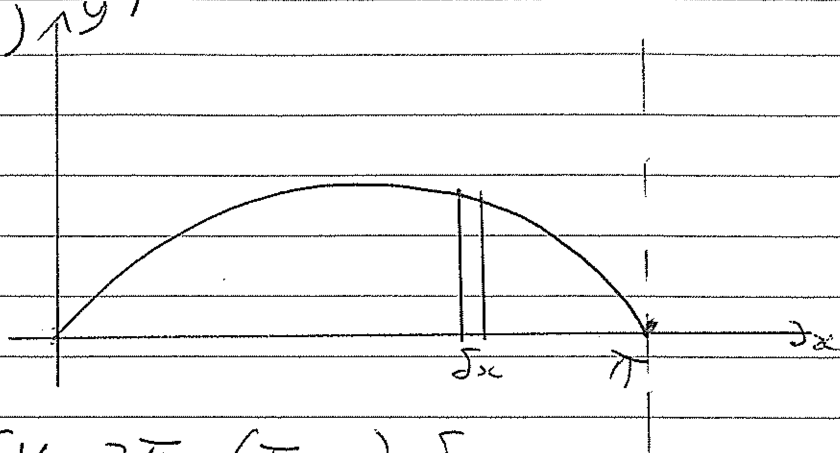
Sub. (2) in (1):

$\int_0^{\pi} x\sin x.dx = \frac{\pi}{2}\times 2$

$= \underline{\pi}.$

p. 11

Q(14)(b)



$$\delta V = 2\pi y(\pi - x).\delta x$$
$$= 2\pi(\pi - x)\sin x.\delta x$$

$$V = \lim_{\delta x \to 0} 2\pi \sum_{x=0}^{x=\pi} (\pi - x)\sin x.\delta x$$

Letting $\delta x \to 0$

3.

$$V = 2\pi \int_0^{\pi} (\pi - x)\sin x.dx$$

$$= 2\pi \int_0^{\pi} (\pi - x)\sin(\pi - x).dx \quad \text{as } \sin(\pi - x) = \sin x.$$

$$= 2\pi \int_0^{\pi} x\sin x.dx \quad \text{as } \int_0^a f(a-x).dx = \int_0^a f(x).dx$$

$$= 2\pi \times \pi$$
$$= 2\pi^2 \text{ c.u.}$$

(c)(i) As all lines in skip bin are straight:

$AB = Fh + G$.	$BC = Jh + K$.
$AB = 3$ when $h = 0 \therefore 3 = G$.	$BC = 3$ when $h = 0 \Rightarrow 3 = k$.
$AB = 5$ when $h = 2.5$.	$BC = 4$ when $h = 2.5$.
$5 = 2.5F + 3$.	$4 = 2.5J + 3$
$\frac{4}{5} = F$	$\frac{2}{5} = J$.
$\therefore AB = \frac{4}{5}h + 3$.	$BC = \frac{2}{5}h + 3$

3

$$\text{Area } ABCD = \left(\frac{4}{5}h + 3\right)\left(\frac{2}{5}h + 3\right)$$

$$= \frac{8h^2}{25} + \frac{18h}{5} + 9.$$

(14)(c)(ii) $\delta V = \left(\frac{8h^2}{25} + \frac{18h}{5} + 9\right).\delta h$

$$V = \lim_{\delta h \to 0} \sum_{h=0}^{h=2.5} \left(\frac{8h^2}{25} + \frac{18h}{5} + 9\right).\delta h$$

Letting $\delta h \to 0$

$$V = \int_0^{2.5} \frac{8h^2}{25} + \frac{18h}{5} + 9.dh$$

$$= \left[\frac{8h^3}{75} + \frac{9h^2}{5} + 9h\right]_0^{2.5}$$

$$= \frac{425}{12} \text{ cubic units.}$$

$$= 35\tfrac{5}{12} \text{ c.u.}$$

2

(d) (i) $I_n = \int_0^{\frac{\pi}{4}} \sin^n x \, .dx$

$$I_n = \int_0^{\frac{\pi}{4}} \sin^{n-1} x . \sin x . dx$$

$u = \sin^{n-1} x \quad v = -\cos x$

$u' = (n-1)\sin^{n-2} x \cos x \quad v' = \sin x$

By $\int uv'.dx = u \quad v \quad - \int vu'.dx$

2

$$I_n = \left[-\cos x \sin^{n-1} x\right]_0^{\frac{\pi}{4}} + (n-1)\int_0^{\frac{\pi}{4}} \sin^{n-2} x \cos^2 x \, .dx$$

$$I_n = \left[-\frac{1}{\sqrt{2}} \times \left(\frac{1}{\sqrt{2}}\right)^{n-1} - 0\right] + (n-1)\int_0^{\frac{\pi}{4}} (1 - \sin^2 x) \sin^{n-2} x.dx$$

$$I_n = -\left(\frac{1}{\sqrt{2}}\right)^n + (n-1)\int_0^{\frac{\pi}{4}} \sin^{n-2} x.dx - (n-1)\int_0^{\frac{\pi}{4}} \sin^n x.dx$$

$$I_n = -\left(\frac{1}{\sqrt{2}}\right)^n + (n-1)I_{n-2} - (n-1)I_n.$$

$$nI_n = -\left(\frac{1}{\sqrt{2}}\right)^n + (n-1)I_{n-2}.$$

$$I_n = \frac{-\left(\frac{1}{\sqrt{2}}\right)^n}{n} + \frac{(n-1)}{n} I_{n-2}$$

= RHS QED.

(14)(d)(ii) $I_0 = \int_0^{\frac{\pi}{4}} 1.dx$

$= \frac{\pi}{4}.$

$I_2 = -\frac{\left(\frac{1}{\sqrt{2}}\right)^2}{2} + \frac{1}{2} I_0$

$= \frac{\pi}{8} - \frac{1}{4}.$

$I_4 = -\frac{\left(\frac{1}{\sqrt{2}}\right)^4}{4} + \frac{3}{4} I_2$

$= -\frac{1}{4} + \frac{3}{4}\left(\frac{\pi}{8} - \frac{1}{4}\right)$

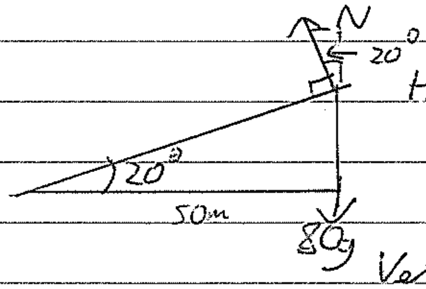
$= \frac{3\pi}{32} - \frac{1}{4}.$

$I_6 = -\frac{\left(\frac{1}{\sqrt{2}}\right)^6}{6} + \frac{5}{6} I_4$ 2

$= -\frac{1}{48} + \frac{5}{6} \times \left(\frac{3\pi}{32} - \frac{1}{4}\right)$

$= \frac{5\pi}{64} - \frac{11}{48}.$

Q.(15)(a)(i)



Horizontal: $N\sin 20^\circ = \frac{mv^2}{r}$

$N\sin 20^\circ = \frac{80v^2}{50}$ (1)

Vertical: $N\cos 20^\circ = 80g$

$N = \frac{80g}{\cos 20^\circ}$ (2)

Sub. (2) in (1): $\frac{80g\sin 20^\circ}{\cos 20^\circ} = \frac{80v^2}{50}$ 3

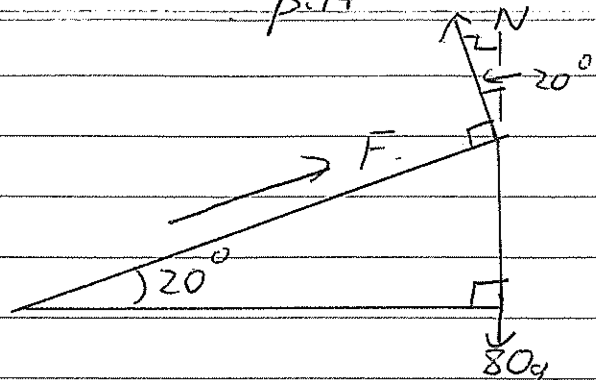
$178.3.. = v^2$

$13.35.. = v$

Optimum speed $\doteq$ 13.35.. m/s.

p.14

(15)(a)(ii)



Finding F

Vertical: $N\cos 20° + F\sin 20° = 80g$ (1) $\times \sin 20° = (3)$

Horizontal: $N\sin 20° - F\cos 20° = \frac{80\times 11^2}{50}$ (2) $\times \cos 20° = (4)$

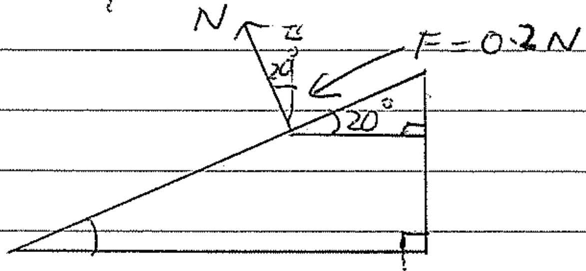
$N\cos 20° \sin 20° + F\sin^2 20° = 80g \sin 20°$ (3) (3)

$N\cos 20° \sin 20° - F\cos^2 20° = \frac{80\times 11^2}{50}\cos 20°$ (4) (4)

$F(\sin^2 20° + \cos^2 20°) = 80g\sin 20° - \frac{80\times 11^2}{50}\cos 20°$ 2

Lateral friction: F = 86.219.. Newtons.

(iii)



Resolving horizontally: $N\sin 20° + 0.2N\cos 20° = \frac{80v^2}{50}$ (1)

Vertically: $N\cos 20° - 0.2N\sin 20° = 80g$ (2) 2

Using (2): $N = \frac{80g}{\cos 20° - 0.2\sin 20°}$ (3)

Sub. (3) in (1): $\frac{80g}{\cos 20° - 0.2\sin 20°}(\sin 20° + 0.2\cos 20°) = \frac{80v^2}{50}$

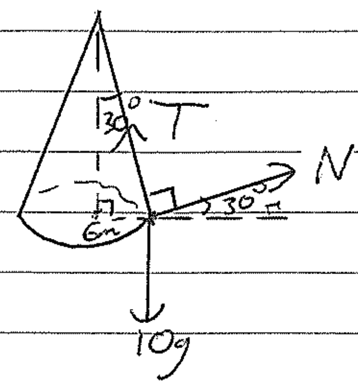
$298.04.. = v^2$

$17.2638.. = v$

The maximum speed before slipping outwards is 17.26 m/s.

2.5h.

(15)(b)(i)



Resolving vertically: $T\cos 30^\circ + N\sin 30^\circ = 10g$ (1) $\times\cos = (3)$

horizontally: $T\sin 30^\circ - N\cos 30^\circ = \frac{10\times 4^2}{6}$ $\left(\frac{mv^2}{r}\right)$ (2) $\times\sin 30^\circ$ (4)

$T\cos^2 30^\circ + N\cos 30^\circ \sin 30^\circ = 10g\cos 30^\circ$ (3)

$T\sin^2 30^\circ - N\cos 30^\circ \sin 30^\circ = \frac{10\times 4^2}{6}\sin 30^\circ$ (4) $+$

$\underline{4}$

$T = 10g\cos 30^\circ + \frac{10\times 4^2}{6}\sin 30^\circ$

$T \doteq 98.2$ Newtons.

Using (1): $N = \frac{10g - T\cos 30^\circ}{\sin 30^\circ}$

$\doteq 25.9$ Newtons.

Tension = 98.2 Newtons, Normal force = 25.9 Newtons.

(ii) If particle loses contact with cone, there is no normal force

$\therefore T\cos 30^\circ = 10g$ (1) [ ].

$T = \frac{10g}{\cos 30^\circ}$

$\underline{4}$

$T\sin 30^\circ = \frac{10v^2}{6}$

$\frac{10g}{\cos 30^\circ} \times \frac{6}{10} \times \sin 30^\circ = v^2$

$33.9\ldots = v^2$

$5.826\ldots = v$

$\rightarrow$ Weight loses contact with cone at 5.83 m/s.

Q. (16)(a)

$$\downarrow mg$$

$$\frac{mv^2}{360} \uparrow$$

(i) $F = ma = mg - \dfrac{mv^2}{360}$

$$a = \ddot{x} = g - \frac{v^2}{360}$$

2

Terminal velocity: $\ddot{x} = 0 : g - \dfrac{v^2}{360} = 0$

$$v^2 = 360g$$

$$v = 59.4 \text{ m/s (1DP)}$$

(ii) $\ddot{x} = v \cdot \dfrac{dv}{dx} = \dfrac{360g - v^2}{360}$

$$\frac{dv}{dx} = \frac{360g - v^2}{360v}$$

$$\frac{dx}{dv} = \frac{360v}{360g - v^2}$$

$$x = -180 \int \frac{-2v}{360g - v^2} \, dv$$

$$= -180 \ln(360g - v^2) + C$$

3

As $x = 0$ when $v = 0$

$$0 = -180 \ln 360g + C$$

$$180 \ln 360g = C$$

$$x = 180 \ln 360g - 180 \ln(360g - v^2)$$

$$x = 180 \ln \left[\frac{360g}{360g - v^2}\right]$$

When $v = 50$, $x = 180 \ln \left[\dfrac{360g}{360g - 50^2}\right]$

$$= 221.96\ldots \text{ m.}$$

She reaches 50 m/s, 221.96 m below the balloon.

Q. (16)(a)(iii) Time taken to reach m/s.

→ Need t in terms of v:

$$\ddot{x} = \frac{dv}{dt} = \frac{360g - v^2}{360}$$

$$\frac{dt}{dv} = \frac{360}{360g - v^2}.$$

Letting $\frac{360}{360g - v^2} = \frac{A}{(\sqrt{360g} - v)} + \frac{B}{(\sqrt{360g} + v)}$

$$360 = A(\sqrt{360g} + v) + B(\sqrt{360g} - v) \quad (1)$$

Sub. in $v = -\sqrt{360g}$ in (1)

$$360 = 2B\sqrt{360g}$$

$$\sqrt{360} = 2B\sqrt{g}$$

$$6\sqrt{10} = 2B\sqrt{g}.$$

$$\frac{3\sqrt{10}}{\sqrt{g}} = B$$

Sub. in $v = \sqrt{360g}$ in (1):

$$360 = 2A\sqrt{360g}$$

$$\frac{3\sqrt{10}}{\sqrt{g}} = A.$$

$$\therefore \frac{dt}{dv} = \frac{3\sqrt{10}}{\sqrt{g}}\left[\frac{1}{\sqrt{360g} - v} + \frac{1}{\sqrt{360g} + v}\right].$$

$$t = \frac{3\sqrt{10}}{\sqrt{g}}\int \frac{-1}{\sqrt{360g} - v}\,dv + \frac{3\sqrt{10}}{\sqrt{g}}\int \frac{1}{\sqrt{360g} + v}\,dv$$

$$= -\frac{3\sqrt{10}}{\sqrt{g}}\ln\left(\sqrt{360g} - v\right) + \frac{3\sqrt{10}}{\sqrt{g}}\ln\left(\sqrt{360g} + v\right)$$

$$t = \frac{3\sqrt{10}}{\sqrt{g}}\ln\left[\frac{\sqrt{360g} + v}{\sqrt{360g} - v}\right].$$

When $v = 50$ m/s

$$t = \frac{3\sqrt{10}}{\sqrt{g}}\ln\left[\frac{\sqrt{360g} + 50}{\sqrt{360g} - 50}\right].$$

$$= 7.4385\ldots$$

The skydiver reaches 50 m/s after 7.44 seconds.

Q.(16)(b)(i) By DeMoivre's theorem:

$$\cos 4\theta + i\sin 4\theta = (\cos\theta + i\sin\theta)^4$$
$$= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta.$$

Equating real parts, $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$
$$= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$$
$$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$$
$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1 \qquad \underline{2}$$
= RHS QED.

(ii) Hence: As $\cos\frac{\pi}{3} = \cos\left(4\times\frac{\pi}{12}\right) = \frac{1}{2}$,

$\cos\frac{\pi}{12}$ is solution to $\cos 4\theta = \cos\frac{\pi}{3} = \frac{1}{2}$.
$\therefore \cos\frac{\pi}{12}$ is solution to $8x^4 - 8x^2 + 1 = \frac{1}{2}$, $x = \cos\frac{\pi}{12}$.
i.e. $16x^4 - 16x^2 + 1 = 0$

Using quadratic formula as equation is quadratic in x^2,
$$x^2 = \frac{16 \pm \sqrt{16^2 - 4\times 16\times 1}}{2\times 16}$$
$$x^2 = \frac{16 \pm 8\sqrt{3}}{32}.$$
$$= \frac{2 \pm \sqrt{3}}{4} \qquad \underline{2}$$

As $\cos\frac{\pi}{12} > \cos\frac{\pi}{4}\left[=\frac{1}{\sqrt{2}}\right]$,
$$\cos^2\frac{\pi}{12} = \frac{2+\sqrt{3}}{4}.$$
$$\cos\frac{\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2}.$$

(iii) Either: $w - 1 = \left(\cos\frac{\pi}{12} - 1\right) + i\sin\frac{\pi}{12}$.
$$\therefore |w-1| = \sqrt{\cos^2\frac{\pi}{12} - 2\cos\frac{\pi}{12} + 1 + \sin^2\frac{\pi}{12}}$$
$$= \sqrt{2 - 2\cos\frac{\pi}{12}}$$
$$= \sqrt{2}\sqrt{1 - \cos\frac{\pi}{12}}$$

PTO →

OR Using cosine rule on $\triangle AOX$: $AX = |w-1|$.

$$(AX)^2 = (OA)^2 + (OX)^2 - 2 \times OA \times OX \times \cos\frac{\pi}{12}$$

$$= 1^2 + 1^2 - 2 \times 1 \times 1 \times \cos\frac{\pi}{12}$$

$$|w-1|^2 = 2 - 2\cos\frac{\pi}{12}$$

1

$$w - 1 = \sqrt{2}\sqrt{1-\cos\frac{\pi}{12}}.$$

(iv) As $AB = BC = CD$, etc.

Perimeter ABCD ... X

$$= 24\sqrt{2}\sqrt{1-\cos\frac{\pi}{12}}.$$

1

(v) The perimeter of a polygon with vertices solution $w^n = 1$ would

$$= n\sqrt{2}\sqrt{1-\cos\frac{2\pi}{n}}.$$

As $n \to \infty$, this shape would approach a circle with perimeter $= 2\pi$.

1

$$\therefore \lim_{n\to\infty} n\sqrt{2}\sqrt{1-\cos\frac{2\pi}{n}} = 2\pi.$$

END OF SOLUTIONS!