Girraween High School

2018

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time: 5 minutes
- Working time: 3 Hours
- Write using a black or blue pen
- Board approved calculators may be used
- Laminated reference sheets are provided
- Answer multiple choice questions by completely colouring in the appropriate circle on your multiple choice answer sheet on the front page of your answer booklet.
- In questions 11-16 start all questions on a separate page in your answer booklet and show all relevant mathematical reasoning and/or calculations.

Total Marks: 100

Section 1 (Pages 2– 5) **10 Marks**

- Attempt Q1 - Q10
- Allow about 15 minutes for this section

Section 2 (Pages 5-13) **90 marks**

- Attempt Q11 – Q16
- Allow about 2 hours and 45 minutes for this section

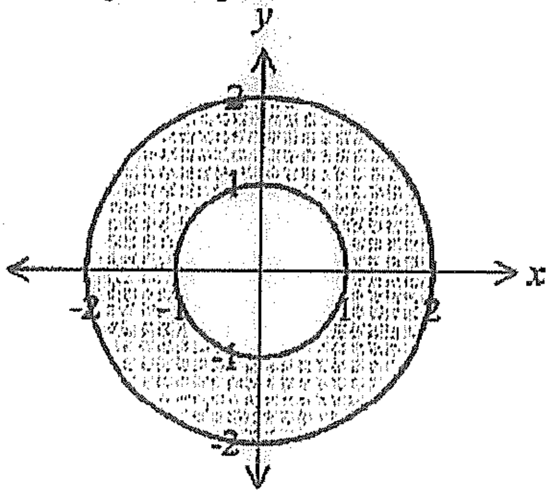
Section 1 (10 marks)

Allow about 15 minutes for this section.
Fill in the appropriate circle in your answer booklet.

1. Given that $z = 1 + i$, what is the value of z^8?

(A) -16 (B) -8 (C) 8 (D) 16

2. Consider the Argand diagram below.



Which inequality could define the shaded area?

(A) $0 \leq |z| \leq 2$

(B) $1 \leq |z| \leq 2$

(C) $0 \leq |z-1| \leq 2$

(D) $1 \leq |z-1| \leq 2$

3. The equation $x^4 + px + q = 0$, where $p \neq 0$ and $q \neq 0$ has roots α, β, γ and δ. What is the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$?

(A) $-4q$ (B) $p^2 - 2q$ (C) $p^4 - 2q$ (D) p^4

4. When $x^y = e$ is implicitly differentiated with respect to x, the result for $\frac{dy}{dx}$ is

(A) $\frac{-y}{x \log_e x}$ (B) $\frac{y}{x \log_e x}$ (C) $\frac{-x \log_e x}{y}$ (D) $\frac{x \log_e x}{y}$

5. Which of the following is an expression for $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,dx$

after the substitution $t = \tan\frac{x}{2}$?

(A) $\displaystyle\int_0^1 \frac{1}{1+2t}\,\mathrm{dt}$ (B) $\displaystyle\int_0^1 \frac{2}{1+2t}\,\mathrm{dt}$ (C) $\displaystyle\int_0^1 \frac{1}{(1+t)^2}\,\mathrm{dt}$ (D) $\displaystyle\int_0^1 \frac{2}{(1+t)^2}\,\mathrm{dt}$

6. What are the equations of the directrices of the hyperbola with equation

$\dfrac{x^2}{144} - \dfrac{y^2}{25} = 1$?

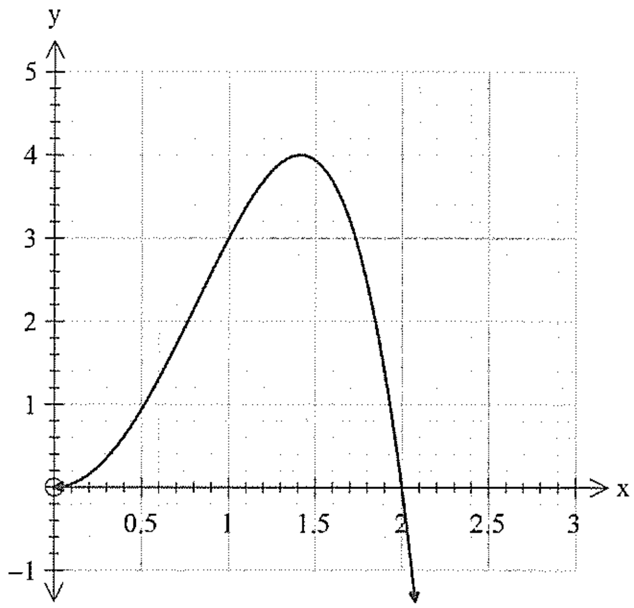
(A) $x = \pm\dfrac{13}{144}$ (B) $x = \pm\dfrac{13}{25}$ (C) $x = \pm\dfrac{25}{13}$ (D) $x = \pm\dfrac{144}{13}$

7. The region enclosed by $y = \sin x$, $y = 0$ and $x = \dfrac{\pi}{2}$ is rotated about the y-axis

to produce a solid. What is the volume of this solid using the method of cylindrical shells?

(A) π cubic units (B) $\dfrac{\pi}{2}$ cubic units (C) $\dfrac{3\pi}{2}$ cubic units (D) 2π cubic units

8. The graph of $y = 4x^2 - x^4$ is given below.



The region in the first quadrant bounded by the curve $y = 4x^2 - x^4$ and the x-axis between $x = 0$ and $x = 2$ is rotated about the y-axis.
Which of the following is an expression for the volume, V, of the solid formed?

(A) $V = 2\pi\int_0^4 \sqrt{4-y}\ dy$

(B) $V = 4\pi\int_0^4 \sqrt{4-y}\ dy$

(C) $V = 8\pi\int_0^4 \sqrt{4-y}\ dy$

(D) $V = 16\pi\int_0^4 \sqrt{4-y}\,dy$

9. A wheel of radius 2 metres rotates at 1200 revolutions per minute.
What is the tangential velocity of a point on the wheel?

(A) 40 m/s (B) 80 m/s (C) 251 m/s (D) 260 m/s

10. A particle of mass m is moving horizontally in a straight line. Its motion is opposed by a force of magnitude $2m(v + v^2)$ Newtons when its speed is v m/s. At time t seconds the particle has a displacement of x metres from a fixed point O on the line and velocity v m/s.
Which of the following is an expression for x in terms of v ?

(A) $-\frac{1}{2}\int \frac{1}{1+v}dv$

(B) $-\frac{1}{2}\int \frac{1}{v(1+v)}dv$

(C) $\frac{1}{2}\int \frac{1}{1+v}dv$

(D) $\frac{1}{2}\int \frac{1}{v(1+v)}dv$

Section 2

Question 11 (15 marks)

a. $z = p + 2i$, where p is a real number, and $w = 1 - 2i$ represent two complex numbers.

(i) Find $\frac{z}{w}$ in the form $a + ib$, where a and b are real numbers. [2]

(ii) Given that $\left|\frac{z}{w}\right| = 13$, find all possible values of p. [2]

b. $z = 1 - \sqrt{3}\,i$

(i) Find the values of $|z|$ and $\arg z$. [2]

(ii) Find the exact value of z^6. [2]

c. (i) On an Argand diagram, sketch the locus of z represented by $|z - 3| = 3$. [2]

(ii) Explain why $\arg(z - 3) = 2\arg z$. [2]

d. If $2 + i$ is a root of $P(x) = x^4 - 6x^3 + 9x^2 + 6x - 20$, resolve $P(x)$ into irreducible factors over the complex field. [3]

Question 12 (15 marks)

a. Find

(i) $\int x\, e^{-x} dx$ [2]

(ii) $\int_0^{\frac{\pi}{2}} \frac{1}{2 - \cos x + 2\sin x}\, dx$ [3]

(iii) $\int_0^{\frac{\pi}{6}} \sec^3\ 2\theta\, d\theta$ [3]

b. (i) Find real numbers a, b, c and d such that:

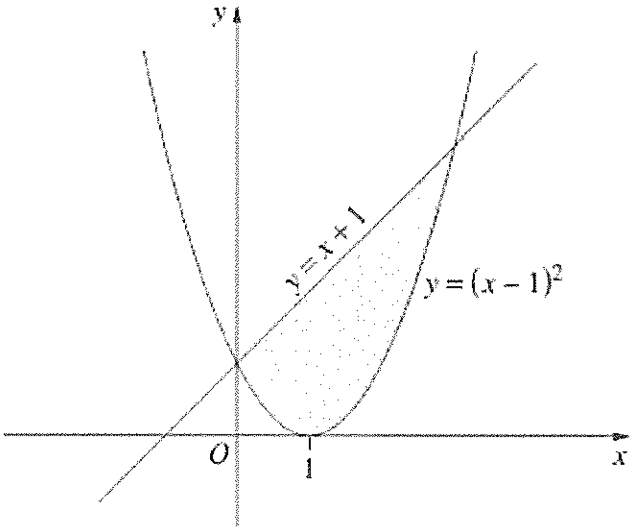
$$\frac{5x^3 - 3x^2 + 2x - 1}{x^2(x^2+1)} = \frac{a}{x} + \frac{b}{x^2} + \frac{cx + d}{x^2 + 1}$$ [2]

(ii) Hence find $\int \frac{5x^3 - 3x^2 + 2x - 1}{x^2(x^2+1)}\, dx$ [2]

c. The diagram shows the region enclosed by the curves $y = x + 1$ and $y = (x - 1)^2$.

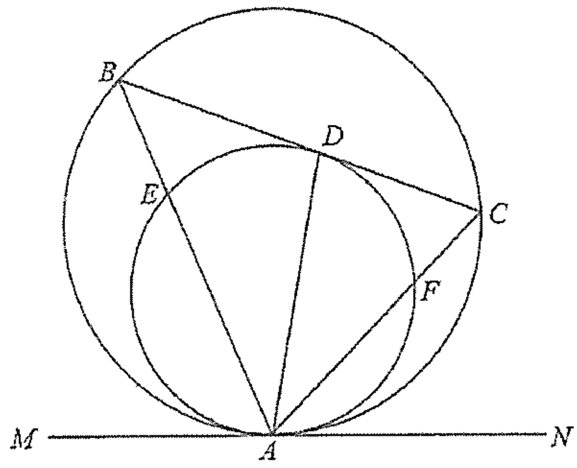
The region is rotated about the y-axis.

Find the volume of the solid using the method of cylindrical shells. [3]



Question 13 (15 marks)

a.



In the diagram, MAN is the common tangent to two circles touching internally at A.

B and C are two points on the larger circle such that BC is a tangent to the smaller circle with point of contact at D. AB and AC cut the smaller circle at E and F respectively.

(i) Copy or trace the into your answer booklet.

(ii) Show that AD bisects $\angle BAC$ [4]

b. An ellipse has the equation $\frac{x^2}{100} + \frac{y^2}{75} = 1$.

(i) Sketch the curve, showing the coordinates of the foci and the equations of the directrices. [2]

(ii) Find the equation of the normal to the ellipse at the point $P\left(5, 7\frac{1}{2}\right)$. [2]

(iii) Find the equation of the circle that is tangential to the ellipse at P and $Q\left(5, -7\frac{1}{2}\right)$. [3]

c. (i) Show that the tangent to the curve $xy = c^2$ at $T\left(ct, \frac{c}{t}\right)$ is given by

$$x + t^2y = 2ct$$ [2]

(ii) The tangent cuts the x and y axes at A and B respectively.

Prove that T is the centre of the circle that passes through O, A and B where O is the origin. [2]

Question 14 (15 marks)

a. (i) Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. [2]

(ii) Hence, find the value of $\int_0^2 x(2-x)^5\,dx$. [2]

b. Given that $I_{2n+1} = \int_0^1 x^{2n+1}e^{x^2}dx$, where n is a positive integer,

(i) Show that $I_{2n+1} = \frac{1}{2}e - nI_{2n-1}$. [3]

(ii) Hence, or otherwise, evaluate $\int_0^1 x^5 e^{x^2}\,dx$. [3]

c. (i) Use the binomial theorem to expand $(\cos\theta + i\sin\theta)^3$. [1]

(ii) Use De Moivre's Theorem and your result from (i) to prove that

$\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$. [2]

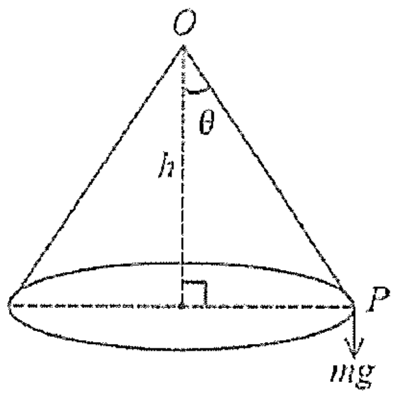
(iii) Hence, or otherwise, find the smallest positive solution of

$4\cos^3\theta - 3\cos\theta = 1$ [2]

Question 15 (15 marks)

a. A mass of m kg at P is suspended by a light inextensible string from point O.

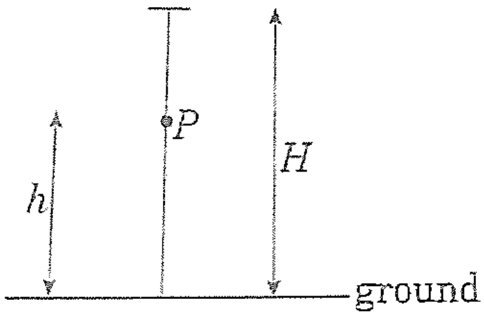
It describes a circle with a constant speed in a horizontal plane whose vertical distance below O is h metres.



(i) Show that $\omega = \sqrt{\frac{g}{h}}$. [2]

(ii) What is the period of the motion? [1]

b.



From a point on the ground an object of mass m kg is projected vertically upward with an initial speed of u. The object reaches a maximum height of H before falling back to the ground. The resistance to motion is equal to mkv^2 and g is the acceleration due to gravity.

(i) Show that $H = \frac{1}{2k}\ln\left(\frac{g + ku^2}{g}\right)$. [2]

(ii) P is a point at height h above the point of projection.

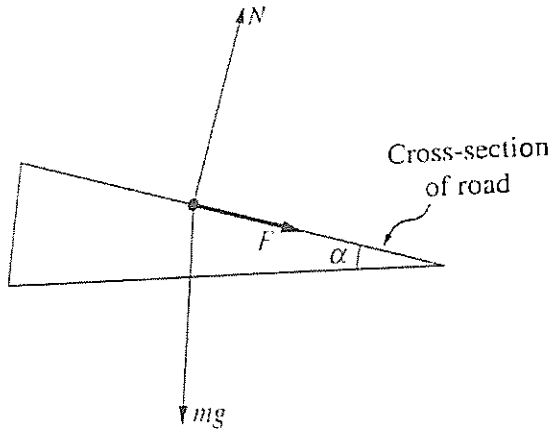
Let V be the speed of the object at P on its upward path when $x = h$.

Show that $h = \frac{1}{2k}\ln\left(\frac{g + ku^2}{g + kV^2}\right)$. [2]

(iii) During the downward path of the object it passes through P with half the speed of when it was first at P.

Show that $V = \sqrt{\frac{3g}{k}}$. [3]

c.



A road contains a bend that is part of a circle of radius, r. At the bend, the road is banked at an angle of α to the horizontal. A car travels around the bend at constant speed, v. Assume that the car is represented by a point of mass m, and that the forces acting on the car are the gravitational force mg, a sideways frictional force F (acting down the road) and a normal reaction N to the road.

(i) By resolving the horizontal and vertical components of force, find expressions for $F\cos\alpha$ and $F\sin\alpha$. [2]

(ii) Show that $F = \dfrac{m(v^2 - gr\tan\alpha)}{r}\cos\alpha$. [2]

(iii) Suppose that the radius of the bend is 200 metres and that the road is banked to allow cars to travel at 100 km/h with no sideways friction force. Take $g = 9.8\ ms^{-2}$.

Find the value of α. [1]

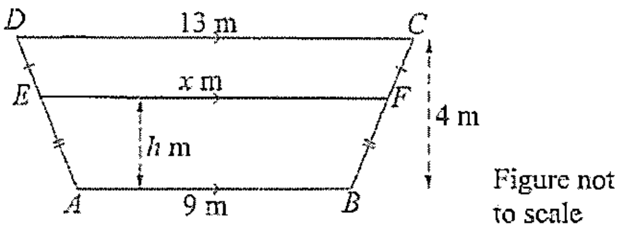
Question 16 (15 marks)

a. $P(x)$ is a polynomial of degree 5 such that $P(x) - 1$ is divisible by $(x-1)^3$ and $P(x)$ itself is divisible by x^3. Derive an expression for $P(x)$. [3]

b. (i) The diagram below shows a trapezium $ABCD$ whose parallel sides AB and DC are 9 metres and 13 metres respectively.

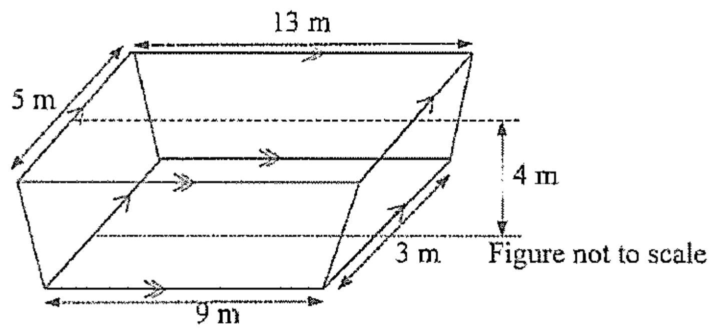
The distance between these sides is 4 metres and $AD = BC$.

EF is parallel to AB and the distance between them is h metres.



Show that $EF = (9 + h)$ metres. [2]

(ii) The trench in the diagram below has a rectangular base with sides 9 metres and 3 metres. Its top is also rectangular with dimensions 13 metres and 5 metres. The trench has a depth of 4 metres and each of its four side faces is an isosceles trapezium.



Find the volume of the trench. [4]

Question 16 continues on Page 13

c. (i) Show that if $y = mx + k$ is a tangent to the hyperbola $xy = c^2$,

then $k^2 + 4mc^2 = 0$. [3]

(ii) Hence, find the equations of the tangents from the point $(-1, -3)$

to the rectangular hyperbola $xy = 4$ and find their points of contact. [3]

End of Examination

GHS 2018 TRIAL HSC MATHEMATICS Ext. 2 SOLUTIONS

MC.

1. $z = 1 + i$, $z^8 = ?$

$= \sqrt{2} \operatorname{cis} \frac{\pi}{4}$

$z^8 = \left(\sqrt{2} \operatorname{cis} \frac{\pi}{4}\right)^8$

$= 2^4 (\cos 2\pi + i \sin 2\pi)$

$= 16$ $\boxed{D}$

2. $1 \leq |z| \leq 2$ $\boxed{B}$

3. $x^4 + px + q = 0$

$\alpha^4 + p\alpha + q = 0$

$\beta^4 + p\beta + q = 0$

$\gamma^4 + p\gamma + q = 0$

$\delta^4 + p\delta + q = 0$

$\alpha^4 + \beta^4 + \gamma^4 + \delta^4 = -4p(\alpha + \beta + \gamma + \delta) - 4q$

$= -4p(0) - 4q$

$= -4q$ $\boxed{A}$

4. $x^y = e$

$\log x^y = \log e$

$y \log x = 1$

$y \cdot \frac{1}{x} + \log x \frac{dy}{dx} = 0$

$\frac{y}{x} + \log x \frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-y}{x \log x}$ $\boxed{A}$

5. $$\int_0^{\pi/2} \frac{1}{1+\sin x}\,dx$$

$t = \tan \frac{x}{2}$

$$= \int_0^{1} \frac{1}{1+\frac{2t}{1+t^2}} \cdot \frac{2\,dt}{1+t^2}$$

$\frac{dt}{dx} = \frac{1}{2}\sec^2 \frac{x}{2}$

$dx = \frac{2dt}{\sec^2 \frac{x}{2}}$

$$= \int_0^{1} \frac{2}{(1+t^2+2t)}\,dt$$

$dx = \frac{2dt}{1+t^2}$

$$= \int_0^{1} \frac{2}{(1+t)^2}\,dt \qquad \boxed{D}$$

when $x = \pi/2$, $t = 1$

when $x = 0$, $t = 0$

6. $$\frac{x^2}{144} - \frac{y^2}{25} = 1$$

$b^2 = a^2(e^2 - 1)$

$e^2 - 1 = \frac{25}{144}$

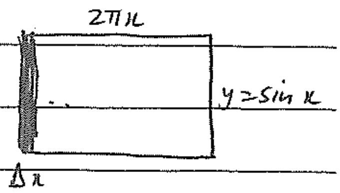
$e^2 = \frac{169}{144}$

$e = \frac{13}{12}$

Directrices : $x = \pm \frac{a}{e}$

$= \pm \frac{144}{13} \qquad \boxed{D}$

7. $$V = \lim_{\Delta x \to 0} \sum_{x=0}^{\pi/2} 2\pi x \sin x \,\Delta x$$



$$= 2\pi \int_0^{\pi/2} x \sin x \,dx$$

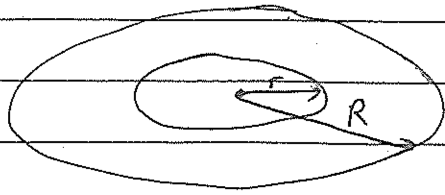
$$= 2\pi \left[(x\cos x)_0^{\pi/2} + \int_0^{\pi/2} \cos x \,dx \right]$$

$$= 2\pi \left[\sin x \right]_0^{\pi/2}$$

$$= 2\pi \qquad \boxed{D}$$

8. $y = 4x^2 - x^4$

$(x^4 - 4x^2 + 4) = -y + 4$



$(x^2 - 2)^2 = 4 - y$

$x^2 - 2 = \pm\sqrt{4-y}$

$x^2 = 2 + \sqrt{4-y} = R^2$; $x^2 = 2 - \sqrt{4-y} = r^2$

$A = \pi(R^2 - r^2)$

$= \pi\left(2 + \sqrt{4-y} - (2 - \sqrt{4-y})\right)$

$= \pi(2\sqrt{4-y})$

$= 2\pi\sqrt{4-y}$

$\Delta V = 2\pi\sqrt{4-y}\ \Delta y$

$V = \lim_{\Delta y \to 0} \sum_{y=0}^{4} 2\pi\sqrt{4-y}\ \Delta y$

$V = 2\pi \int_0^4 \sqrt{4-y}\ dy$ $\boxed{A}$

9. $\omega = 1200$ rpm

$= \frac{1200 \times 2\pi}{60}$

$= 40\pi$ radians / sec

$v = r\omega$

$= 2 \times 40\pi$

$= 80\pi$

≈ 251 m/s $\boxed{C}$

10. $m\ddot{x} = -2m(v + v^2)$

$\ddot{x} = -2(v + v^2)$

$v\frac{dv}{dx} = -2(v + v^2)$

$\frac{dx}{dv} = \frac{-1}{2(1+v)}$

$x = -\frac{1}{2}\int \frac{1}{1+v}\, dv$ $\boxed{A}$

Question 11

a) $z = p + 2i \quad ; \quad w = 1 - 2i$

i) $\frac{z}{w} = \frac{p+2i}{1-2i} \times \frac{1+2i}{1+2i}$

$= \frac{p + 2pi + 2i - 4}{1+4}$

$= \frac{p-4}{5} + \frac{2(p+1)}{5} i$ (2)

ii) $\left|\frac{z}{w}\right| = 13$

$\frac{|z|}{|w|} = 13$

$\frac{p^2+4}{5} = 169$

$p^2 = 841$

$p = \pm\sqrt{841} = \pm 29$ (2)

b) $z = 1 - \sqrt{3}i$

i) $|z| = \sqrt{1+3} = 2$

$\arg z = \frac{-\pi}{3}$ (2)

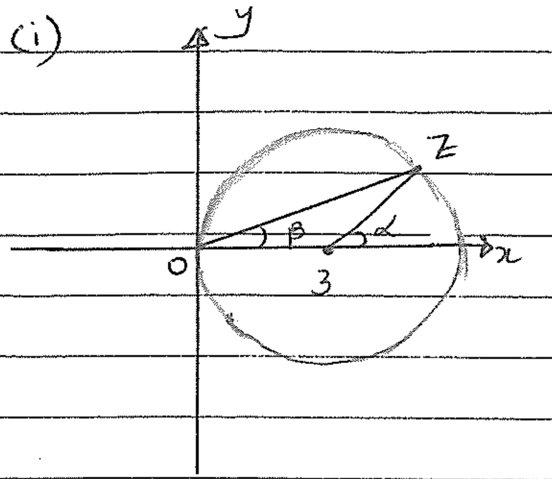
ii) $z = 2 \operatorname{cis} \frac{-\pi}{3}$

$z^6 = \left(2 \operatorname{cis} \frac{-\pi}{3}\right)^6$

$= 2^6 (\cos -2\pi + i\sin -2\pi)$

$= 64$ (2)

c. (i)



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ii) $\alpha = \arg(z-3)$; $\beta = \arg z$

$\alpha = 2\beta$ (< at the centre is twice the < at the circumference subtended on same arc)

$\therefore \arg(z-3) = 2\arg z$

2

d. $2+i$ is a root

$\therefore$ $2-i$ is also a root ($P(x)$ has real coefficients)

$\therefore$ x^2-4x+5 is a factor.

$$
\begin{array}{r|l}
 & x^2 - 2x - 4 \\
x^2-4x+5 & x^4 - 6x^3 + 9x^2 + 6x - 20 \\
 & x^4 - 4x^3 + 5x^2 \\
 & -2x^3 + 4x^2 + 6x \\
 & -2x^2 + 8x^2 - 10x \\
 & -4x^2 + 16x - 20 \\
 & -4x^2 + 16x - 20
\end{array}
$$

$P(x) = (x^2-4x+5)(x^2-2x-4)$

Roots of $P(x)$ are $2+i$, $2-i$, $1+\sqrt{5}$, $1-\sqrt{5}$

Factors : $(x-(2+i))$, $(x-(2-i))$, $(x-(1+\sqrt{5}))$, $(x-(1-\sqrt{5}))$

3

Question 12

a) i)

$$\int x e^{-x}\,dx$$

$u = x \qquad v = -e^{-x}$

$u' = 1 \qquad v' = e^{-x}$

$$= -xe^{-x} - \int -e^{-x}\,dx$$

$$= -xe^{-x} + \int e^{-x}\,dx$$

$$= -xe^{-x} - e^{-x} + C$$

2

ii)

$$\int_0^{\pi/2} \frac{1}{2 - \cos x + 2\sin x}\,dx$$

let $t = \tan \frac{x}{2}$

$dt = \frac{1}{2}\sec^2 \frac{x}{2}\,dx$

$= \frac{1}{2}(1+t^2)\,dx$

$dx = \frac{2dt}{1+t^2}$

When $x = 0$, $t = 0$

$x = \pi/2 \qquad t = 1$

$$= \int \frac{1}{2 - \left(\frac{1-t^2}{1+t^2}\right) + 2\left(\frac{2t}{1+t^2}\right)} \cdot \frac{2dt}{1+t^2}$$

$$= \int_0^1 \frac{2\,dt}{2(1+t^2) - (1-t^2) + 4t}$$

$$= \int_0^1 \frac{2}{1 + 3t^2 + 4t}\,dt$$

$$= \int_0^1 \frac{2}{(3t+1)(t+1)}\,dt$$

$\frac{A}{3t+1} + \frac{B}{t+1} = \frac{2}{(3t+1)(t+1)}$

$A(t+1) + B(3t+1) = 2$

substitute $t = -1$

$-2B = 2 \Rightarrow B = -1$

substitute $t = -\frac{1}{3}$

$\frac{2}{3}A = 2$

$A = 3$

$$= \int_0^1 \left(\frac{3}{3t+1} - \frac{1}{t+1}\right)dt$$

$$= \Big[\log(3t+1) - \log(t+1)\Big]_0^1$$

$$= (\log 4 - \log 2) - (\log 1 - \log 1)$$

$$= \log 2$$

3

iii) $\int_0^{\pi/6} \sec^3 2\theta \, d\theta$

$= \int_0^{\pi/6} \sec^2 2\theta \cdot \sec 2\theta \, d\theta$

$= \left[\frac{\tan 2\theta}{2} \cdot \sec 2\theta\right]_0^{\pi/6} - \int_0^{\pi/6} \sec 2\theta \tan^2 2\theta \, d\theta$

$= \left(\frac{\tan \pi/3}{2} \cdot \sec \pi/3\right) - \int_0^{\pi/6} \sec 2\theta(\sec^2 2\theta - 1) \, d\theta$

$= \frac{\sqrt{3}}{2} \cdot 2 - \int_0^{\pi/6} (\sec^3 2\theta + \sec 2\theta) \, d\theta$

$= \sqrt{3} - \int_0^{\pi/6} \sec^3 2\theta \, d\theta + \int_0^{\pi/6} \sec 2\theta \, d\theta$

let $u = \sec 2\theta$

$= \frac{1}{\cos 2\theta}$

$= (\cos 2\theta)^{-1}$

$u' = -1(\cos 2\theta)^{-2} \cdot -2\sin 2\theta$

$= \frac{2\sin 2\theta}{\cos^2 2\theta}$

$= 2\sec 2\theta \cdot \tan 2\theta$

$v' = \sec^2 2\theta$

$v = \frac{\tan 2\theta}{2}$

$2\int_0^{\pi/6} \sec^3 2\theta \, d\theta = \sqrt{3} + \int_0^{\pi/6} \sec 2\theta \, d\theta$

$= \sqrt{3} + \int_0^{\pi/6} \frac{\sec 2\theta(\sec 2\theta + \tan 2\theta)}{(\sec 2\theta + \tan 2\theta)} \, d\theta$

$= \sqrt{3} + \frac{1}{2}\int_0^{\pi/6} \frac{2\sec^2 2\theta + 2\sec 2\theta \tan 2\theta}{\sec 2\theta + \tan 2\theta} \, d\theta$

$= \sqrt{3} + \frac{1}{2} \ln \left[\sec 2\theta + \tan 2\theta\right]_0^{\pi/6}$

$= \sqrt{3} + \frac{1}{2} \ln (2 + \sqrt{3})$

$\therefore \int_0^{\pi/6} \sec^3 2\theta \, d\theta = \frac{\sqrt{3}}{2} + \frac{1}{4} \ln (2 + \sqrt{3})$

3

b) i) $\dfrac{5x^3-3x^2+2x-1}{x^2(x^2+1)} = \dfrac{a}{x} + \dfrac{b}{x^2} + \dfrac{cx+d}{x^2+1}$

$5x^3-3x^2+2x-1 = ax(x^2+1)+b(x^2+1)+(cx+d)x^2$
$= (a+c)x^3+(b+d)x^2+ax+b$

comparing :

Constant term $\Rightarrow$ $b=-1$

Coefficients of x : $a=2$

Coefficient of x^2 : $b+d=-3$
$d=-2$

Coefficient of x^3 : $a+c=5$
$c=3$

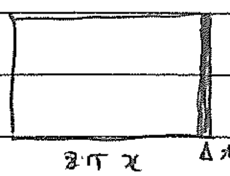
$\therefore a=2,\ b=-1,\ c=3,\ d=-2$ 2

ii) $\displaystyle\int \frac{5x^3-3x^2+2x-1}{x^2(x^2+1)}\,dx = \int\left(\frac{2}{x}-\frac{1}{x^2}+\frac{3x-2}{x^2+1}\right)dx$

$\displaystyle = \int\left(\frac{2}{x}-\frac{1}{x^2}+\frac{3x}{x^2+1}-\frac{2}{x^2+1}\right)dx$

$\displaystyle = 2\log x + \frac{1}{x} + \frac{3}{2}\log(x^2+1) - 2\tan^{-1}x + c$

c) 2



$y_2 - y_1 = (x+1-(x-1)^2)$
$= -x^2+3x$

Point of intersection :

$(x-1)^2 = x+1$

$x^2-2x+1 = x+1$

$x^2-3x=0$

$x=0,3$

$\Delta V = 2\pi x(-x^2+3x)\Delta x$

$\displaystyle V = \lim_{\Delta x\to 0}\sum_{x=0}^{3} 2\pi(-x^3+3x^2)\Delta x$

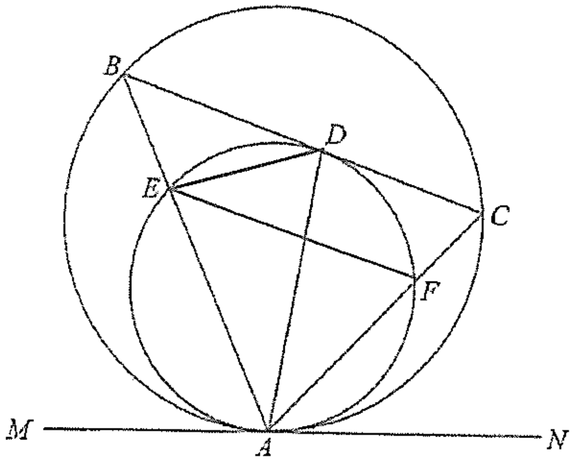
$\displaystyle = 2\pi\int_0^3(-x^3+3x^2)\,dx$

$\displaystyle = 2\pi\left[\frac{-x^4}{4}+x^3\right]_0^3$ 3

$\displaystyle = \frac{27\pi}{2}\ u^3$

Question 13

a) i)



ii) Construct EF, ED.

$\angle ABC = \angle NAC$ ($\angle$ in the alternate segment)

$\angle AEF = \angle NAC$ ($\angle$ in the alternate segment)

$\therefore EF \parallel BC$ (corresponding $\angle$s on transversal AB)

$\angle DEF = \angle BDE$ (alternate $\angle$s, $EF \parallel BC$)

$\angle DEF = \angle DAF$ ($\angle$s subtended by arc DF at the circumference of circle AEF)

$\angle BDE = \angle DAE$ ($\angle$ in alternate segment in circle AEF)

$\therefore \angle DAF = \angle DAE$

$\therefore$ AD bisects $\angle BAC$.

b) $\frac{x^2}{100} + \frac{y^2}{75} = 1$ $\qquad a = 10, \quad b = 5\sqrt{3}$

i) $b^2 = a^2(1-e^2)$

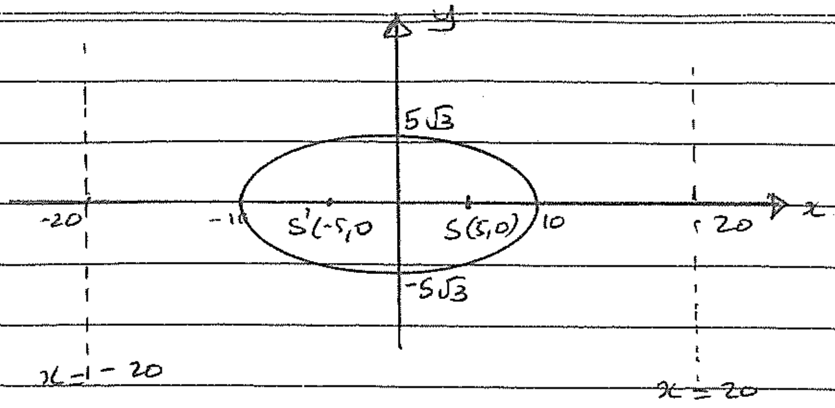
$75 = 100(1-e^2)$

$e^2 = \frac{1}{4}$

$e = \frac{1}{2}$ $\qquad$ Foci : $(\pm ae, 0)$

Foci : $(\pm 5, 0)$; Directrices : $x = \pm \frac{a}{e}$

Directrices : $x = \pm 20$



ii) $\frac{x^2}{100} + \frac{y^2}{75} = 1$

Differentiating implicitly,

$$\frac{2x}{100} + \frac{2y}{75}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{3x}{4y}.$$

At $(5, 7\frac{1}{2})$

$$m_{tangent} = \frac{-3(5)}{4(7.5)} = \frac{-1}{2}$$

$$\therefore m_{normal} = 2$$

E_{normal} : $y - 7.5 = 2(x-5)$

$y = 2x - 2.5$

iii) Circle and ellipse have common tangent at P

$\therefore$ Normal at P also at right $\angle$s to circle at P

$\therefore$ Normal at P passes through centre, C, of circle.

$\therefore$ C lies on $y = 2x - 2.5$ — 1

Tangent at Q $(5, -7.5)$ has gradient, $m = \frac{-3(5)}{4(-7.5)} = \frac{1}{2}$

$$\therefore m_{normal\ at\ Q} = -2$$

$E_{normal\ at\ Q}$: $y = -2x + 2.5$

$\Rightarrow$ also passes through centre, C, of circle

$\therefore$ C lies on line $y = -2x + 2.5$ — 2

Solving ① & ② simultaneously

$$2x - 2.5 = -2x + 2.5$$

$$4x = 5$$

$$x = \frac{5}{4} \quad , y = 0$$

$$\therefore C\left(\frac{5}{4}, 0\right)$$

Radius of circle is CP

$$\therefore r = \sqrt{\left(\frac{15}{4}\right)^2 + \left(\frac{15}{2}\right)^2}$$

$$= \sqrt{\frac{1125}{16}}$$

$\therefore$ Equation of circle : $\left(x - \frac{5}{4}\right)^2 + y^2 = \frac{1125}{16}$

c) i) $xy = c^2$

differentiating; $y + x\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{-y}{x} \qquad \text{At } T\left(ct, \frac{c}{t}\right)$$

$$m_{tangent} = \frac{-1}{t^2}$$

$$E_{tangent} : y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$$

$$t^2y - ct = x + ct$$

$$x + t^2y = 2ct$$

ii) x int $\Rightarrow y = 0$ $\qquad$ y int $\Rightarrow x = 0$

$\therefore x = 2ct$ $\qquad$ $t^2y = 2ct$

$\therefore A(2ct, 0)$ $\qquad$ $y = \frac{2c}{t}$

Midpoint of AB $= \left(ct, \frac{c}{t}\right) = T$ $\qquad$ $B\left(0, \frac{2c}{t}\right)$

$\angle BOA = 90^\circ$ $\therefore$ AB is the diameter of a circle passing through O. T is the midpoint of this diameter and therefore the centre of the circle passing through O, A and B.

Question 14

a) i) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

RHS $= \int_0^a f(a-x)\,dx$ let $u = a - x$

$du = -dx$

$= -\int_a^0 f(u)\,du$ when $x = 0, u = a$

$x = a, u = 0$

$= \int_0^a f(u)\,du$

$= \int_0^a f(x)\,dx$

$=$ LHS.

ii) $\int_0^2 x(2-x)^5\,dx$

$= \int_0^2 (2-x)\,x^5\,dx$ (from (i))

$= \int_0^2 \left(2x^5 - x^6\right) dx$

$= \left[\frac{x^6}{3} - \frac{x^7}{7}\right]_0^2$

$= \frac{64}{3} - \frac{128}{7}$

$= \frac{64}{21}$

b) i) $I_{2n+1} = \int_0^1 x^{2n+1} e^{x^2} dx$

$= \int_0^1 x^{2n} \cdot x e^{x^2} dx$

$u = x^{2n}$

$u' = 2n\, e^{2n-1}$

$v = \frac{1}{2} e^{x^2}$

$v' = x e^{x^2}$

$= \left[\frac{1}{2} x^{2n} e^{x^2}\right]_0^1 - \int_0^1 \frac{1}{2} e^{x^2} \cdot 2n\, x^{2n-1} dx$

$= \frac{1}{2} e^1 - n \int_0^1 x^{2n-1} e^{x^2} dx$

$I_{2n+1} = \frac{1}{2} e - n I_{2n-1}$

ii) $I_5 = \frac{1}{2} e - 2 I_3$

$= \frac{1}{2} e - 2\left(\frac{1}{2} e - I_1\right)$

$= \frac{1}{2} e - e + 2\int_0^1 x e^{x^2} dx$

$= -\frac{e}{2} + \left[e^{x^2}\right]_0^1$

$= \frac{-e}{2} + e^1 - e^0$

$= \frac{e}{2} - 1$

c) i) $(\cos\theta + i\sin\theta)^3$

$= \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3$

$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta.$

ii) Using De Moivre's Thm,

$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$

Equating real parts,

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$

$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$4\cos^3\theta = \cos 3\theta + 3\cos\theta$

$\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$

iii) $4\cos^3\theta - 3\cos\theta = 1$

$\therefore \cos 3\theta = 1$ (from (i))

$3\theta = 2k\pi$

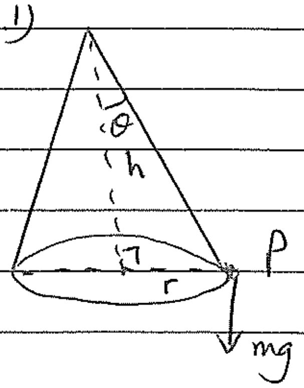
$\theta = \frac{2k\pi}{3}$

Smallest value occurs when $k=1$

$\therefore \theta = \frac{2\pi}{3}$

Question 15

a) i)



Resolving forces vertically & horizontally

$T\cos\theta = mg$ —— ①

$T\sin\theta = mr\omega^2$ —— ②

② ÷ ①

$\frac{T\sin\theta}{T\cos\theta} = \frac{mr\omega^2}{mg}$

$\tan\theta = \frac{r\omega^2}{g}$

$\tan\theta = \frac{r}{h}$

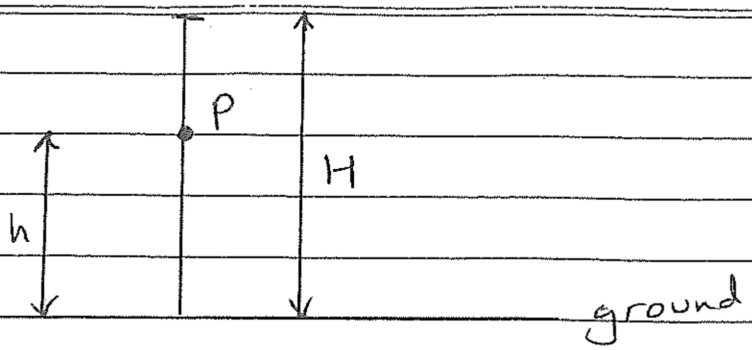
$\therefore \frac{r\omega^2}{g} = \frac{r}{h}$

$\omega^2 = \frac{g}{h}$

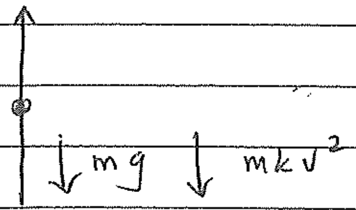
$\omega = \sqrt{\frac{g}{h}}$

ii) Period $= \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\frac{g}{h}}} = \frac{2\pi\sqrt{h}}{\sqrt{g}}$

b)



i)



$$m\ddot{x} = -mg - mkv^2$$

$$\ddot{x} = -(g + kv^2)$$

$$v\frac{dv}{dx} = -(g + kv^2)$$

$$dx = \frac{-v}{g + kv^2}\,dv$$

at $x = H$, $v = 0$

$x = 0$, $v = u$

$$\therefore \int_0^H dx = -\frac{1}{2k}\int_u^0 \frac{2kv}{g + kv^2}\,dv$$

$$H = \frac{1}{2k}\int_0^u \frac{2kv}{g + kv^2}\,dv$$

$$= \frac{1}{2k}\Big[\ln(g + kv^2)\Big]_0^u$$

$$= \frac{1}{2k}\left(\ln(g + ku^2) - \ln g\right)$$

$$H = \frac{1}{2k}\ln\left(\frac{g + ku^2}{g}\right)$$

ii) when $x = h$, $v = V$

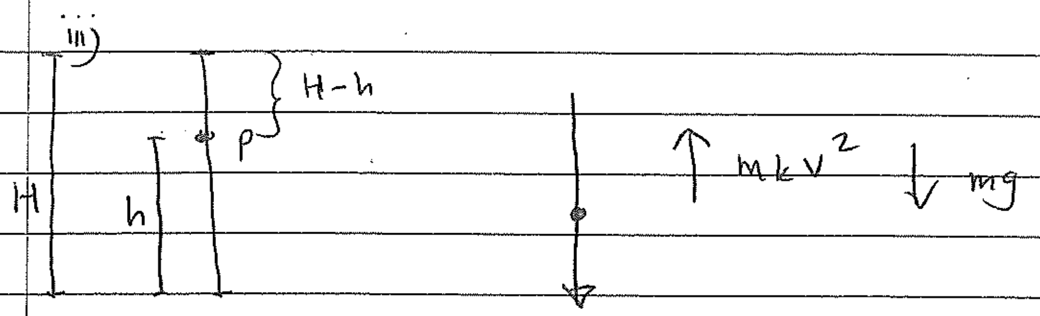
$$\int_0^h dx = -\frac{1}{2k}\int_u^V \frac{2kv}{g+kv^2}\,dv \qquad \text{(from (i))}$$

$$h = \frac{1}{2k}\Big[\ln(g+kv^2)\Big]_V^u$$

$$= \frac{1}{2k}\Big[\ln(g+ku^2) - \ln(g+kV^2)\Big]$$

$$= \frac{1}{2k}\ln\left(\frac{g+ku^2}{g+kV^2}\right)$$

iii)



$$m\ddot{x} = mg - mkv^2$$

$$\ddot{x} = g - kv^2$$

$$v\frac{dv}{dx} = g - kv^2$$

$$dx = \frac{v}{g-kv^2}\,dv$$

at $x = 0$, $v = 0$

$x = H - h$, $v = \frac{1}{2}V$

$$\int_0^{H-h} dx = \frac{-1}{2k}\int_0^{V/2}\left(\frac{-2kv}{g-kv^2}\right)dv$$

$$H - h = \frac{-1}{2k}\Big[\ln(g-kv^2)\Big]_0^{V/2}$$

$$= \frac{-1}{2k}\left[\ln\left(g - \frac{kV^2}{4}\right) - \ln g\right] = \frac{-1}{2k}\left(\ln\left(\frac{g-\frac{kV^2}{4}}{g}\right)\right)$$

$$H - h = \frac{-1}{2k}\ln\left(\frac{4g-kV^2}{4g}\right) \quad \text{—— } 7$$

Also, using H from (i) and h from (ii)

$$H-h = \frac{1}{2k}\ln\left(\frac{g+ku^2}{g}\right) - \frac{1}{2k}\ln\left(\frac{g+ku^2}{g+kV^2}\right)$$

$$H-h = \frac{1}{2k}\ln\left(\frac{g+kV^2}{g}\right) \quad \text{——— (2)}$$

From (1) & (2)

$$\ln\left(\frac{g+kV^2}{g}\right) = -\ln\left(\frac{4g-kV^2}{4g}\right) = \ln\left(\frac{4g-kV^2}{4g}\right)^{-1}$$

$$\frac{g+kV^2}{g} = \frac{4g}{4g-kV^2}$$

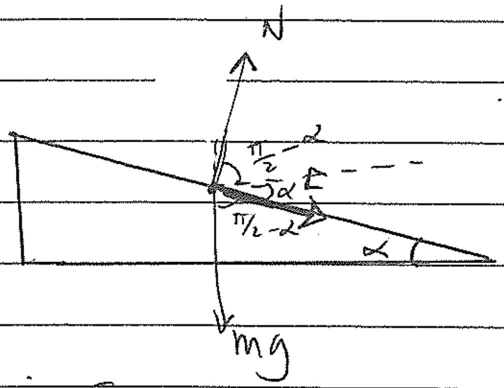
$$4g^2+4gkV^2-gkV^2-k^2V^4 = 4g^2$$

$$3gkV^2 = k^2V^4$$

$$V^2 = \frac{3g}{k}$$

$$V = \sqrt{\frac{3g}{k}}$$

c)



Resolving Forces

Horizontally

$$\frac{mV^2}{r} = F\cos\alpha + N\cos\left(\frac{\pi}{2}-\alpha\right)$$

$$= F\cos\alpha + N\sin\alpha$$

$$\therefore F\cos\alpha = \frac{mV^2}{r} - N\sin\alpha \quad \text{——— (1)}$$

Vertically

$$N\cos\alpha - mg - F\cos\left(\tfrac{\pi}{2}-\alpha\right) = 0$$

$$N\cos\alpha - mg - F\sin\alpha = 0$$

$$\therefore F\sin\alpha = N\cos\alpha - mg \quad \text{——— (2)}$$

ii) ① $\times \cos\alpha$ + ② $\times \sin\alpha$

$$F\cos^2\alpha = \frac{mv^2}{r}\cos\alpha - N\sin\alpha\cos\alpha$$

$$+$$

$$F\sin^2\alpha = N\sin\alpha\cos\alpha - mg\sin\alpha$$

$$= F(\cos^2\alpha + \sin^2\alpha) = \frac{mv^2}{r}\cos\alpha - mg\sin\alpha$$

$$F = \frac{m}{r}\cos\alpha\left(v^2 - \frac{gr\sin\alpha}{\cos\alpha}\right)$$

$$= \frac{m(v^2 - gr\tan\alpha)}{r}\cos\alpha$$

iii) $r = 200\text{m}$, $g = 9.8\text{ m/s}^2$, $F = 0$

$$v = 100\text{ km/h} = \frac{100 \times 1000}{3600} = \frac{250}{9}\text{ m/s}$$

From (ii) $0 = \frac{m(v^2 - gr\tan\alpha)}{r}\cos\alpha$

$$v^2 - gr\tan\alpha = 0$$

$$\tan\alpha = \frac{v^2}{gr}$$

$$= \frac{(250/9)^2}{9.8 \times 200}$$

$$= 21.486$$

$$\alpha \approx 21°29'$$

Question 16

a) $P(x) = x^3(ax^2+bx+c) = ax^5+bx^4+cx^3$

Let $Q(x) = P(x) - 1 = ax^5+bx^4+cx^3 - 1$

Since $Q(x)$ is divisible by $(x-1)^3$, x is a triple root

$Q'(x) = 5ax^4 + 4bx^3 + 3cx^2$

$Q'(1) = 5a + 4b + 3c = 0$ —— ①

$Q''(x) = 20ax^3 + 12bx^2 + 6cx$

$Q''(1) = 20a + 12b + 6c = 0$ —— ②

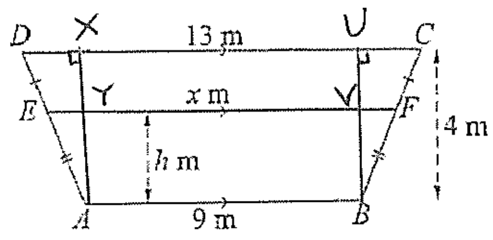
$Q(1) = a + b + c = 0$ —— ③

Solving ①, ② & ③ simultaneously,

$a = 6, \ b = -15, \ c = 10$

$\therefore P(x) = 6x^5 - 15x^4 + 10x^3$

b)



Draw AX and $BU \perp DC$

$\triangle AXD \equiv \triangle BUC$ (RHS)

$DX = CU = 2$

$\dfrac{EY}{AY} = \dfrac{DX}{AX}$ (ratio of matching sides of $\equiv \Delta s$)

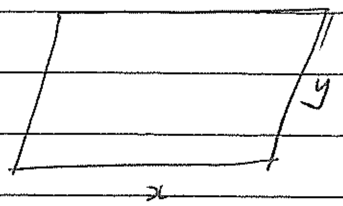
$\dfrac{EY}{h} = \dfrac{2}{4}$

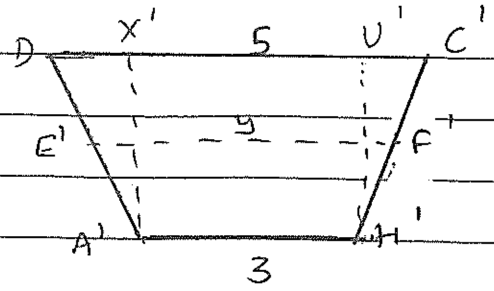
$EY = \dfrac{h}{2}$; Similarly, $VF = \dfrac{h}{2}$

$EF = EY + 9 + VF$

$x = 9 + h$

ii) Cross-sections // to the base will be rectangles





From (i) $x = 9 + h$

$D'X' = C'U' = 1$

Using ||| Δs

$\frac{E'Y'}{h} = \frac{1}{4}$

$E'Y' = \frac{h}{4}$

$y = 3 + 2 \times \frac{h}{4}$

$y = 3 + \frac{h}{2}$

Area of cross-section

$= (9+h)\left(3 + \frac{h}{2}\right)$

$= \frac{h^2}{2} + \frac{15h}{2} + 27$

$$\Delta V = \lim_{h \to 0} \sum_{h=0}^{4} \left(\frac{h^2}{2} + \frac{15h}{2} + 27\right)\Delta h$$

$$V = \int_0^4 \left(\frac{h^2}{2} + \frac{15h}{2} + 27\right) dh$$

$$= \left[\frac{h^3}{6} + \frac{15h^2}{4} + 27h\right]_0^4$$

$$V = 178\tfrac{2}{3} \text{ m}^3$$

c) i) $y = mx + k$ —— ①

$xy = c^2$ —— ②

Solving ① & ②

$x(mx+k) = c^2$

$mx^2 + kx = c^2$

$mx^2 + kx - c^2 = 0$

$\Delta = k^2 - 4mc^2$

Since $y = mx + k$ is tangent $\Delta = 0$

$\therefore k^2 - 4mc^2 = 0$

ii) The equation of the line through $(-1, -3)$ with gradient m is:

$y + 3 = m(x+1)$

$y = mx + m - 3$

$y = mx + k$ where $k = m - 3$.

This is tangent to $xy = 4$ if $k^2 + 16m = 0$

$(m-3)^2 + 16m = 0$

$m^2 + 10m + 9 = 0$

$(m+9)(m+1) = 0$

$m = -9$ or $m = -1$

Equation of tangent $y = mx + m - 3$

$m = -9$; $y = -9x - 12$

$m = -1$, $y = -x - 4$

Solving $y = -9x - 12$ and $xy = 4 \Rightarrow x(-9x-12) = 4$

$9x^2 + 12x + 4 = 0$

$(3x+2)^2 = 0$

$x = -\frac{2}{3}$, $y = -6$

$(-\frac{2}{3}, -6)$

Solving $y = -x - 4$ and $xy = 4$,

$x(-x-4) = 4$

$x^2 + 4x + 4 = 0$

$(x+2)^2 = 0$

$x = -2$, $y = -2 \Rightarrow (-2, -2)$